IX. Exercises on Vectors
vectorall.tex

A. DAY1

1) Evaluate

$$\sqrt{(-6)^2+5^2}$$

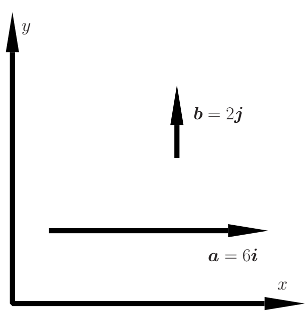
$$\begin{aligned}\sqrt{(-6)^2+5^2} &= \sqrt{36+25}\\ &= \sqrt{61}\end{aligned}$$

2) Evaluate

$$8\cdot(-4)+12\cdot(2)-(-1)\cdot(3)$$

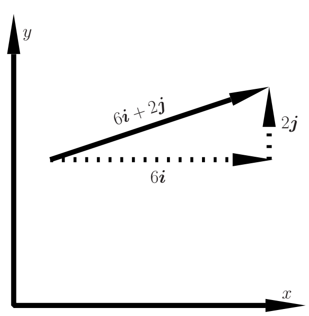
$$\begin{aligned}&8\cdot(-4)+12\cdot(2)-(-1)\cdot(3)\\ &= -32+24+3\\ &= -5\end{aligned}$$

3) Draw the vectors $\boldsymbol{a} = 6\boldsymbol{i}$ and $\boldsymbol{b} = 2\boldsymbol{j}$.

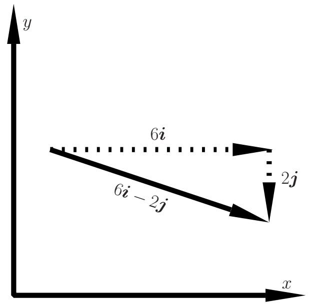


Then find and draw

a) $\boldsymbol{a}+\boldsymbol{b}$



b) $\boldsymbol{a}-\boldsymbol{b}$



4) If $\boldsymbol{p} = 9\boldsymbol{i} - 7\boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{q} = -8\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ express

a) $\boldsymbol{p} + \boldsymbol{q}$ in terms of $\boldsymbol{i}$,$\boldsymbol{j}$, and $\boldsymbol{k}$

$$\boldsymbol{p} + \boldsymbol{q} = \begin{pmatrix} 9 \\ -7 \\ 5 \end{pmatrix} + \begin{pmatrix} -8 \\ 3 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 9-8 \\ -7+3 \\ 5-2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$$

Therefore $\boldsymbol{p} + \boldsymbol{q} = \boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}$

b) $\boldsymbol{p} - \boldsymbol{q}$ in terms of $\boldsymbol{i}$,$\boldsymbol{j}$, and $\boldsymbol{k}$

$$\boldsymbol{p} - \boldsymbol{q} = \begin{pmatrix} 9 \\ -7 \\ 5 \end{pmatrix} - \begin{pmatrix} -8 \\ 3 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 9-(-8) \\ -7-(+3) \\ 5-(-2) \end{pmatrix} = \begin{pmatrix} 9+8 \\ -7-3 \\ 5+2 \end{pmatrix} = \begin{pmatrix} 17 \\ -10 \\ 7 \end{pmatrix}$$

Therefore $\boldsymbol{p} - \boldsymbol{q} = 17\boldsymbol{i} - 10\boldsymbol{j} + 7\boldsymbol{k}$

5) Sketch the position vectors

$$\boldsymbol{p} = 3\boldsymbol{i} + 4\boldsymbol{j}$$

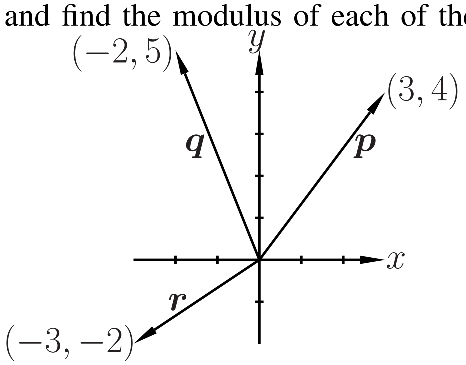
and

$$\boldsymbol{q} = -2\boldsymbol{i} + 5\boldsymbol{j}$$

and

$$\boldsymbol{r} = -3\boldsymbol{i} - 2\boldsymbol{j}$$

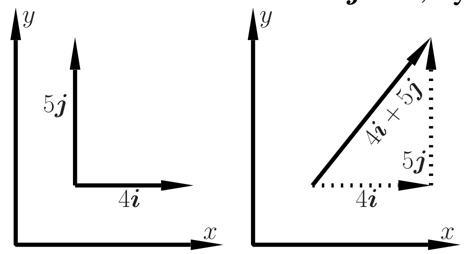
and find the modulus of each of the vectors.



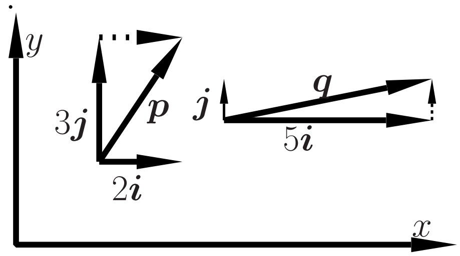
$$|\boldsymbol{p}| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$|\boldsymbol{q}| = \sqrt{(-2)^2 + 5^2} = \sqrt{4+25} = \sqrt{29}$$
$$|\boldsymbol{r}| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$$

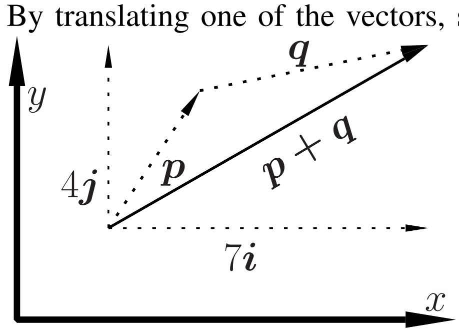
6) Draw the vectors $4\boldsymbol{i}$ and $5\boldsymbol{j}$ and, by translating the vectors so that they lie head to tail, the vector sum $4\boldsymbol{i}+5\boldsymbol{j}$.



7) Answer the following set of problems
   a) Draw an xy plane and show the vectors
$$\boldsymbol{p} = 2\boldsymbol{i} + 3\boldsymbol{j}$$
      and
$$\boldsymbol{q} = 5\boldsymbol{i} + \boldsymbol{j}$$
.

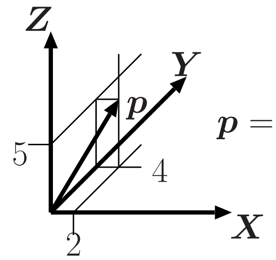


   b) Express $\boldsymbol{p}$ and $\boldsymbol{q}$ using column vector notation.
$$\boldsymbol{p} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \boldsymbol{q} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$
   c) By translating one of the vectors, show the sum $\boldsymbol{p}+\boldsymbol{q}$ on an xy plane.



   d) Express the resultant $\boldsymbol{p}+\boldsymbol{q}$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$
$$\boldsymbol{p}+\boldsymbol{q} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+5 \\ 3+1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$
      Therefore $\boldsymbol{p}+\boldsymbol{q} = 7\boldsymbol{i} + 4\boldsymbol{j}$

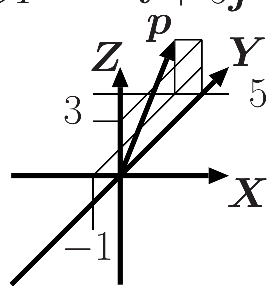
8) State the position vectors of the points with the coordinates
   a) P(2,4,5)
      $\boldsymbol{p} = \overrightarrow{OP} = 2\boldsymbol{i} + 4\boldsymbol{j} + 5\boldsymbol{k}$



$$\boldsymbol{p} = \vec{OP} = 2\ \boldsymbol{i} + 4\ \boldsymbol{j} + 5\ \boldsymbol{k}$$

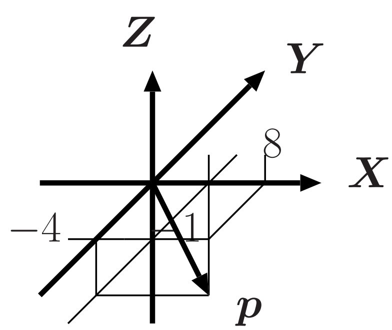
b) P(-1,5,3)
   $\boldsymbol{p} = \overrightarrow{OP} = -\boldsymbol{i} + 5\boldsymbol{j} + 3\boldsymbol{k}$

$$\boldsymbol{p} = \vec{OP} = -\boldsymbol{i} + 5\boldsymbol{j} + 3\boldsymbol{k}$$



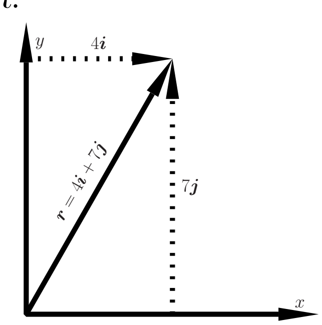
c) P(-2,-1,4)
   $\boldsymbol{p} = \overrightarrow{OP} = -2\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$

d) P(8,-4,-1)
   $\boldsymbol{p} = \overrightarrow{OP} = 8\boldsymbol{i} - 4\boldsymbol{j} - \boldsymbol{k}$

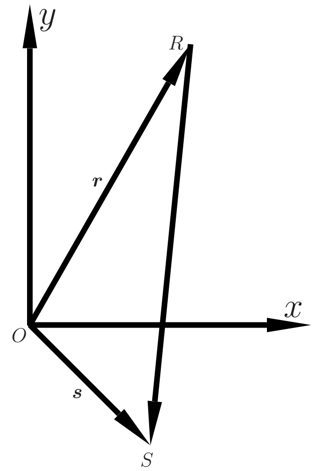


$$\boldsymbol{p} = \vec{OP} = 8\boldsymbol{i} - 4\ \boldsymbol{j} - \boldsymbol{k}$$

9) Consider the points $R = (4, 7)$ and $S = (3, -3)$.
   Find
   a) and draw the position vector of point R.



   b) the vector $\overrightarrow{RS}$ expressed in column notation



$$\begin{aligned}\overrightarrow{RS} &= \overrightarrow{RO} + \overrightarrow{OS} \\ &= -\overrightarrow{OR} + \overrightarrow{OS} \\ &= -\boldsymbol{r} + \boldsymbol{s} \\ &= -\begin{pmatrix} 4 \\ 7 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} -4+3 \\ -7-3 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ -10 \end{pmatrix}\end{aligned}$$

10) Consider the vectors $\boldsymbol{c} = 4\boldsymbol{i} - 5\boldsymbol{j} + 10\boldsymbol{k}$ and $\boldsymbol{d} = -6\boldsymbol{i} + \boldsymbol{j} - 7\boldsymbol{k}$ together with the scalar $\lambda = 3$. Find

   a) $\boldsymbol{c} - \lambda\boldsymbol{d}$ expressed in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$

$$\begin{aligned}\boldsymbol{c} - \lambda\boldsymbol{d} &= \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix} - 3 \cdot \begin{pmatrix} -6 \\ 1 \\ -7 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix} - \begin{pmatrix} 3\cdot(-6) \\ 3\cdot 1 \\ 3\cdot(-7) \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix} - \begin{pmatrix} -18 \\ 3 \\ -21 \end{pmatrix} \\ &= \begin{pmatrix} 4-(-18) \\ -5-3 \\ 10-(-21) \end{pmatrix} \\ &= \begin{pmatrix} 22 \\ -8 \\ 31 \end{pmatrix} \\ &= 22\boldsymbol{i} - 8\boldsymbol{j} + 31\boldsymbol{k}\end{aligned}$$

b) the magnitude of $\boldsymbol{c}$

$$\begin{aligned}|\boldsymbol{c}| &= \sqrt{4^2 + (-5)^2 + 10^2} \\ &= \sqrt{16 + 25 + 100} \\ &= \sqrt{141}\end{aligned}$$

c) a unit vector parallel to $\boldsymbol{c}$

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{c}}{|\boldsymbol{c}|} = \frac{1}{\sqrt{141}}(4\boldsymbol{i} - 5\boldsymbol{j} + 10\boldsymbol{k})$$

11) Point P has coordinates $(7, -4, -2)$. Point Q has coordinates $(-2, -5, -1)$

a) State the position vectors of P and Q

$$\boldsymbol{p} = 7\boldsymbol{i} - 4\boldsymbol{j} - 2\boldsymbol{k}$$

$$\boldsymbol{q} = -2\boldsymbol{i} - 5\boldsymbol{j} - \boldsymbol{k}$$

b) Find an expression for $\overrightarrow{PQ}$

$$\begin{aligned}&\overrightarrow{PQ} \\ &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= -\overrightarrow{OP} + \overrightarrow{OQ} \\ &= -\boldsymbol{p} + \boldsymbol{q} \\ &= -\begin{pmatrix} 7 \\ -4 \\ -2 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -7-2 \\ 4-5 \\ 2-1 \end{pmatrix} \\ &= \begin{pmatrix} -9 \\ -1 \\ 1 \end{pmatrix} \\ &= -9\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}\end{aligned}$$

Or alternatively,

$$\begin{aligned}&\overrightarrow{PQ} \\ &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= -\overrightarrow{OP} + \overrightarrow{OQ} \\ &= -\boldsymbol{p} + \boldsymbol{q} \\ &= -(7\boldsymbol{i} - 4\boldsymbol{j} - 2\boldsymbol{k}) + (-2\boldsymbol{i} - 5\boldsymbol{j} - \boldsymbol{k}) \\ &= -7\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k} - 2\boldsymbol{i} - 5\boldsymbol{j} - \boldsymbol{k} \\ &= (-7-2)\boldsymbol{i} + (4-5)\boldsymbol{j} + (2-1)\boldsymbol{k} \\ &= (-7-2)\boldsymbol{i} + (-1)\boldsymbol{j} + (1)\boldsymbol{k} \\ &= -9\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}\end{aligned}$$

c) Find $\left|\overrightarrow{PQ}\right|$
Since

$$\overrightarrow{PQ} = -9\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k},$$

$$\begin{aligned} \left|\overrightarrow{PQ}\right| \\ = \sqrt{9^2 + (-1)^2 + 1^2} \\ = \sqrt{81 + 1 + 1} \\ = \sqrt{83} \end{aligned}$$

B. DAY2

12) Simplify

$$2^8 \cdot 2^8$$

$$\begin{aligned} & 2^8 \cdot 2^8 \\ &= 2^{8+8} \\ &= 2^{16} \end{aligned}$$

13) Simplify

$$3x + 2y - z + t(7x + 5y + z - (3x + 2y - z))$$

$$\begin{aligned} & 3x + 2y - z + t(7x + 5y + z - (3x + 2y - z)) \\ &= 3x + 2y - z + t(7x + 5y + z - 3x - 2y + z) \\ &= 3x + 2y - z + t(4x + 3y + 2z) \\ &= 3x + 2y - z + 4tx + 3ty + 2tz \\ &= x(4t + 3) + y(3t + 2) + z(2t - 1) \end{aligned}$$

14) Evaluate

$$9 - 7(12 - 5^2)$$

$$\begin{aligned} & 9 - 7(12 - 5^2) \\ &= 9 - 7(12 - 25) \\ &= 9 - 7(-13) \\ &= 9 + 7 \cdot 13 \\ &= 9 + 91 \\ &= 100 \end{aligned}$$

15) Find the angle between the vectors

$$\boldsymbol{p} = -3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$$

and

$$\boldsymbol{q} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$$

$$\boldsymbol{p} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}, \boldsymbol{q} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$
$$= (-3) \cdot 2 + (-1) \cdot (3) + (2) \cdot (1)$$
$$= -6 - 3 + 2 = -7$$

On the other hand

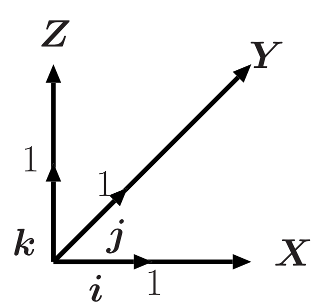
$$|\boldsymbol{p}|$$
$$= \sqrt{(-3)^2 + (-1)^2 + 2^2}$$
$$= \sqrt{9 + 1 + 4}$$
$$= \sqrt{14}$$

$$|\boldsymbol{q}|$$
$$= \sqrt{(2)^2 + (3)^2 + 1^2}$$
$$= \sqrt{4 + 9 + 1}$$
$$= \sqrt{14}$$

Therefore

$$\boldsymbol{p} \cdot \boldsymbol{q} = |\boldsymbol{p}| \cdot |\boldsymbol{q}| \cdot \cos\theta$$
$$\therefore -7 = \sqrt{14} \cdot \sqrt{14} \cos\theta$$
$$\therefore -7 = 14 \cos\theta$$
$$\therefore \frac{-7}{14} = \frac{14 \cos\theta}{14}$$
$$\therefore \frac{-1}{2} = \cos\theta$$
$$\therefore \theta = 120^o = \frac{2\pi}{3}$$

16) Find $\boldsymbol{i} \cdot \boldsymbol{i}$ and $\boldsymbol{i} \cdot \boldsymbol{j}$ and $\boldsymbol{i} \cdot \boldsymbol{k}$



$$\boldsymbol{i} \cdot \boldsymbol{i}$$
$$= |\boldsymbol{i}| \cdot |\boldsymbol{i}| \cdot \cos 0^o$$
$$= 1 \cdot 1 \cdot 1$$
$$= 1$$

$$\boldsymbol{i} \cdot \boldsymbol{j}$$
$$= |\boldsymbol{i}| \cdot |\boldsymbol{j}| \cdot \cos 90^o$$
$$= 1 \cdot 1 \cdot 0$$
$$= 0$$

$$
\begin{aligned}
&\boldsymbol{i}\cdot\boldsymbol{k}\\
&=|\boldsymbol{i}|\cdot|\boldsymbol{k}|\cdot\cos 90^o\\
&=1\cdot 1\cdot 0\\
&=0
\end{aligned}
$$

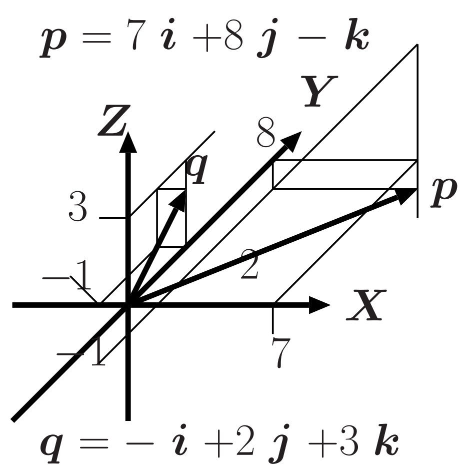
17) If

$$\boldsymbol{p}=7\boldsymbol{i}+8\boldsymbol{j}-\boldsymbol{k}$$

and

$$\boldsymbol{q}=-\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$$

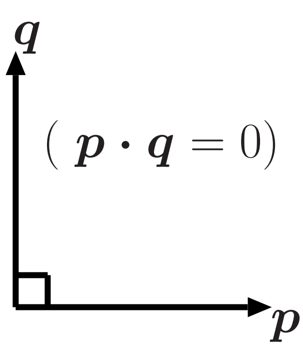
find the scalar product $\boldsymbol{p}\cdot\boldsymbol{q}$



$$\boldsymbol{p}=\begin{pmatrix}7\\8\\-1\end{pmatrix},\boldsymbol{q}=\begin{pmatrix}-1\\2\\3\end{pmatrix}$$

$$
\begin{aligned}
&\begin{pmatrix}7\\8\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\3\end{pmatrix}\\
&=7\cdot(-1)+8\cdot(2)+(-1)\cdot(3)\\
&=-7+16-3=6
\end{aligned}
$$

18) If $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular, simplify $(\boldsymbol{p}-2\boldsymbol{q})\cdot(3\boldsymbol{p}+5\boldsymbol{q})$



$$\begin{aligned}
&(\boldsymbol{p}-2\boldsymbol{q})\cdot(3\boldsymbol{p}+5\boldsymbol{q})\\
&=3\boldsymbol{p}\cdot\boldsymbol{p}+5\boldsymbol{q}\cdot\boldsymbol{p}-2\boldsymbol{q}\cdot 3\boldsymbol{p}-2\boldsymbol{q}\cdot 5\boldsymbol{q}\\
&=3|\boldsymbol{p}|^2+5\boldsymbol{q}\cdot\boldsymbol{p}-6\boldsymbol{q}\cdot\boldsymbol{p}-10|\boldsymbol{q}|^2\\
&=3|\boldsymbol{p}|^2-\boldsymbol{q}\cdot\boldsymbol{p}-10|\boldsymbol{q}|^2\\
&=3|\boldsymbol{p}|^2-10|\boldsymbol{q}|^2\\
&(\because \boldsymbol{p}\cdot\boldsymbol{q}=0)
\end{aligned}$$

19) Points R, S, and T have coordinates $(-4,0,-1)$, $(5,3,-5)$ and $(2,-7,-3)$ respectively. Find

a) the scalar product $\overrightarrow{RS}\cdot\overrightarrow{RT}$.

$$\begin{aligned}
\overrightarrow{RS}&=\overrightarrow{RO}+\overrightarrow{OS}\\
&=-\overrightarrow{OR}+\overrightarrow{OS}\\
&=-\boldsymbol{r}+\boldsymbol{s}\\
&=-\begin{pmatrix}-4\\0\\-1\end{pmatrix}+\begin{pmatrix}5\\3\\-5\end{pmatrix}\\
&=\begin{pmatrix}-(-4)+5\\3\\-(-1)-5\end{pmatrix}\\
&=\begin{pmatrix}9\\3\\-4\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{RT}&=\overrightarrow{RO}+\overrightarrow{OT}\\
&=-\overrightarrow{OR}+\overrightarrow{OT}\\
&=-\boldsymbol{r}+\boldsymbol{t}\\
&=-\begin{pmatrix}-4\\0\\-1\end{pmatrix}+\begin{pmatrix}2\\-7\\-3\end{pmatrix}\\
&=\begin{pmatrix}-(-4)+2\\-7\\-(-1)-3\end{pmatrix}\\
&=\begin{pmatrix}6\\-7\\-2\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\therefore \overrightarrow{RS}\cdot\overrightarrow{RT}&=\begin{pmatrix}9\\3\\-4\end{pmatrix}\cdot\begin{pmatrix}6\\-7\\-2\end{pmatrix}\\
&=9\cdot 6+3\cdot(-7)+(-4)\cdot(-2)\\
&=54-21+8=41
\end{aligned}$$

b) the vector product $\overrightarrow{RS} \times \overrightarrow{RT}$.

$$\begin{pmatrix} \overrightarrow{RS} & \overrightarrow{RT} \end{pmatrix} = \begin{pmatrix} 9 & 6 \\ 3 & -7 \\ -4 & -2 \end{pmatrix}$$

$$\begin{aligned} \overrightarrow{RS} \times \overrightarrow{RT} &= \begin{vmatrix} 3 & -7 \\ -4 & -2 \end{vmatrix} \boldsymbol{i} \\ &+ \begin{vmatrix} -4 & -2 \\ 9 & 6 \end{vmatrix} \boldsymbol{j} \\ &+ \begin{vmatrix} 9 & 6 \\ 3 & -7 \end{vmatrix} \boldsymbol{k} \\ &= \{3 \cdot (-2) - (-4) \cdot (-7)\}\boldsymbol{i} \\ &+ \{(-4) \cdot 6 - 9 \cdot (-2)\}\boldsymbol{j} \\ &+ \{9 \cdot (-7) - 3 \cdot 6\}\boldsymbol{k} \\ &= \{-6 - 28\}\boldsymbol{i} \\ &+ \{-24 - (-18)\}\boldsymbol{j} \\ &+ \{-63 - 18)\}\boldsymbol{k} \\ &= -34\boldsymbol{i} - 6\boldsymbol{j} - 81\boldsymbol{k} \end{aligned}$$

c) the angle between the vectors $\overrightarrow{RS}$ and $\overrightarrow{RT}$.

$$\begin{aligned} \left|\overrightarrow{RS}\right| &= \sqrt{9^2 + 3^2 + (-4)^2} \\ &= \sqrt{81 + 9 + 16} \\ &= \sqrt{106} \\ \left|\overrightarrow{RT}\right| &= \sqrt{6^2 + (-7)^2 + (-2)^2} \\ &= \sqrt{36 + 49 + 4} \\ &= \sqrt{89} \end{aligned}$$

$$\begin{aligned} \therefore \overrightarrow{RS} \cdot \overrightarrow{RT} &= \left|\overrightarrow{RS}\right| \cdot \left|\overrightarrow{RT}\right| \cdot \cos\theta \\ \therefore 41 &= \sqrt{106} \cdot \sqrt{89} \cdot \cos\theta \\ \therefore \frac{41}{\sqrt{106} \cdot \sqrt{89}} &= \cos\theta \\ \therefore \theta = \cos^{-1} \frac{41}{\sqrt{106} \cdot \sqrt{89}} &= 1.12363 \text{radians} \end{aligned}$$

20) Find a vector which is perpendicular to both of the vectors

$$\boldsymbol{c} = 5\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$$

and

$$\boldsymbol{d} = \boldsymbol{i} - 4\boldsymbol{j} - 6\boldsymbol{k}.$$

A vector which is orthogonal to $\boldsymbol{c}$ and $\boldsymbol{d}$ is $\boldsymbol{c} \times \boldsymbol{d}$.

$$\boldsymbol{c} = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}, \boldsymbol{d} = \begin{pmatrix} 1 \\ -4 \\ -6 \end{pmatrix}$$

$$\therefore \begin{pmatrix} \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & -4 \\ -3 & -6 \end{pmatrix}$$

Thus

$$\begin{aligned} & \boldsymbol{c} \times \boldsymbol{d} \\ &= \begin{vmatrix} 1 & -4 \\ -3 & -6 \end{vmatrix} \boldsymbol{i} \\ &+ \begin{vmatrix} -3 & -6 \\ 5 & 1 \end{vmatrix} \boldsymbol{j} \\ &+ \begin{vmatrix} 5 & 1 \\ 1 & -4 \end{vmatrix} \boldsymbol{k} \\ &= \{1 \cdot (-6) - (-3) \cdot (-4)\} \boldsymbol{i} \\ &+ \{-3 \cdot (1) - 5 \cdot (-6)\} \boldsymbol{j} \\ &+ \{5 \cdot (-4) - 1 \cdot 1\} \boldsymbol{k} \\ &= \{-6 - 12\} \boldsymbol{i} \\ &+ \{-3 + 30\} \boldsymbol{j} \\ &+ \{-20 - 1\} \boldsymbol{k} \\ &= -18\boldsymbol{i} + 27\boldsymbol{j} - 21\boldsymbol{k} \end{aligned}$$

21) Points P, Q, and R have coordinates $(9, 1, -2)$, $(3, 1, 3)$, and $(1, 0, -1)$ respectively. Find $\overrightarrow{PQ} \times \overrightarrow{PR}$.

$$\begin{aligned} & \overrightarrow{PQ} \\ &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= -\overrightarrow{OP} + \overrightarrow{OQ} \\ &= -\begin{pmatrix} 9 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -9 + 3 \\ -1 + 1 \\ +2 + 3 \end{pmatrix} \\ &= \begin{pmatrix} -6 \\ 0 \\ 5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} & \overrightarrow{PR} \\ &= \overrightarrow{PO} + \overrightarrow{OR} \\ &= -\overrightarrow{OP} + \overrightarrow{OR} \\ &= -\begin{pmatrix} 9 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} -9+1 \\ -1+0 \\ +2-1 \end{pmatrix}$$

$$= \begin{pmatrix} -8 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} \overrightarrow{PQ} & \overrightarrow{PR} \end{pmatrix} = \begin{pmatrix} -6 & -8 \\ 0 & -1 \\ 5 & 1 \end{pmatrix}$$

Thus

$$\begin{aligned} &\overrightarrow{PQ} \times \overrightarrow{PR} \\ &= \begin{vmatrix} 0 & -1 \\ 5 & 1 \end{vmatrix} \boldsymbol{i} \\ &+ \begin{vmatrix} 5 & 1 \\ -6 & -8 \end{vmatrix} \boldsymbol{j} \\ &+ \begin{vmatrix} -6 & -8 \\ 0 & -1 \end{vmatrix} \boldsymbol{k} \\ &= \{0 \cdot (1) - (-1) \cdot 5\}\boldsymbol{i} \\ &+\{5 \cdot (-8) - (1) \cdot (-6)\}\boldsymbol{j} \\ &+\{-6 \cdot (-1) - (-8) \cdot (0)\}\boldsymbol{k} \\ &= \{0+5\}\boldsymbol{i} \\ &+\{-40+6\}\boldsymbol{j} \\ &+\{6+0\}\boldsymbol{k} \\ &= 5\boldsymbol{i} - 34\boldsymbol{j} + 6\boldsymbol{k} \end{aligned}$$

22) Evaluate the vector product $\boldsymbol{p} \times \boldsymbol{q}$ if

$$\boldsymbol{p} = 3\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$$

and

$$\boldsymbol{q} = 7\boldsymbol{i} + 4\boldsymbol{j} - 8\boldsymbol{k}$$

$$\boldsymbol{p} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}, \boldsymbol{q} = \begin{pmatrix} 7 \\ 4 \\ -8 \end{pmatrix}$$

$$\therefore \begin{pmatrix} \boldsymbol{p} & \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ -2 & 4 \\ 5 & -8 \end{pmatrix}$$

Thus

$$\begin{aligned} &\boldsymbol{p}\times\boldsymbol{q} \\ &= \begin{vmatrix} -2 & 4 \\ 5 & -8 \end{vmatrix}\boldsymbol{i} \\ &+\begin{vmatrix} 5 & -8 \\ 3 & 7 \end{vmatrix}\boldsymbol{j} \\ &+\begin{vmatrix} 3 & 7 \\ -2 & 4 \end{vmatrix}\boldsymbol{k} \\ &= \{-2\cdot(-8)-4\cdot 5\}\boldsymbol{i} \\ &+\{5\cdot(7)-(-8)\cdot 3\}\boldsymbol{j} \\ &+\{3\cdot(4)-(7)\cdot(-2)\}\boldsymbol{k} \\ &= \{16-20\}\boldsymbol{i} \\ &+\{35+24\}\boldsymbol{j} \\ &+\{12+14\}\boldsymbol{k} \\ &= -4\boldsymbol{i}+59\boldsymbol{j}+26\boldsymbol{k} \end{aligned}$$

23) Find the vector product of

$$\boldsymbol{p} = -2\boldsymbol{i} - 3\boldsymbol{j}$$

and

$$\boldsymbol{q} = 4\boldsymbol{i} + 7\boldsymbol{j}$$

$$\boldsymbol{p} = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}, \boldsymbol{q} = \begin{pmatrix} 4 \\ 7 \\ 0 \end{pmatrix}$$

$$\therefore \left(\ \boldsymbol{p} \quad \boldsymbol{q}\ \right) = \begin{pmatrix} -2 & 4 \\ -3 & 7 \\ 0 & 0 \end{pmatrix}$$

Thus

$$\begin{aligned} &\boldsymbol{p}\times\boldsymbol{q} \\ &= \begin{vmatrix} -3 & 7 \\ 0 & 0 \end{vmatrix}\boldsymbol{i} \\ &+\begin{vmatrix} 0 & 0 \\ -2 & 4 \end{vmatrix}\boldsymbol{j} \\ &+\begin{vmatrix} -2 & 4 \\ -3 & 7 \end{vmatrix}\boldsymbol{k} \\ &= \{-3\cdot(0)-7\cdot 0\}\boldsymbol{i} \\ &+\{0\cdot(4)-(0)\cdot(-2)\}\boldsymbol{j} \\ &+\{-2\cdot(7)-(4)\cdot(-3)\}\boldsymbol{k} \\ &= \{0\}\boldsymbol{i} \end{aligned}$$

$$+\{0\}\boldsymbol{j}$$
$$+\{-14+12\}\boldsymbol{k}$$
$$=-2\boldsymbol{k}$$

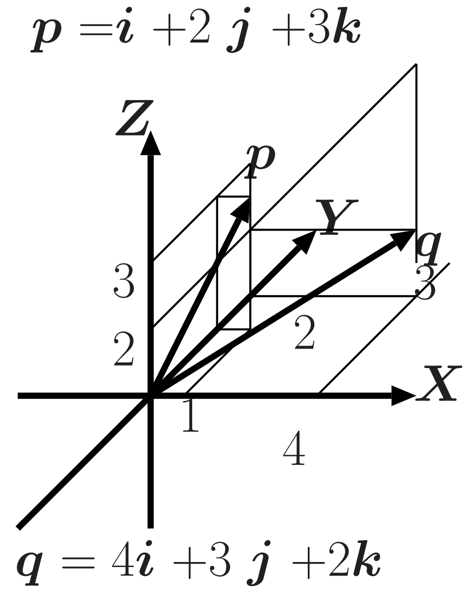
24) If

$$\boldsymbol{p}=\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$$

and

$$\boldsymbol{q}=4\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k},$$

find $\boldsymbol{p}\times\boldsymbol{q}$ and $\boldsymbol{q}\times\boldsymbol{p}$



$$\boldsymbol{p}=\begin{pmatrix}1\\2\\3\end{pmatrix},\boldsymbol{q}=\begin{pmatrix}4\\3\\2\end{pmatrix}$$

$$\therefore\left(\ \boldsymbol{p}\quad\boldsymbol{q}\ \right)=\begin{pmatrix}1&4\\2&3\\3&2\end{pmatrix}$$

Thus

$$\boldsymbol{p}\times\boldsymbol{q}$$
$$=\begin{vmatrix}2&3\\3&2\end{vmatrix}\boldsymbol{i}$$
$$+\begin{vmatrix}3&2\\1&4\end{vmatrix}\boldsymbol{j}$$
$$+\begin{vmatrix}1&4\\2&3\end{vmatrix}\boldsymbol{k}$$
$$=\{2\cdot(2)-3\cdot3\}\boldsymbol{i}$$
$$+\{3\cdot(4)-(1)\cdot(2)\}\boldsymbol{j}$$

$$+\{1\cdot(3)-(2)\cdot(4)\}\boldsymbol{k}$$
$$=\{4-9\}\boldsymbol{i}$$
$$+\{12-2\}\boldsymbol{j}$$
$$+\{3-8\}\boldsymbol{k}$$
$$=-5\boldsymbol{i}+10\boldsymbol{j}-5\boldsymbol{k}$$

$$\therefore \begin{pmatrix} \boldsymbol{q} & \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \\ 2 & 3 \end{pmatrix}$$

Thus

$$\boldsymbol{p}\times\boldsymbol{q}$$
$$=\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}\boldsymbol{i}$$
$$+\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}\boldsymbol{j}$$
$$+\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}\boldsymbol{k}$$
$$=\{3\cdot(3)-2\cdot 2\}\boldsymbol{i}$$
$$+\{2\cdot(1)-(4)\cdot(3)\}\boldsymbol{j}$$
$$+\{4\cdot(2)-(1)\cdot(3)\}\boldsymbol{k}$$
$$=\{9-4\}\boldsymbol{i}$$
$$+\{2-12\}\boldsymbol{j}$$
$$+\{8-3\}\boldsymbol{k}$$
$$=5\boldsymbol{i}-10\boldsymbol{j}+5\boldsymbol{k}$$

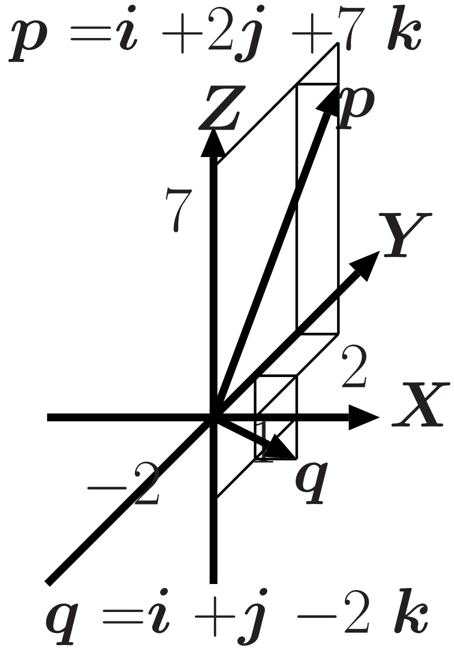
25) Find a vector which is perpendicular to both of the vectors

$$\boldsymbol{p}=\boldsymbol{i}+2\boldsymbol{j}+7\boldsymbol{k}$$

and

$$\boldsymbol{q}=\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}.$$

Hence find a unit vector which is perpendicular to both $\boldsymbol{p}$ and $\boldsymbol{q}$.



A vector which is orthogonal to $\boldsymbol{p}$ and $\boldsymbol{q}$ is $\boldsymbol{p} \times \boldsymbol{q}$.

$$\boldsymbol{p} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}, \boldsymbol{q} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} \boldsymbol{p} & \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 7 & -2 \end{pmatrix}$$

Thus

$$\begin{aligned}
& \boldsymbol{p} \times \boldsymbol{q} \\
&= \begin{vmatrix} 2 & 1 \\ 7 & -2 \end{vmatrix} \boldsymbol{i} \\
&+ \begin{vmatrix} 7 & -2 \\ 1 & 1 \end{vmatrix} \boldsymbol{j} \\
&+ \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \boldsymbol{k} \\
&= \{2 \cdot (-2) - 1 \cdot 7\} \boldsymbol{i} \\
&+ \{7 \cdot (1) - (-2) \cdot (1)\} \boldsymbol{j} \\
&+ \{1 \cdot (1) - (1) \cdot (2)\} \boldsymbol{k} \\
&= \{-4 - 7\} \boldsymbol{i} \\
&+ \{7 + 2\} \boldsymbol{j} \\
&+ \{1 - 2\} \boldsymbol{k} \\
&= -11\boldsymbol{i} + 9\boldsymbol{j} - \boldsymbol{k}
\end{aligned}$$

In order to get the unit vector, we need the modulus of $-11\boldsymbol{i}+9\boldsymbol{j}-\boldsymbol{k}$. The modulus is

$$\begin{aligned}&\sqrt{11^2+9^2+(-1)^2}\\&=\sqrt{121+81+1}\\&=\sqrt{203}\end{aligned}$$

Thus the unit vector is

$$\frac{-11\boldsymbol{i}+9\boldsymbol{j}-\boldsymbol{k}}{\sqrt{203}}.$$

26) For the vectors

$$\boldsymbol{p}=4\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$$

$$\boldsymbol{q}=\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k},$$

and

$$\boldsymbol{r}=3\boldsymbol{i}-3\boldsymbol{j}+4\boldsymbol{k},$$

evaluate both $\boldsymbol{p}\times(\boldsymbol{q}\times\boldsymbol{r})$ and $(\boldsymbol{p}\times\boldsymbol{q})\times\boldsymbol{r}$.

$$\boldsymbol{p}=\begin{pmatrix}4\\2\\1\end{pmatrix},\boldsymbol{q}=\begin{pmatrix}1\\-2\\1\end{pmatrix},\boldsymbol{r}=\begin{pmatrix}3\\-3\\4\end{pmatrix}$$

$$\therefore\begin{pmatrix}\boldsymbol{q} & \boldsymbol{r}\end{pmatrix}=\begin{pmatrix}1 & 3\\-2 & -3\\1 & 4\end{pmatrix}$$

Thus

$$\begin{aligned}&\boldsymbol{q}\times\boldsymbol{r}\\&=\begin{vmatrix}-2 & -3\\1 & 4\end{vmatrix}\boldsymbol{i}\\&+\begin{vmatrix}1 & 4\\1 & 3\end{vmatrix}\boldsymbol{j}\\&+\begin{vmatrix}1 & 3\\-2 & -3\end{vmatrix}\boldsymbol{k}\\&=\{-2\cdot(4)-(-3)\cdot 1\}\boldsymbol{i}\\&+\{1\cdot(3)-(4)\cdot(1)\}\boldsymbol{j}\\&+\{1\cdot(-3)-(3)\cdot(-2)\}\boldsymbol{k}\\&=\{-8+3\}\boldsymbol{i}\\&+\{3-4\}\boldsymbol{j}\\&+\{-3+6\}\boldsymbol{k}\\&=-5\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}\\&=\begin{pmatrix}-5\\-1\\3\end{pmatrix}\end{aligned}$$

$$\therefore \begin{pmatrix} \boldsymbol{p} & \boldsymbol{q}\times\boldsymbol{r} \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 2 & -1 \\ 1 & 3 \end{pmatrix}$$

$$\begin{aligned} &\therefore \boldsymbol{p}\times(\boldsymbol{q}\times\boldsymbol{r}) \\ &= \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}\boldsymbol{i} \\ &+\begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix}\boldsymbol{j} \\ &+\begin{vmatrix} 4 & -5 \\ 2 & -1 \end{vmatrix}\boldsymbol{k} \\ &= \{2\cdot(3)-(-1)\cdot 1\}\boldsymbol{i} \\ &+\{1\cdot(-5)-(3)\cdot(4)\}\boldsymbol{j} \\ &+\{4\cdot(-1)-(-5)\cdot(2)\}\boldsymbol{k} \\ &= \{6+1\}\boldsymbol{i} \\ &+\{-5-12\}\boldsymbol{j} \\ &+\{-4+10\}\boldsymbol{k} \\ &= 7\boldsymbol{i}-17\boldsymbol{j}+6\boldsymbol{k} \end{aligned}$$

$$\therefore \begin{pmatrix} \boldsymbol{p} & \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & -2 \\ 1 & 1 \end{pmatrix}$$

Thus

$$\begin{aligned} &\boldsymbol{p}\times\boldsymbol{q} \\ &= \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}\boldsymbol{i} \\ &+\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix}\boldsymbol{j} \\ &+\begin{vmatrix} 4 & 1 \\ 2 & -2 \end{vmatrix}\boldsymbol{k} \\ &= \{2\cdot(1)-(-2)\cdot 1\}\boldsymbol{i} \\ &+\{1\cdot(1)-(1)\cdot(4)\}\boldsymbol{j} \\ &+\{4\cdot(-2)-(1)\cdot(2)\}\boldsymbol{k} \\ &= \{2+2\}\boldsymbol{i} \\ &+\{1-4\}\boldsymbol{j} \\ &+\{-8-2\}\boldsymbol{k} \\ &= 4\boldsymbol{i}-3\boldsymbol{j}-10\boldsymbol{k} \\ &= \begin{pmatrix} 4 \\ -3 \\ -10 \end{pmatrix} \end{aligned}$$

$$\therefore \begin{pmatrix} \boldsymbol{p}\times\boldsymbol{q} & \boldsymbol{r} \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ -3 & -3 \\ -10 & 4 \end{pmatrix}$$

$$\therefore (\boldsymbol{p} \times \boldsymbol{q}) \times \boldsymbol{r}$$
$$= \begin{vmatrix} -3 & -3 \\ -10 & 4 \end{vmatrix} \boldsymbol{i}$$
$$+ \begin{vmatrix} -10 & 4 \\ 4 & 3 \end{vmatrix} \boldsymbol{j}$$
$$+ \begin{vmatrix} 4 & 3 \\ -3 & -3 \end{vmatrix} \boldsymbol{k}$$
$$= \{-3 \cdot (4) - (-3) \cdot (-10)\}\boldsymbol{i}$$
$$+\{-10 \cdot (3) - (4) \cdot (4)\}\boldsymbol{j}$$
$$+\{4 \cdot (-3) - (3) \cdot (-3)\}\boldsymbol{k}$$
$$= \{-12 - 30\}\boldsymbol{i}$$
$$+\{-30 - 16\}\boldsymbol{j}$$
$$+\{-12 + 9\}\boldsymbol{k}$$
$$= -42\boldsymbol{i} - 46\boldsymbol{j} - 3\boldsymbol{k}$$

Thus, in general, the vector product is not associative.