On avoiding dependent choices in Formal Topology

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Why avoid countable/dependent choice?

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- *"The theory should be compatible with known theories such as classical set theory and the theory of a generic topos, ... and hence it should be minimal with respect to such more expressive existing theories"* - from "Toward a minimalist foundation for constructive mathematics", by Maria Emilia Maietti and Giovanni Sambin.

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- I suggest: *A good core (minimal?) theory for constructive set theory should, at least, be preserved by the Heyting valued model construction.*

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Plan of talk

- Review
- Relation Reflection
- Coinductive Definitions
- If time: a slide on characterising the collection of positivity relations

Some axiom systems for Constructive Set Theory

These are formulated in the language of ZF, but use intuitionistic logic.

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- $CZF^+ \equiv CZF^o +$ Regular Extension Axiom.

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- Restricted Separation: For each restricted class X, if a is a set the class $a \cap X$ is a set.
- A class is restricted if it is defined by a restricted formula; i.e. a formula all of whose quantifiers have one of the forms $(\forall x \in t)$or $(\exists x \in t)$ where t is a variable or parameter.

The Collection Principles

- Definition: *For classes* X, Y, R

$$R : X \succ\!\!- Y \quad \textit{iff} \quad (\forall x \in X)(\exists y \in Y)\ (x, y) \in R$$

$$R : X \succ\!\!-\!\!\prec Y \quad \textit{iff} \quad [R : X \succ\!\!- Y\ ] \&\ (\forall y \in Y)(\exists x \in X)\ (x, y) \in R$$

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- or: For all sets a, b there is a set c of subsets of $a \times b$ such that for every set $r : a \rightarrowtail b$ there is a set $r' \in c$ such that $r \supseteq r' : a \rightarrowtail b$.

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Countable Choice (CC): For all sets X, R, if $R : \mathbb{N} \rightarrowtail X$ then there is a $f : \mathbb{N} \to X$ such that

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Dependent Choices (DC): For all sets X, R, if $R : X \succ\!\!- X$ and $a \in X$ then there is $f : \mathbb{N} \to X$ such that $f(0) = a$ and

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Inductive Definitions

- Let C be a covering system on a class S; i.e. an operation $C : S \to Pow(Pow(S))$.
- X/a is a C-step if $a \in S$ and $X \in C(a)$.
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Theorem ($CZF^o + REA$): If S is a set, for each set $U \subseteq S$ the class $\mathcal{A}U$ is a set. Moreover there is a covering system D on S such that such that, for all $U \in Pow(S)$, if $a \in S$ then

$$a \in \mathcal{A}U \;\Leftrightarrow\; (\exists V \in D(a))[V \subseteq U].$$

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Let C be a covering system on a class S. A class $U \subseteq S$ is C-progressive if, for all C-steps X/a,

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Theorem ($CZF^o + uREA + DC$): If S is a set then, for each set $U \subseteq S$, the class $\mathcal{J}U$ exists and is a set.

- $uREA + DC$ can be replaced by $*REA$.

An example avoiding Dependent Choices, 1

Theorem [CZF^o+?]: Let C be a covering system on a class S and let

$$J = \bigcup \{V \in Pow(S) \mid V \text{ is } C\text{-progressive}\}.$$

Then J is the largest C-progressive class; i.e. (i) J is C-progressive, (ii) If B is a C-progressive class then $B \subseteq J$.

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Proof: (i) is easy. For (ii), for classes B_1, B_2 let

$$B_1 \mapsto B_2 \quad \equiv \quad \forall a{\in}B_1 \ \forall X{\in}C(a) \ X \between B_2.$$

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So $\forall a{\in}B \;\; \forall X{\in}C(a) \;\; \exists y{\in}B \; y \in X$, and so, by Strong Collection,

$$(*) \quad \forall a{\in}B \; \exists Y{\in}Pow(B) \;\; \forall X{\in}C(a) \; \exists y{\in}Y \;\; y \in X.$$

An example avoiding Dependent Choices, 1

Theorem [$CZF^o+?$]: Let C be a covering system on a class S and let

$$J = \bigcup \{V \in Pow(S) \mid V \text{ is } C\text{-progressive}\}.$$

Then J is the largest C-progressive class; i.e. (i) J is C-progressive, (ii) If B is a C-progressive class then $B \subseteq J$.
Proof: (i) is easy. For (ii), for classes B_1, B_2 let

$$B_1 \mapsto B_2 \quad \equiv \quad \forall a{\in}B_1 \ \forall X{\in}C(a) \ X \between B_2.$$

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We must show that $\forall a{\in}B \ \exists V{\in}Pow(S)[a \in V \mapsto V]$.

An example avoiding Dependent Choices, 2

Lemma: $\forall U \in Pow(B)\ \exists V \in Pow(B)\ \ U \mapsto V$.

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An example avoiding Dependent Choices, 2

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To prove: $\forall a \in B\ \exists V \in Pow(S)[a \in V \mapsto V]$. Let $a \in B$.
Then $\{a\} \in Pow(B)$ so that, by the lemma and RDC there is $f : \mathbb{N} \to Pow(B)$ such that $f(0) = \{a\}$ and, for $n \in \mathbb{N}$,

$$f(n) \mapsto f(n+1).$$

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Let $V = \bigcup_{n \in \mathbb{N}} f(n) \in Pow(S)$. Then $a \in V$. Also $V \mapsto V$ as

$$\begin{aligned} b \in V \quad &\Rightarrow b \in f(n) \text{ for some } n \in \mathbb{N} \\ &\Rightarrow \forall X \in C(b)\ \ X \between f(n+1) \\ &\Rightarrow \forall X \in C(b)\ \ X \between V, \end{aligned}$$

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Using RRS instead of RDC: there is a set $\mathcal{X} \subseteq Pow(B)$ such that $\{a\} \in \mathcal{X}$ and $\forall U \in \mathcal{X}\ \exists V \in \mathcal{X}\ U \mapsto V$. Let $V = \cup \mathcal{X} \in Pow(S)$. Then $a \in V \mapsto V$.

An example avoiding Dependent Choices, 3

Proof of Lemma: $\forall U \in Pow(B)\ \exists V \in Pow(B)\ \ U \mapsto V$.

Recall that

$$(*) \quad \forall a \in B\ \exists Y \in Pow(B)\ \forall X \in C(a)\ \ Y \between X.$$

An example avoiding Dependent Choices, 3

Proof of Lemma: $\forall U{\in}Pow(B)\ \exists V{\in}Pow(B)\ \ U \mapsto V$.

Recall that

$$(*) \quad \forall a{\in}B\ \exists Y{\in}Pow(B)\ \forall X{\in}C(a)\ \ Y \between X.$$

Let $U \in Pow(B)$. Then

$$\forall a{\in}U\ \exists Y{\in}Pow(B)\ \forall X{\in}C(a)\ \ Y \between X.$$

An example avoiding Dependent Choices, 3

Proof of Lemma: $\forall U{\in}Pow(B)\ \exists V{\in}Pow(B)\ \ U \mapsto V$.

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$$(*)\quad \forall a{\in}B\ \exists Y{\in}Pow(B)\ \forall X{\in}C(a)\ \ Y \between X.$$

Let $U \in Pow(B)$. Then

$$\forall a{\in}U\ \exists Y{\in}Pow(B)\ \forall X{\in}C(a)\ \ Y \between X.$$

By Strong Collection there is a set $\mathcal{Z} \subseteq Pow(B)$ such that

$$\forall a{\in}U\ \exists Y{\in}\mathcal{Z}\ \forall X{\in}C(a)\ \ Y \between X.$$

Let $V = \cup\mathcal{Z}$. Then $U \mapsto V$.

Conclusion

Recall:

$$\begin{aligned} CZF^{+} &\equiv CZF^{o} + REA \\ CZF^{u} &\equiv CZF^{o} + uREA \\ CZF^{*} &\equiv CZF^{o} + *REA \\ CZF^{*_2} &\equiv CZF^{o} + *_2REA \end{aligned}$$

- CZF^{*} and CZF^{*_2} can be used to prove certain results that had been previously proved in $CZF^{u} + DC$.

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- CZF^{*} and CZF^{*_2} can be used to prove certain results that had been previously proved in $CZF^{u} + DC$.
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- CZF^* and CZF^{*_2} can be used to prove certain results that had been previously proved in $CZF^u + DC$.
- Conjecture: CZF^* (and CZF^{*_2}) are preserved in Heyting valued models over set-presented cHa's.
- But DC does not generally hold in Heyting valued models.

A characterisation of the positivity relations $\ltimes$ on $(S, \lhd)$

Let $\mathcal{S} = (S, \lhd)$ be a fixed formal topology. I prefer to work with the positivity operators $\mathcal{J} : Pow(S) \to Pow(S)$ where, for all a, U, $a \in \mathcal{J}U \;\equiv\; a \ltimes U$.

- Call $F \subseteq S$ completely prime if, whenever $a \lhd U$,
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- Call $F \subseteq S$ completely prime if, whenever $a \lhd U$,
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- Let I be a class of completely prime sets. For all U let
$$\mathcal{J}_I U = \cup\{F \in I \mid F \subseteq U\},$$
and call I standard if
$(\forall I' \in Pow(I))\ \cup I' \in I$ and $(\forall U \in Pow(S))\ \mathcal{J}_I U$ is a set.

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- Let I be a class of completely prime sets. For all U let
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and call I standard if
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Proposition: The assignments $\mathcal{J} \mapsto \{F \mid \mathcal{J}F = F\}$ and $I \mapsto \mathcal{J}_I$ are inverse one-one correspondences between the standard classes of completely prime sets and the positivity operators.

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Note: $\mathcal{J}_I$ is a set-presented positivity operator for any set I of completely prime sets.

x0

x1

- x2

x0

x1

- x2
- x3

x0

x1

- x2
- x3

x4

x0

x1

- x2
- x3

x4