University of Manchester

MATHEMATICAL FORMULA TABLES

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Other than this front cover these tables are identical to the UMIST, version 2.0 tables.

CONTENTS

GREEK ALPHABET

A α	alpha	N ν	nu
B β	beta	Ξ ξ	xi
Γ γ	gamma	O o	omicron
Δ δ	delta	Π π	pi
E ϵ, ε	epsilon	P ρ	rho
Z ζ	zeta	Σ σ	sigma
H η	eta	T τ	tau
Θ θ, ϑ	theta	Υ υ	upsilon
I ι	iota	Φ ϕ, φ	phi
K κ	kappa	X χ	chi
Λ λ	lambda	Ψ ψ	psi
M μ	mu	Ω ω	omega

INDICES AND LOGARITHMS

$$a^m \times a^n = a^{m+n}$$

$$(a^m)^n = a^{mn}$$

$$\begin{aligned} \log(AB) &= \log A + \log B \\ \log(A/B) &= \log A - \log B \\ \log(A^n) &= n \log A \\ \log_b a &= \frac{\log_c a}{\log_c b} \end{aligned}$$

TRIGONOMETRIC IDENTITIES

$\tan A = \sin A / \cos A$

$\sec A = 1/ \cos A$

$\text{cosec } A = 1/ \sin A$

$\cot A = \cos A/ \sin A = 1/ \tan A$

$\sin^2 A + \cos^2 A = 1$

$\sec^2 A = 1 + \tan^2 A$

$\text{cosec } {}^2 A = 1 + \cot^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\qquad = 2\cos^2 A - 1$

$\qquad = 1 - 2\sin^2 A$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

$\sin 3A = 3 \sin A - 4 \sin^3 A$

$\cos 3A = 4 \cos^3 A - 3 \cos A$

$\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3\tan^2 A}$

$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$-2\sin A\sin B = \cos(A+B) - \cos(A-B)$

$a\sin x + b\cos x = R\sin(x+\phi)$, where $R = \sqrt{a^2+b^2}$ and $\cos\phi = a/R$, $\sin\phi = b/R$.

If $t = \tan\frac{1}{2}x$ then $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$.

$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$; $\quad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$

$e^{ix} = \cos x + i\sin x$; $\quad e^{-ix} = \cos x - i\sin x$

COMPLEX NUMBERS

$i = \sqrt{-1}$ Note:- 'j' often used rather than 'i'.

Exponential Notation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

De Moivre's theorem

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

nth roots of complex numbers

$$\text{If } z = re^{i\theta} = r(\cos\theta + i\sin\theta) \text{ then}$$

$$z^{1/n} = \sqrt[n]{r}e^{i(\theta+2k\pi)/n}, \quad k = 0, \pm1, \pm2, ...$$

HYPERBOLIC IDENTITIES

$\cosh x = (e^x + e^{-x})/2$ $\sinh x = (e^x - e^{-x})/2$

$\tanh x = \sinh x / \cosh x$

$\text{sech} x = 1/\cosh x$ $\text{cosech} x = 1/\sinh x$

$\coth x = \cosh x/\sinh x = 1/\tanh x$

$\cosh ix = \cos x$ $\sinh ix = i\sin x$

$\cos ix = \cosh x$ $\sin ix = i\sinh x$

$$\cosh^2 A - \sinh^2 A = 1$$

$$\text{sech}^2 A = 1 - \tanh^2 A$$

$$\text{cosech}^2 A = \coth^2 A - 1$$

SERIES

Powers of Natural Numbers

$$\sum_{k=1}^{n} k = \frac{1}{2}\, n(n+1)\ ;\ \ \sum_{k=1}^{n} k^2 = \frac{1}{6}\, n(n+1)(2n+1);\sum_{k=1}^{n} k^3 = \frac{1}{4}\, n^2(n+1)^2$$

Arithmetic $\qquad S_n = \sum_{k=0}^{n-1}(a+kd) = \frac{n}{2}\,\{2a+(n-1)d\}$

Geometric (convergent for $-1 < r < 1$)

$$S_n = \sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{1-r}\ ,\ S_\infty = \frac{a}{1-r}$$

Binomial (convergent for $|x| < 1$)

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + ... + \binom{n}{r}x^r + ...$$

where $\binom{n}{r} = \frac{n(n-1)(n-2)...(n-r+1)}{r!}$

Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^k}{k!}f^{(k)}(0) + R_{k+1}$$

where $R_{k+1} = \frac{x^{k+1}}{(k+1)!}f^{(k+1)}(\theta x), \quad 0 < \theta < 1$

Taylor series

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + ... + \frac{h^k}{k!}f^{(k)}(a) + R_{k+1}$$

where $R_{k+1} = \frac{h^{k+1}}{(k+1)!}f^{(k+1)}(a+\theta h)\ ,\ 0 < \theta < 1.$

OR

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + ... + \frac{(x-x_0)^k}{k!}f^{(k)}(x_0) + R_{k+1}$$

where $R_{k+1} = \frac{(x-x_0)^{k+1}}{(k+1)!}f^{(k+1)}(x_0+(x-x_0)\theta), \quad 0 < \theta < 1$

Special Power Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^r}{r!} + ... \qquad (\text{all } x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + ... \qquad (\text{all } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... + \frac{(-1)^r x^{2r}}{(2r)!} + ... \qquad (\text{all } x)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + ... \qquad (|x| < \tfrac{\pi}{2})$$

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} +$$

$$... + \frac{1.3.5....(2n-1)}{2.4.6....(2n)}\frac{x^{2n+1}}{2n+1} + ... \qquad (|x| < 1)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... + (-1)^n \frac{x^{2n+1}}{2n+1} + ... \qquad (|x| < 1)$$

$$\ell n(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + (-1)^{n+1}\frac{x^n}{n} + ... \qquad (-1 < x \leq 1)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ... + \frac{x^{2n+1}}{(2n+1)!} + ... \qquad (\text{all } x)$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ... + \frac{x^{2n}}{(2n)!} + ... \qquad (\text{all } x)$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + ... \qquad (|x| < \tfrac{\pi}{2})$$

$$\sinh^{-1} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} - \frac{1.3.5}{2.4.6}\frac{x^7}{7} +$$

$$... + (-1)^n \frac{1.3.5...(2n-1)}{2.4.6...2n}\frac{x^{2n+1}}{2n+1} + ... \qquad (|x| < 1)$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + ...\frac{x^{2n+1}}{2n+1} + ... \qquad (|x| < 1)$$

DERIVATIVES

function	derivative
x^n	nx^{n-1}
e^x	e^x
$a^x (a > 0)$	$a^x \ell n a$
$\ell n x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \ell n a}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\text{cosec } x$	$-\text{ cosec } x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\text{ cosec }^2 x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech }^2 x$
$\text{cosech } x$	$-\text{ cosech } x \coth x$
$\text{sech } x$	$-\text{ sech } x \tanh x$
$\coth x$	$-\text{ cosech}^2 x$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x (x > 1)$	$\dfrac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x (\|x\| < 1)$	$\dfrac{1}{1-x^2}$
$\coth^{-1} x (\|x\| > 1)$	$-\dfrac{1}{x^2-1}$

Product Rule

$$\frac{d}{dx}(u(x)\ v(x)) = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v(x)\frac{du}{dx} - u(x)\frac{dv}{dx}}{[v(x)]^2}$$

Chain Rule

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \times g'(x)$$

Leibnitz's theorem

$$\frac{d^n}{dx^n}(f.g) = f^{(n)}.g + nf^{(n-1)}.g^{(1)} + \frac{n(n-1)}{2!}f^{(n-2)}.g^{(2)} + ... + \binom{n}{r}f^{(n-r)}.g^{(r)} + ... + f.g^{(n)}$$

INTEGRALS

function	integral
$f(x)\dfrac{dg(x)}{dx}$	$f(x)g(x) - \int \dfrac{df(x)}{dx} g(x)dx$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ell n\|x\|$ Note:- $\ell n\|x\| + K = \ell n\|x/x_0\|$
e^x	e^x
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$\ell n\|\sec x\|$
$\text{cosec } x$	$-\ell n\|\text{ cosec } x + \cot x\|$ or $\ell n \left\|\tan \frac{x}{2}\right\|$
$\sec x$	$\ell n\|\sec x + \tan x\| = \ell n \left\|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\|$
$\cot x$	$\ell n\|\sin x\|$
$\dfrac{1}{a^2 + x^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$
$\dfrac{1}{a^2 - x^2}$	$\dfrac{1}{2a}\ell n\dfrac{a+x}{a-x}$ or $\dfrac{1}{a}\tanh^{-1}\dfrac{x}{a}$ $(\|x\| < a)$
$\dfrac{1}{x^2 - a^2}$	$\dfrac{1}{2a}\ell n\dfrac{x-a}{x+a}$ or $-\dfrac{1}{a}\coth^{-1}\dfrac{x}{a}$ $(\|x\| > a)$
$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\dfrac{x}{a}$ $(a > \|x\|)$
$\dfrac{1}{\sqrt{a^2 + x^2}}$	$\sinh^{-1}\dfrac{x}{a}$ or $\ell n\left(x + \sqrt{x^2 + a^2}\right)$
$\dfrac{1}{\sqrt{x^2 - a^2}}$	$\cosh^{-1}\dfrac{x}{a}$ or $\ell n\|x + \sqrt{x^2 - a^2}\|$ $(\|x\| > a)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ell n \cosh x$
$\text{cosech } x$	$-\ell n\|\text{cosech } x + \text{coth } x\|$ or $\ell n \left\|\tanh \frac{x}{2}\right\|$
$\text{sech } x$	$2 \tan^{-1} e^x$
$\coth x$	$\ell n\|\sinh x\|$

Double integral

$$\int\int f(x,y)\ dx\ dy = \int\int g(r,s)\ J\ dr\ ds$$

where

$$J = \frac{\partial(x,y)}{\partial(r,s)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix}$$

LAPLACE TRANSFORMS

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt$$

function	transform
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$H_a(t) = H(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t)$	1
$e^{at} t^n$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{at} \sinh \omega t$	$\frac{\omega}{(s-a)^2-\omega^2}$
$e^{at} \cosh \omega t$	$\frac{s-a}{(s-a)^2-\omega^2}$

Let $\tilde{f}(s) = \mathcal{L}\{f(t)\}$ then

$$\begin{aligned}
\mathcal{L}\left\{e^{at}f(t)\right\} &= \tilde{f}(s-a), \\
\mathcal{L}\{tf(t)\} &= -\frac{d}{ds}(\tilde{f}(s)), \\
\mathcal{L}\left\{\frac{f(t)}{t}\right\} &= \int_{x=s}^{\infty} \tilde{f}(x)dx \quad \text{if this exists.}
\end{aligned}$$

Derivatives and integrals

Let $y = y(t)$ and let $\tilde{y} = \mathcal{L}\{y(t)\}$ then

$$\begin{aligned}
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= s\tilde{y} - y_0, \\
\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2\tilde{y} - sy_0 - y_0', \\
\mathcal{L}\left\{\int_{\tau=0}^{t} y(\tau)d\tau\right\} &= \frac{1}{s}\tilde{y}
\end{aligned}$$

where y_0 and y_0' are the values of y and dy/dt respectively at $t = 0$.

Time delay

Let $$g(t) = H_a(t)f(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t > a \end{cases}$$

then $$\mathcal{L}\{g(t)\} = e^{-as}\tilde{f}(s).$$

Scale change

$$\mathcal{L}\{f(kt)\} = \frac{1}{k}\tilde{f}\left(\frac{s}{k}\right).$$

Periodic functions

Let $f(t)$ be of period T then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_{t=0}^{T} e^{-st}f(t)dt.$$

Convolution

Let $f(t) * g(t) = \int_{x=0}^{t} f(x)g(t-x)dx = \int_{x=0}^{t} f(t-x)g(x)dx$

then $\mathcal{L}\{f(t) * g(t)\} = \tilde{f}(s)\tilde{g}(s)$.

RLC circuit

For a simple RLC circuit with initial charge q_0 and initial current i_0,

$$\tilde{E} = \left(r + Ls + \frac{1}{Cs}\right)\tilde{i} - Li_0 + \frac{1}{Cs}q_0.$$

Limiting values

initial value theorem

$$\lim_{t \to 0+} f(t) = \lim_{s \to \infty} s\tilde{f}(s),$$

final value theorem

$$\begin{aligned} \lim_{t \to \infty} f(t) &= \lim_{s \to 0+} s\tilde{f}(s), \\ \int_0^\infty f(t)dt &= \lim_{s \to 0+} \tilde{f}(s) \end{aligned}$$

provided these limits exist.

Z TRANSFORMS

$$Z\{f(t)\} = \tilde{f}(z) = \sum_{k=0}^{\infty} f(kT)z^{-k}$$

function	transform
$\delta_{t,nT}$	$z^{-n}(n \geq 0)$
e^{-at}	$\dfrac{z}{z-e^{-aT}}$
te^{-at}	$\dfrac{Tze^{-aT}}{(z-e^{-aT})^2}$
t^2e^{-at}	$\dfrac{T^2ze^{-aT}(z+e^{-aT})}{(z-e^{-aT})^3}$
$\sinh at$	$\dfrac{z\sinh aT}{z^2-2z\cosh aT+1}$
$\cosh at$	$\dfrac{z(z-\cosh aT)}{z^2-2z\cosh aT+1}$
$e^{-at}\sin\omega t$	$\dfrac{ze^{-aT}\sin\omega T}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
$e^{-at}\cos\omega t$	$\dfrac{z(z-e^{-aT}\cos\omega T)}{z^2-2ze^{-aT}\cos\omega T+e^{-2aT}}$
$te^{-at}\sin\omega t$	$\dfrac{Tze^{-aT}(z^2-e^{-2aT})\sin\omega T}{(z^2-2ze^{-aT}\cos\omega T+e^{-2aT})^2}$
$te^{-at}\cos\omega t$	$\dfrac{Tze^{-aT}(z^2\cos\omega T-2ze^{-aT}+e^{-2aT}\cos\omega T)}{(z^2-2ze^{-aT}\cos\omega T+e^{-2aT})^2}$

Shift Theorem

$$Z\{f(t+nT)\} = z^n\tilde{f}(z) - \sum_{k=0}^{n-1} z^{n-k}f(kT) \quad (n>0)$$

Initial value theorem

$$f(0) = \lim_{z\to\infty}\tilde{f}(z)$$

Final value theorem

$$f(\infty) = \lim_{z \to 1} \left[(z-1)\tilde{f}(z) \right] \qquad \text{provided} \quad f(\infty) \quad \text{exists.}$$

Inverse Formula

$$f(kT) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} \tilde{f}(e^{i\theta}) d\theta$$

FOURIER SERIES AND TRANSFORMS

Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\omega t + b_n \sin n\omega t\} \qquad \text{(period } T = 2\pi/\omega)$$

where

$$\begin{aligned} a_n &= \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega t \, dt \\ b_n &= \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega t \, dt \end{aligned}$$

Half range Fourier series

sine series $$a_n = 0,\ b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\,dt$$

cosine series $$b_n = 0,\ a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt$$

Finite Fourier transforms

sine transform

$$\begin{aligned}\tilde{f}_s(n) &= \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\,dt \\ f(t) &= \sum_{n=1}^{\infty} \tilde{f}_s(n)\sin n\omega t\end{aligned}$$

cosine transform

$$\begin{aligned}\tilde{f}_c(n) &= \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt \\ f(t) &= \frac{1}{2}\tilde{f}_c(0) + \sum_{n=1}^{\infty} \tilde{f}_c(n)\cos n\omega t\end{aligned}$$

Fourier integral

$$\frac{1}{2}\left(\lim_{t\nearrow 0} f(t) + \lim_{t\searrow 0} f(t)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\int_{-\infty}^{\infty} f(u)e^{-i\omega u}\,du\,d\omega$$

Fourier integral transform

$$\begin{aligned}\tilde{f}(\omega) &= F\{f(t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega u} f(u)\,du \\ f(t) &= F^{-1}\left\{\tilde{f}(\omega)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t}\tilde{f}(\omega)\,d\omega\end{aligned}$$

NUMERICAL FORMULAE

Iteration

Newton Raphson method for refining an approximate root x_0 of $f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Particular case to find $\sqrt{N}$ use $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$.

Secant Method

$$x_{n+1} = x_n - f(x_n)/\left(\frac{f\left(x_n\right) - f\left(x_{n-1}\right)}{x_n - x_{n-1}}\right)$$

Interpolation

$$\Delta f_n = f_{n+1} - f_n \quad , \quad \delta f_n = f_{n+\frac{1}{2}} - f_{n-\frac{1}{2}}$$
$$\nabla f_n = f_n - f_{n-1} \quad , \quad \mu f_n = \frac{1}{2}\left(f_{n+\frac{1}{2}} + f_{n-\frac{1}{2}}\right)$$

Gregory Newton Formula

$$f_p = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + ... + \binom{p}{r}\Delta^r f_0$$

$$\text{where } p = \frac{x - x_0}{h}$$

Lagrange's Formula for n points

$$y = \sum_{i=1}^{n} y_i \ell_i(x)$$

where

$$\ell_i(x) = \frac{\Pi_{j=1,j\neq i}^{n}(x - x_j)}{\Pi_{j=1,j\neq i}^{n}(x_i - x_j)}$$

Numerical differentiation

Derivatives at a tabular point

$$\begin{aligned}
hf_0' &= \mu\,\delta f_0 - \frac{1}{6}\mu\,\delta^3 f_0 + \frac{1}{30}\mu\,\delta^5 f_0 - ... \\
h^2 f_0'' &= \delta^2 f_0 - \frac{1}{12}\delta^4 f_0 + \frac{1}{90}\delta^6 f_0 - ... \\
hf_0' &= \Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0 - \frac{1}{4}\Delta^4 f_0 + \frac{1}{5}\Delta^5 f_0 - ... \\
h^2 f_0'' &= \Delta^2 f_0 - \Delta^3 f_0 + \frac{11}{12}\Delta^4 f_0 - \frac{5}{6}\Delta^5 f_0 + ...
\end{aligned}$$

Numerical Integration

Trapezium Rule
$$\int_{x_0}^{x_0+h} f(x)dx \simeq \frac{h}{2}(f_0 + f_1) + E$$

where
$$f_i = f(x_0 + ih),\ E = -\frac{h^3}{12}f''(a),\ x_0 < a < x_0 + h$$

Composite Trapezium Rule

$$\int_{x_0}^{x_0+nh} f(x)dx \simeq \frac{h}{2}\{f_0 + 2f_1 + 2f_2 + ...2f_{n-1} + f_n\} - \frac{h^2}{12}(f_n' - f_0') + \frac{h^4}{720}(f_n''' - f_0''')...$$

where $f_0' = f'(x_0)$, $f_n' = f'(x_0 + nh)$, etc

Simpson's Rule
$$\int_{x_0}^{x_0+2h} f(x)dx \simeq \frac{h}{3}(f_0+4f_1+f_2)+E$$

where
$$E = -\frac{h^5}{90}f^{(4)}(a) \qquad x_0 < a < x_0 + 2h.$$

Composite Simpson's Rule (n even)

$$\int_{x_0}^{x_0+nh} f(x)dx \simeq \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + ... + 2f_{n-2} + 4f_{n-1} + f_n) + E$$

where
$$E = -\frac{nh^5}{180}f^{(4)}(a). \qquad x_0 < a < x_0 + nh$$

Gauss order 1. (Midpoint)

$$\int_{-1}^{1} f(x)dx = 2f(0) + E$$

where $E = \frac{2}{3} f''(a). \qquad -1 < a < 1$

Gauss order 2.

$$\int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) + E$$

where $E = \frac{1}{135} f'^{v}(a). \qquad -1 < a < 1$

Differential Equations

To solve $y' = f(x, y)$ given initial condition y_0 at $x_0, x_n = x_0 + nh$.

Euler's forward method

$$y_{n+1} = y_n + hf(x_n, y_n) \qquad n = 0, 1, 2, ...$$

Euler's backward method

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}) \qquad n = 0, 1, 2, ...$$

Heun's method (Runge Kutta order 2)

$$y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))).$$

Runge Kutta order 4.

$$y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

where

$$\begin{aligned} K_1 &= f(x_n, y_n) \\ K_2 &= f\left(x_n + \frac{h}{2}, y_n + \frac{hK_1}{2}\right) \\ K_3 &= f\left(x_n + \frac{h}{2}, y_n + \frac{hK_2}{2}\right) \\ K_4 &= f(x_n + h, y_n + hK_3) \end{aligned}$$

Chebyshev Polynomials

$$T_n(x) = \cos\left[n(\cos^{-1} x)\right]$$

$$T_o(x) = 1 \qquad T_1(x) = x$$

$$U_{n-1}(x) = \frac{T_n'(x)}{n} = \frac{\sin\left[n(\cos^{-1} x)\right]}{\sqrt{1-x^2}}$$

$$\begin{aligned}
T_m(T_n(x)) &= T_{mn}(x). \\
T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \\
U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x) \\
\int T_n(x)dx &= \frac{1}{2}\left\{\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right\} + \text{constant}, \qquad n \geq 2
\end{aligned}$$

$$f(x) = \frac{1}{2}a_0T_0(x) + a_1T_1(x)...a_jT_j(x) + ...$$

where $$a_j = \frac{2}{\pi}\int_0^\pi f(\cos\theta)\cos j\theta d\theta \qquad j \geq 0$$

and $\int f(x)dx =$ constant $+A_1T_1(x) + A_2T_2(x) + ...A_jT_j(x) + ...$

where $A_j = (a_{j-1} - a_{j+1})/2j \qquad j \geq 1$

VECTOR FORMULAE

Scalar product $\mathbf{a}.\mathbf{b} = ab\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$

Vector product $\mathbf{a}\times\mathbf{b} = ab\sin\theta\hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Triple products

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = (\mathbf{a}\times\mathbf{b}).\mathbf{c} = \mathbf{a}.(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$$

Vector Calculus

$$\nabla \;\equiv \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$$

grad $\phi \equiv \nabla\phi$, div $\mathbf{A} \equiv \nabla.\mathbf{A}$, curl $\mathbf{A} \equiv \nabla\times\mathbf{A}$

div grad $\phi \equiv \nabla.(\nabla\ \phi) \equiv \nabla^2\phi$ (for scalars only)

div curl $\mathbf{A} = 0$ curl grad $\phi \equiv \mathbf{0}$

$\nabla^2\mathbf{A} =$ grad div $\mathbf{A} -$ curl curl $\mathbf{A}$

$\nabla(\alpha\beta) = \alpha\,\nabla\beta + \beta\,\nabla\alpha$

div $(\alpha\mathbf{A}) = \alpha$ div $\mathbf{A} + \mathbf{A}.(\nabla\alpha)$

curl $(\alpha\mathbf{A}) = \alpha$ curl $\mathbf{A} - \mathbf{A}\times(\nabla\alpha)$

div $(\mathbf{A}\times\mathbf{B}) = \mathbf{B}.$ curl $\mathbf{A} - \mathbf{A}.$ curl $\mathbf{B}$

curl $(\mathbf{A}\times\mathbf{B}) = \mathbf{A}$ div $\mathbf{B} - \mathbf{B}$ div $\mathbf{A} + (\mathbf{B}.\nabla\ )\mathbf{A} - (\mathbf{A}.\nabla\ )\mathbf{B}$

$$\text{grad}\,(\mathbf{A}.\mathbf{B}) = \mathbf{A}\times\ \text{curl}\ \mathbf{B} + \mathbf{B}\times\ \text{curl}\ \mathbf{A} + (\mathbf{A}.\nabla\ )\mathbf{B} + (\mathbf{B}.\nabla\ )\mathbf{A}$$

Integral Theorems

Divergence theorem

$$\int_{\text{surface}} \mathbf{A}.d\mathbf{S} = \int_{\text{volume}}\ \text{div}\ \mathbf{A}\ dV$$

Stokes' theorem

$$\int_{\text{surface}} (\ \text{curl}\ \mathbf{A}).d\mathbf{S} = \oint_{\text{contour}} \mathbf{A}.d\mathbf{r}$$

Green's theorems

$$\int_{\text{volume}} (\psi\nabla^{\ 2}\ \phi - \phi\nabla^{\ 2}\ \psi)dV \quad = \quad \int_{\text{surface}} \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right)|d\mathbf{S}|$$

$$\int_{\text{volume}} \left\{\psi\nabla^{\ 2}\phi + (\nabla\phi)(\nabla\psi)\right\} dV \quad = \quad \int_{\text{surface}} \psi\frac{\partial\phi}{\partial n}|d\mathbf{S}|$$

where

$$d\mathbf{S} = \hat{\mathbf{n}}|d\mathbf{S}|$$

Green's theorem in the plane

$$\oint (Pdx + Qdy) = \int\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\ dy$$

MECHANICS

Kinematics

Motion constant acceleration

$$\mathbf{v} = \mathbf{u} + \mathbf{f}t, \qquad \mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{f}t^2 = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$$

$$\mathbf{v}^2 = \mathbf{u}^2 + 2\mathbf{f}.\mathbf{s}$$

General solution of $\frac{d^2x}{dt^2} = -\omega^2 x$ is

$$x = a\cos\omega t + b\sin\omega t = R\sin(\omega t + \phi)$$

where $R = \sqrt{a^2 + b^2}$ and $\cos\phi = a/R$, $\sin\phi = b/R$.

In polar coordinates the velocity is $(\dot{r}, r\dot{\theta}) = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$ and the acceleration is $\left[\ddot{r} - r\dot{\theta}^2, r\ddot{\theta} + 2\dot{r}\dot{\theta}\right] = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$.

Centres of mass

The following results are for uniform bodies:

hemispherical shell, radius r	$\frac{1}{2}r$	from centre
hemisphere, radius r	$\frac{3}{8}r$	from centre
right circular cone, height h	$\frac{3}{4}h$	from vertex
arc, radius r and angle 2θ	$(r\sin\theta)/\theta$	from centre
sector, radius r and angle 2θ	$(\frac{2}{3}r\sin\theta)/\theta$	from centre

Moments of inertia

i. The moment of inertia of a body of mass m about an axis $= I + mh^2$, where I is the moment of inertial about the parallel axis through the mass-centre and h is the distance between the axes.

ii. If I_1 and I_2 are the moments of inertia of a lamina about two perpendicular axes through a point 0 in its plane, then its moment of inertia about the axis through 0 perpendicular to its plane is $I_1 + I_2$.

iii. The following moments of inertia are for uniform bodies about the axes stated:

rod, length ℓ, through mid-point, perpendicular to rod	$\frac{1}{12}m\ell^2$
hoop, radius r, through centre, perpendicular to hoop	mr^2
disc, radius r, through centre, perpendicular to disc	$\frac{1}{2}mr^2$
sphere, radius r, diameter	$\frac{2}{5}mr^2$

Work done

$$W = \int_{t_A}^{t_B} \mathbf{F}.\frac{d\mathbf{r}}{dt}dt.$$

ALGEBRAIC STRUCTURES

A group G is a set of elements $\{a, b, c, \ldots\}$ — with a binary operation $*$ such that

i. $a * b$ is in G for all a, b in G

ii. $a * (b * c) = (a * b) * c$ for all a, b, c in G

iii. G contains an element e, called the identity element, such that $e * a = a = a * e$ for all a in G

iv. given any a in G, there exists in G an element a^{-1}, called the element inverse to a, such that $a^{-1} * a = e = a * a^{-1}$.

A commutative (or Abelian) group is one for which $a * b = b * a$ for all a, b, in G.

A field F is a set of elements $\{a, b, c, \ldots\}$ — with two binary operations $+$ and $.$ such that

i. F is a commutative group with respect to $+$ with identity 0

ii. the non-zero elements of F form a commutative group with respect to $.$ with identity 1

iii. $a.(b + c) = a.b + a.c$ for all a, b, c, in F.

A vector space V over a field F is a set of elements $\{\underline{a}, \underline{b}, \underline{c}, \ldots\}$ — with a binary operation $+$ such that

i. they form a commutative group under $+$;

and, for all λ, μ in F and all $\underline{a}, \underline{b}$, in V,

ii. $\lambda \underline{a}$ is defined and is in V

iii. $\lambda(\underline{a} + \underline{b}) = \lambda \underline{a} + \lambda \underline{b}$

iv. $(\lambda + \mu)\underline{a} = \lambda\underline{a} + \mu\underline{a}$

v. $(\lambda.\mu)\underline{a} = \lambda(\mu\underline{a})$

vi. if 1 is an element of F such that $1.\lambda = \lambda$ for all λ in F, then $1\underline{a} = \underline{a}$.

An equivalence relation R between the elements $\{a, b, c, \ldots\}$ — of a set C is a relation such that, for all a, b, c in C

i. aRa (R is reflextive)

ii. $aRb \Rightarrow bRa$ (R is symmetric)

iii. (aRb and bRc) $\Rightarrow aRc$ (R is transitive).

PROBABILITY DISTRIBUTIONS

Name	Parameters	Probability distribution / density function	Mean	Variance
Binomial	n, p	$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$, $r = 0, 1, 2, ..., n$	np	$np(1-p)$
Poisson	λ	$P(X = n) = \frac{e^{-\lambda}\lambda^n}{n!}$, $n = 0, 1, 2, ...$	λ	λ
Normal	μ, σ	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\}$, $-\infty < x < \infty$	μ	σ^2
Exponential	λ	$f(x) = \lambda e^{-\lambda x}$, $x > 0, \quad \lambda > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

THE F-DISTRIBUTION

The function tabulated on the next page is the inverse cumulative distribution function of Fisher's F-distribution having ν_1 and ν_2 degrees of freedom. It is defined by

$$P = \frac{\Gamma\left(\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2\right)}{\Gamma\left(\frac{1}{2}\nu_1\right)\Gamma\left(\frac{1}{2}\nu_2\right)} \nu_1^{\frac{1}{2}\nu_1} \nu_2^{\frac{1}{2}\nu_2} \int_0^x u^{\frac{1}{2}\nu_1 - 1} (\nu_2 + \nu_1 u)^{-\frac{1}{2}(\nu_1+\nu_2)} du.$$

If X has an F-distribution with ν_1 and ν_2 degrees of freedom then $Pr.(X \leq x) = P$. The table lists values of x for $P = 0.95$, $P = 0.975$ and $P = 0.99$, the upper number in each set being the value for $P = 0.95$.

ν_2	$\nu_1 : 1$	$\nu_1 : 2$	3	4	5	6	7	8	9	10	12	15	20	25	50	100	
1	161 648 4052	199 799 5000	216 864 5403	225 900 5625	230 922 5764	234 937 5859	237 948 5928	239 957 5981	241 963 6022	242 969 6056	244 977 6106	246 985 6157	248 993 6209	249 998 6240	252 1008 6303	253 1013 6334	1
2	18.51 38.51 98.50	19.00 39.00 99.00	19.16 39.17 99.17	19.25 39.25 99.25	19.30 39.30 99.30	19.33 39.33 99.33	19.35 39.36 99.36	19.37 39.37 99.37	19.38 39.39 99.39	19.40 39.40 99.40	19.41 39.41 99.42	19.43 39.43 99.43	19.45 39.45 99.45	19.46 39.46 99.46	19.48 39.48 99.48	19.49 39.49 99.49	2
3	10.13 17.44 34.12	9.55 16.04 30.82	9.28 15.44 29.46	9.12 15.10 28.71	9.01 14.88 28.24	8.94 14.73 27.91	8.89 14.62 27.67	8.85 14.54 27.49	8.81 14.47 27.35	8.79 14.42 27.23	8.74 14.34 27.05	8.70 14.25 26.87	8.66 14.17 26.69	8.63 14.12 26.58	8.58 14.01 26.35	8.55 13.96 26.24	3
4	7.71 12.22 21.20	6.94 10.65 18.00	6.59 9.98 16.69	6.39 9.60 15.98	6.26 9.36 15.52	6.16 9.20 15.21	6.09 9.07 14.98	6.04 8.98 14.80	6.00 8.90 14.66	5.96 8.84 14.55	5.91 8.75 14.37	5.86 8.66 14.20	5.80 8.56 14.02	5.77 8.50 13.91	5.70 8.38 13.69	5.66 8.32 13.58	4
5	6.61 10.01 16.26	5.79 8.43 13.27	5.41 7.76 12.06	5.19 7.39 11.39	5.05 7.15 10.97	4.95 6.98 10.67	4.88 6.85 10.46	4.82 6.76 10.29	4.77 6.68 10.16	4.74 6.62 10.05	4.68 6.52 9.89	4.62 6.43 9.72	4.56 6.33 9.55	4.52 6.27 9.45	4.44 6.14 9.24	4.41 6.08 9.13	5
6	5.99 8.81 13.75	5.14 7.26 10.92	4.76 6.60 9.78	4.53 6.23 9.15	4.39 5.99 8.75	4.28 5.82 8.47	4.21 5.70 8.26	4.15 5.60 8.10	4.10 5.52 7.98	4.06 5.46 7.87	4.00 5.37 7.72	3.94 5.27 7.56	3.87 5.17 7.40	3.83 5.11 7.30	3.75 4.98 7.09	3.71 4.92 6.99	6
7	5.59 8.07 12.25	4.74 6.54 9.55	4.35 5.89 8.45	4.12 5.52 7.85	3.97 5.29 7.46	3.87 5.12 7.19	3.79 4.99 6.99	3.73 4.90 6.84	3.68 4.82 6.72	3.64 4.76 6.62	3.57 4.67 6.47	3.51 4.57 6.31	3.44 4.47 6.16	3.40 4.40 6.06	3.32 4.28 5.86	3.27 4.21 5.75	7
8	5.32 7.57 11.26	4.46 6.06 8.65	4.07 5.42 7.59	3.84 5.05 7.01	3.69 4.82 6.63	3.58 4.65 6.37	3.50 4.53 6.18	3.44 4.43 6.03	3.39 4.36 5.91	3.35 4.30 5.81	3.28 4.20 5.67	3.22 4.10 5.52	3.15 4.00 5.36	3.11 3.94 5.26	3.02 3.81 5.07	2.97 3.74 4.96	8
9	5.12 7.21 10.56	4.26 5.71 8.02	3.86 5.08 6.99	3.63 4.72 6.42	3.48 4.48 6.06	3.37 4.32 5.80	3.29 4.20 5.61	3.23 4.10 5.47	3.18 4.03 5.35	3.14 3.96 5.26	3.07 3.87 5.11	3.01 3.77 4.96	2.94 3.67 4.81	2.89 3.60 4.71	2.80 3.47 4.52	2.76 3.40 4.41	9
10	4.96 6.94 10.04	4.10 5.46 7.56	3.71 4.83 6.55	3.48 4.47 5.99	3.33 4.24 5.64	3.22 4.07 5.39	3.14 3.95 5.20	3.07 3.85 5.06	3.02 3.78 4.94	2.98 3.72 4.85	2.91 3.62 4.71	2.85 3.52 4.56	2.77 3.42 4.41	2.73 3.35 4.31	2.64 3.22 4.12	2.59 3.15 4.01	10
12	4.75 6.55 9.33	3.89 5.10 6.93	3.49 4.47 5.95	3.26 4.12 5.41	3.11 3.89 5.06	3.00 3.73 4.82	2.91 3.61 4.64	2.85 3.51 4.50	2.80 3.44 4.39	2.75 3.37 4.30	2.69 3.28 4.16	2.62 3.18 4.01	2.54 3.07 3.86	2.50 3.01 3.76	2.40 2.87 3.57	2.35 2.80 3.47	12
15	4.54 6.20 8.68	3.68 4.77 6.36	3.29 4.15 5.42	3.06 3.80 4.89	2.90 3.58 4.56	2.79 3.41 4.32	2.71 3.29 4.14	2.64 3.20 4.00	2.59 3.12 3.89	2.54 3.06 3.80	2.48 2.96 3.67	2.40 2.86 3.52	2.33 2.76 3.37	2.28 2.69 3.28	2.18 2.55 3.08	2.12 2.47 2.98	15
20	4.35 5.87 8.10	3.49 4.46 5.85	3.10 3.86 4.94	2.87 3.51 4.43	2.71 3.29 4.10	2.60 3.13 3.87	2.51 3.01 3.70	2.45 2.91 3.56	2.39 2.84 3.46	2.35 2.77 3.37	2.28 2.68 3.23	2.20 2.57 3.09	2.12 2.46 2.94	2.07 2.40 2.84	1.97 2.25 2.64	1.91 2.17 2.54	20
25	4.24 5.69 7.77	3.39 4.29 5.57	2.99 3.69 4.68	2.76 3.35 4.18	2.60 3.13 3.85	2.49 2.97 3.63	2.40 2.85 3.46	2.34 2.75 3.32	2.28 2.68 3.22	2.24 2.61 3.13	2.16 2.51 2.99	2.09 2.41 2.85	2.01 2.30 2.70	1.96 2.23 2.60	1.84 2.08 2.40	1.78 2.00 2.29	25
50	4.03 5.34 7.17	3.18 3.97 5.06	2.79 3.39 4.20	2.56 3.05 3.72	2.40 2.83 3.41	2.29 2.67 3.19	2.20 2.55 3.02	2.13 2.46 2.89	2.07 2.38 2.78	2.03 2.32 2.70	1.95 2.22 2.56	1.87 2.11 2.42	1.78 1.99 2.27	1.73 1.92 2.17	1.60 1.75 1.95	1.52 1.66 1.82	50
100	3.94 5.18 6.90	3.09 3.83 4.82	2.70 3.25 3.98	2.46 2.92 3.51	2.31 2.70 3.21	2.19 2.54 2.99	2.10 2.42 2.82	2.03 2.32 2.69	1.97 2.24 2.59	1.93 2.18 2.50	1.85 2.08 2.37	1.77 1.97 2.22	1.68 1.85 2.07	1.62 1.77 1.97	1.48 1.59 1.74	1.39 1.48 1.60	100

NORMAL DISTRIBUTION

The function tabulated is the cumulative distribution function of a standard $N(0,1)$ random variable, namely

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt.$$

If X is distributed $N(0,1)$ then $\Phi(x) = Pr.(X \leq x)$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

THE t-DISTRIBUTION

The function tabulated is the inverse cumulative distribution function of Student's t-distribution having ν degrees of freedom. It is defined by

$$P = \frac{1}{\sqrt{\nu\pi}} \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2})}{\Gamma(\frac{1}{2}\nu)} \int_{-\infty}^{x} (1 + t^2/\nu)^{-\frac{1}{2}(\nu+1)} dt.$$

If X has Student's t-distribution with ν degrees of freedom then $Pr.(X \leq x) = P$.

ν	P=0.90	P=0.95	0.975	0.990	0.995	0.999	0.9995
1	3.078	6.314	12.706	31.821	63.657	318.302	636.619
2	1.886	2.920	4.303	6.965	9.925	22.327	31.598
3	1.638	2.353	3.182	4.541	5.841	10.215	12.941
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.894	6.859
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.405
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.611	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
50	1.299	1.676	2.009	2.403	2.678	3.261	3.496
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	1.290	1.660	1.984	2.364	2.626	3.174	3.391
200	1.286	1.653	1.972	2.345	2.601	3.131	3.340
∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291

THE χ^2 (CHI-SQUARED) DISTRIBUTION

The function tabulated is the inverse cumulative distribution function of a Chi-squared distribution having ν degrees of freedom. It is defined by

$$P = \frac{1}{2^{\nu/2}\Gamma\left(\frac{1}{2}\nu\right)} \int_0^x u^{\frac{1}{2}\nu-1} e^{-\frac{1}{2}u} du.$$

If X has an χ^2 distribution with ν degrees of freedom then $Pr.(X \leq x) = P$. For $\nu > 100$, $\sqrt{2X}$ is approximately normally distributed with mean $\sqrt{2\nu - 1}$ and unit variance.

ν	P = 0.005	P = 0.01	0.025	0.05	0.950	0.975	0.990	0.995	0.999
1.0	0.0^4393	0.0^3157	0.0^3982	0.00393	3.841	5.024	6.635	7.879	10.828
2.0	0.010003	0.02010	0.05064	0.1026	5.991	7.378	9.210	10.597	13.816
3.0	0.07172	0.1148	0.2158	0.3518	7.815	9.348	11.345	12.838	16.266
4.0	0.2070	0.2971	0.4844	0.7107	9.488	11.143	13.277	14.860	18.467
5.0	0.4117	0.5543	0.8312	1.145	11.070	12.832	15.086	16.750	20.515
6.0	0.6757	0.8721	1.237	1.635	12.592	14.449	16.812	18.548	22.458
7.0	0.9893	1.239	1.690	2.167	14.067	16.013	18.475	20.278	24.322
8.0	1.344	1.646	2.180	2.733	15.507	17.535	20.090	21.955	26.124
9.0	1.735	2.088	2.700	3.325	16.919	19.023	21.666	23.589	27.877
10.0	2.156	2.558	3.247	3.940	18.307	20.483	23.209	25.188	29.588
11.0	2.603	3.053	3.816	4.575	19.675	21.920	24.725	26.757	31.264
12.0	3.074	3.571	4.404	5.226	21.026	23.337	26.217	28.300	32.909
13.0	3.565	4.107	5.009	5.892	22.362	24.736	27.688	29.819	34.528
14.0	4.075	4.660	5.629	6.571	23.685	26.119	29.141	31.319	36.123
15.0	4.601	5.229	6.262	7.261	24.996	27.488	30.578	32.801	37.697
16.0	5.142	5.812	6.908	7.962	26.296	28.845	32.000	34.267	39.252
17.0	5.697	6.408	7.564	8.672	27.587	30.191	33.409	35.718	40.790
18.0	6.265	7.015	8.231	9.390	28.869	31.526	34.805	37.156	42.312
19.0	6.844	7.633	8.907	10.117	30.144	32.852	36.191	38.582	43.820
20.0	7.434	8.260	9.591	10.851	31.410	34.170	37.566	39.997	45.315
21.0	8.034	8.897	10.283	11.591	32.671	35.479	38.932	41.401	46.797
22.0	8.643	9.542	10.982	12.338	33.924	36.781	40.289	42.796	48.268
23.0	9.260	10.196	11.689	13.091	35.172	38.076	41.638	44.181	49.728
24.0	9.886	10.856	12.401	13.848	36.415	39.364	42.980	45.559	51.179
25.0	10.520	11.524	13.120	14.611	37.652	40.646	44.314	46.928	52.620
26.0	11.160	12.198	13.844	15.379	38.885	41.923	45.642	48.290	54.052
27.0	11.808	12.879	14.573	16.151	40.113	43.195	46.963	49.645	55.476
28.0	12.461	13.565	15.308	16.928	41.337	44.461	48.278	50.993	56.892
29.0	13.121	14.256	16.047	17.708	42.557	45.722	49.588	52.336	58.301
30.0	13.787	14.953	16.791	18.493	43.773	46.979	50.892	53.672	59.703
40.0	20.707	22.164	24.433	26.509	55.758	59.342	63.691	66.766	73.402
50.0	27.991	29.707	32.357	34.764	67.505	71.420	76.154	79.490	86.661
60.0	35.534	37.485	40.482	43.188	79.082	83.298	88.379	91.952	99.607
70.0	43.275	45.442	48.758	51.739	90.531	95.023	100.425	104.215	112.317
80.0	51.172	53.540	57.153	60.391	101.879	106.629	112.329	116.321	124.839
90.0	59.196	61.754	65.647	69.126	113.145	118.136	124.116	128.299	137.208
100.0	67.328	70.065	74.222	77.929	124.342	129.561	135.807	140.169	149.449

PHYSICAL AND ASTRONOMICAL CONSTANTS

c	Speed of light in vacuo	2.998×10^{8}m s^{-1}
e	Elementary charge	1.602×10^{-19} C
m_n	Neutron rest mass	1.675×10^{-27} kg
m_p	Proton rest mass	1.673×10^{-27} kg
m_e	Electron rest mass	9.110×10^{-31} kg
h	Planck's constant	6.626×10^{-34} J s
$\hbar$	Dirac's constant $(= h/2\pi)$	1.055×10^{-34} J s
k	Boltzmann's constant	1.381×10^{-23} J K^{-1}
G	Gravitational constant	6.673×10^{-11} N m^{2} kg^{-2}
σ	Stefan-Boltzmann constant	5.670×10^{-8} J m^{-2} K^{-4} s^{-1}
c_1	First Radiation Constant $(= 2\pi hc^2)$	3.742×10^{-16} J m^{2} s^{-1}
c_2	Second Radiation Constant $(= hc/k)$	1.439×10^{-2} m K
ε_o	Permittivity of free space	8.854×10^{-12} C^{2} N^{-1} m^{-2}
μ_o	Permeability of free scpae	$4\pi \times 10^{-7}$ H m^{-1}
N_A	Avogadro constant	6.022×10^{23} mol^{-1}
R	Gas constant	8.314 J K^{-1} mol^{-1}
a_0	Bohr radius	5.292×10^{-11} m
μ_B	Bohr magneton	9.274×10^{-24} J T^{-1}
α	Fine structure constant $(= 1/137.0)$	7.297×10^{-3}
$M_\odot$	Solar Mass	1.989×10^{30} kg
$R_\odot$	Solar radius	6.96×10^{8} m
$L_\odot$	Solar luminosity	3.827×10^{26} J s^{-1}
$M_\oplus$	Earth Mass	5.976×10^{24} kg
$R_\oplus$	Mean earth radius	6.371×10^{6} m
1 light year		9.461×10^{15} m
1 AU	Astronomical Unit	1.496×10^{11} m
1 pc	Parsec	3.086×10^{16} m
1 year		3.156×10^{7} s

ENERGY CONVERSION : 1 joule (J) = 6.2415×10^{18} electronvolts (eV)