A Decidable Dynamic Logic for Agents with Motivational Attitudes*

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Abstract. The present paper defines a multi-modal logic for modelling and verifying the behaviour of intelligent and rational agents. The agents can perform actions, can reason about their own knowledge and their motivational attitudes. We give a Hilbert-style axiomatisation which is proved sound and complete with respect to a Kripke-style semantics. We also show the small model property, decidability, and give lower and upper bounds for the complexity of the satisfiability problem in the logic.

1 Introduction

Modal and temporal logics are popular for modelling agent systems. Among the more well-known agent formalisms with a modal or temporal flavour are the BDI model [20, 22], the KARO framework [15, 24], and temporal logics of knowledge and belief [7, 10, 11]. Examples of more recent work includes the proposal of a modal logic framework of belief and change in [14], and an epistemic dynamic logic in [13]. Such theories are meant to formalise the representation and reasoning about various aspects of the behaviour and mental attitudes of agents as well as the state of the environment the agents live in.

The present paper is very much based on the work of Meyer, van der Hoek and van Linder [15, 24] who have proposed a powerful agent theory, called the KARO framework. This framework develops an approach based on propositional dynamic logic for describing and reasoning about the actions of agents. Enriched by additional modal operators for the agents' knowledge, their wishes and goals, and operators for the agents' abilities and commitments, KARO provides a very expressive framework in which the informational and motivational attitudes of agents can be modelled. The informational attitudes include the agents' knowledge and beliefs. An agent's motivational attitudes determine her willingness or unwillingness to commit to actions in order to fulfil her wishes and goals. Commitments can be considered as representations of the agents' duties, that is, actions which agents must perform sooner or later. The formalisation of the motivational attitudes of agents has been a main contribution of the KARO framework [15, 24].

KARO is a purely semantic framework. Although it can be seen to incorporate deterministic propositional dynamic logic, at this moment it not known whether the whole of KARO can be formalised as a logical systems. This paper is an attempt to address this issue. Because KARO contains some sophisticated operators for managing the agents' goals and commitments, KARO is not directly expressible in (propositional) modal logic. This prevents us from using the powerful techniques developed in the field of modal logic. Our approach has been to omit some operators from KARO and try to give a modal logic axiomatisation which

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characterises the properties of the remaining operators. Although this necessarily reduces the expressivity of the logic, we obtain a logic with a sound and complete axiomatisation, and some other nice properties. For example, the logic is shown to be decidable, which is obviously an important property for an agent logic if it is to be used in practice. Moreover, we believe the logic remains expressive enough to be useful in applications. From a technical point of view, as our results suggest, the logic is at least as expressive and complex as propositional dynamic logic and also converse propositional dynamic logic.

The logic introduced in this paper is called *agent dynamic logic* (ADL). As in the KARO framework, in ADL the dynamic activities of agents are formulated in a propositional dynamic logic (PDL). This formalises the specification of the results of the agents' actions. For example, $[\alpha]_i \phi$ says property ϕ holds always after agent i has performed the action α. ADL also allows for the definition of the opportunity ($\langle\alpha\rangle_i \top$) and ability to perform actions ($\mathbf{A}_i\alpha$). In order to describe formally the commitments of an agent the ADL logic has a static commitment constructor which states the commitment of an agent to an action ($\mathbf{Comm}_i\alpha$), and two dynamic constructors which represent the actions of committing or undoing a commitment ($\mathbf{c}\alpha$ and $\mathbf{u}\alpha$).

To illustrate the capabilities of the ADL-operators consider the scenario of an examination situation for two agents, a teacher t and a student s.

- $\mathbf{A}_t\mathtt{give\_task}$ represents that 'the teacher is able to give a task to the student'.
- $\langle\mathtt{give\_task}\rangle_t \top$ represents that 'the teacher has an opportunity to give a task'.
- $\langle\mathtt{perform\_task}\rangle_s \top$ represents that 'the student has an opportunity to perform the task'.
- $[\mathtt{give\_task}]_t \mathbf{K}_s \mathbf{Comm}_s \mathtt{perform\_task}$ represents that 'the student knows he is committed to performing the task whenever it is given by the teacher'.

The last formula gives an example of a contract agreement between the two agents, which suggests a potential area of application for ADL. Although this example is rather primitive, we believe that it is possible to formally describe legal contracts in terms of the ADL language, and then to use the logic for analysis and verification purposes.

The specification of teamwork might be another use for ADL. In the development of a complex project, for instance, the duties of the individual developers are usually fixed at the outset. So, if we can formally represent the duties and abilities of the developers in the team, as well as the goals of the project then we can reason about the stages during project development and their interdependencies. (In the setting of this paper the wishes of agents may be regarded as first approximations of the goals of agents.) On the other hand, as soon as an optimal sequence of development has been determined one can assign tasks to developers and fix conditions of teamwork such that the optimum can be realised. Further examples and discussion of the fundamental nature of commitments, abilities and wishes can be found in [15, 24].

The ADL logic and the KARO framework are similar in many respects but there are some important differences. First, in order to reason about actions KARO extends the strictly deterministic version of PDL with deterministic 'while' and 'if' action constructors while ADL uses non-deterministic PDL with the test operator. We did not want to constrain ourselves to the more narrow language if decidability or other good properties are not lost in the non-deterministic case. Non-determinism is a natural property in real world applications as it allows the exploration of alternative paths of execution of actions. Therefore, ADL cannot be reduced to KARO, at least, not in a straightforward way. Second, some of the KARO-operators are not included in ADL. This concerns the operators for representing goals, making choices, and an implementability operator. Although the first two operators could be added to ADL in future, it is not clear how best to handle the planning element of the implementability operator which involves quantifying over atomic actions. This shows there is no straightforward translation of KARO into ADL. Third, as mentioned above, full KARO is defined only by its semantics whereas ADL is a logic with both semantics and Hilbert-style axiomatisation. This means, because ADL has a purely relational semantics, almost all what is known in modal logic can be applied to ADL. Finally, one of the main features of ADL, distinguishing it from KARO, is the treatment of the abilities and commitments with respect to non-determinism. Our definition is an attempt to merge the 'internal' (angelic) and 'external' (demonic) approaches considered by [23] for the ability operator, which can be seen from comparing the defining axioms and the desirable properties identified in [23, Theorems 4 and 9]. Our approach does not lead to inconsistency because we establish ADL is sound.

Despite the many difference, ADL and KARO have some common properties. In constructing ADL, it was our intention to keep most of the important properties of agents which are discussed and formalised in the KARO framework, possibly, slightly modifying them. For example, the following properties of KARO are also true in ADL.

$$\langle \mathbf{c}\alpha \rangle_i \top \leftrightarrow \langle \mathbf{c}\alpha \rangle_i \mathbf{Comm}_i \alpha \qquad \mathbf{Comm}_i \mathbf{while}(\phi, \alpha) \wedge \mathbf{K}_i \phi \rightarrow \mathbf{Comm}_i (\phi?;\alpha;\mathbf{while}(\phi, \alpha))$$

The structure of the paper is as follows. Sections 2 and 3 give an axiomatisation and define a Kripke-style semantics for the logic ADL. Soundness is proved in Section 3. In Section 4 we analyse the properties of ADL and show completeness with respect to standard and non-standard models, the small model property, decidability and discuss the complexity of ADL.

2 Axiomatisation of ADL

The language $\mathcal{L}$ of the logic ADL considered in this paper is defined over the following primitive types: a countable set $\mathbf{Var} = \{p, q, r, \ldots\}$ of propositional variables, a countable set $\mathbf{AtAc} = \{a, b, c, \ldots\}$ of atomic action variables, and a finite set $\mathbf{Ag}$ of agents. $\mathcal{L}$ is constructed over the usual Boolean connectives, $\rightarrow$ and $\perp$, the standard PDL connectives, $\cup$, ;, $*$, ? and $[\_]_i$, where i denotes any agent, and modal connectives $\mathbf{K}_i$ (for knowledge) and $\mathbf{W}_i$ (for wishes). In addition, there are four unary operations on actions, $\mathbf{A}_i$ (for 'has the ability to perform'), $\mathbf{Comm}_i$ (for 'committed to'), $\mathbf{c}$ (for 'commit to') and $\mathbf{u}$ (for 'uncommit'). The former two are formula forming, and the latter two are action forming. The set **For** of formulae and the set **Ac** of actions in $\mathcal{L}$ are the smallest sets that satisfy the following conditions ($i \in \mathbf{Ag}$).

- $\mathbf{AtAc} \subseteq \mathbf{Ac}$, $\mathbf{Var} \cup \{\perp\} \subseteq \mathbf{For}$.
- If $\phi \in \mathbf{For}$ and $\alpha, \beta \in \mathbf{Ac}$ then $\phi?, \alpha^*, \mathbf{c}\alpha, \mathbf{u}\alpha, \alpha \cup \beta, \alpha;\beta \in \mathbf{Ac}$.
- If $\phi, \psi \in \mathbf{For}$, $\alpha \in \mathbf{Ac}$ then $\phi \rightarrow \psi, \mathbf{K}_i\phi, \mathbf{W}_i\phi, [\alpha]_i\phi, \mathbf{A}_i\alpha, \mathbf{Comm}_i\alpha \in \mathbf{For}$.

As usual we use the following abbreviations: $\neg\phi$ for $\phi \rightarrow \perp$, $\phi \vee \psi$ for $\neg\phi \rightarrow \psi$, $\phi \wedge \psi$ for $\neg(\neg\phi \vee \neg\psi)$, $\langle\alpha\rangle_i\phi$ for $\neg[\alpha]_i\neg\phi$, and $\phi \leftrightarrow \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

By definition, an *atomic action* is an action variable, and a *semi-atomic* action is an atomic action or a test $\phi?$ action.

ADL is a combination of PDL, S5, and K with some extra axioms for the ability and commitment operators. Formally, the axioms of ADL are:

A1. All classical tautologies over $\mathcal{L}$

A2. The PDL axioms (see [16]) for each $[\alpha]_i$ ($\alpha \in \mathbf{Ac}$, $i \in \mathbf{Ag}$):

1. $[\alpha]_i(\phi \to \psi) \to ([\alpha]_i\phi \to [\alpha]_i\psi)$
2. $[\alpha \cup \beta]_i\phi \leftrightarrow [\alpha]_i\phi \wedge [\beta]_i\phi$
3. $[\alpha;\beta]_i\phi \leftrightarrow [\alpha]_i[\beta]_i\phi$
4. $[\alpha^*]_i\phi \to \phi \wedge [\alpha]_i\phi$
5. $[\alpha^*]_i\phi \to [\alpha]_i[\alpha^*]_i\phi$
6. $\phi \wedge [\alpha^*]_i(\phi \to [\alpha]_i\phi) \to [\alpha^*]_i\phi$
7. $[\phi?]_i\psi \leftrightarrow (\phi \to \psi)$

A3. The S5 axioms for each $\mathbf{K}_i$ ($i \in \mathbf{Ag}$):

1. $\mathbf{K}_i(\phi \to \psi) \to (\mathbf{K}_i\phi \to \mathbf{K}_i\psi)$
2. $\mathbf{K}_i\phi \to \mathbf{K}_i\mathbf{K}_i\phi$
3. $\mathbf{K}_i\phi \to \phi$
4. $\neg\mathbf{K}_i\phi \to \mathbf{K}_i\neg\mathbf{K}_i\phi$

A4. The K axiom for each $\mathbf{W}_i$ ($i \in \mathbf{Ag}$):
$\mathbf{W}_i(\phi \to \psi) \to (\mathbf{W}_i\phi \to \mathbf{W}_i\psi)$

A5. The following axioms for each $\mathbf{A}_i$ ($i \in \mathbf{Ag}$):

1. $\mathbf{A}_i\alpha \cup \beta \leftrightarrow \mathbf{A}_i\alpha \vee \mathbf{A}_i\beta$
2. $\mathbf{A}_i\alpha;\beta \leftrightarrow \mathbf{A}_i\alpha \wedge [\alpha]_i\mathbf{A}_i\beta$
3. $\mathbf{A}_i\alpha^* \to \mathbf{A}_i\alpha \wedge [\alpha]_i\mathbf{A}_i\alpha^*$
4. $\mathbf{A}_i\alpha \wedge [\alpha^*]_i(\mathbf{A}_i\alpha \to [\alpha]_i\mathbf{A}_i\alpha) \to \mathbf{A}_i\alpha^*$

The first axiom of the last group expresses the fact that to be able to perform a non-deterministic choice of two actions is exactly the same as to be able to do the first action or to be able to do the second one. The other three axioms of this group express the persistence and sequential behaviour of the agents' abilities.

A similar group of axioms for the static commitment operators, $\mathbf{Comm}_i$ ($i \in \mathbf{Ag}$), are:

A6.
1. $\mathbf{Comm}_i\alpha \cup \beta \leftrightarrow \mathbf{Comm}_i\alpha \vee \mathbf{Comm}_i\beta$
2. $\mathbf{Comm}_i\alpha;\beta \leftrightarrow \mathbf{Comm}_i\alpha \wedge [\alpha]_i\mathbf{Comm}_i\beta$
3. $\mathbf{Comm}_i\alpha^* \to \mathbf{Comm}_i\alpha \wedge [\alpha]_i\mathbf{Comm}_i\alpha^*$
4. $\mathbf{Comm}_i\alpha \wedge [\alpha^*]_i(\mathbf{Comm}_i\alpha \to [\alpha]_i\mathbf{Comm}_i\alpha) \to \mathbf{Comm}_i\alpha^*$

The axioms for the dynamic commit and uncommit operators, $\mathbf{c}$ and $\mathbf{u}$, are:

A7.
1. $[\mathbf{c}\alpha]_i\mathbf{Comm}_i\alpha$
2. $[\mathbf{u}\alpha]_i\neg\mathbf{Comm}_i\alpha$
3. $\mathbf{A}_i\mathbf{c}\alpha$
4. $\mathbf{A}_i\mathbf{u}\alpha$
5. $\mathbf{Comm}_i\mathbf{c}\alpha \leftrightarrow \mathbf{Comm}_i\alpha$
6. $\mathbf{Comm}_i\mathbf{u}\alpha \leftrightarrow \neg\mathbf{Comm}_i\alpha$

The first two axioms express that an agent cannot be uncommitted after committing to an action and, vice versa, cannot be committed just after uncommitting. The axioms 3 and 4 say that any agent is always able to perform actions of committing and uncommitting. The last two axioms are forms of simplification to being committed. They say that to be committed to (un)committing to an action is exactly the same as to be (un)committed to this action.

The inference rules of ADL are modus ponens for formulae, $\phi, \phi \to \psi \vdash \psi$, and the necessitation rules for the modal connectives,

$$\phi \vdash [\alpha]_i\phi, \quad \phi \vdash \mathbf{K}_i\phi, \quad \text{and} \quad \phi \vdash \mathbf{W}_i\phi, \quad \text{for any action } \alpha \text{ and agent } i.$$

Let L be any logic in the language $\mathcal{L}$ and $\Gamma \cup \{\phi\}$ is a set of formulae. We will write $\Gamma \vdash_L \phi$ if ϕ is deducible (in usual sense) by modus ponens and the above necessitation rules from Γ and substitution instances of the axioms of L. We will also write $L \vdash \phi$ if $\emptyset \vdash_L \phi$.

3 Semantics

The semantics of our agent dynamic logic is defined in the familiar Kripke-style where models are defined over relational structures called frames. We define an *ADL-frame* to be a tuple

$$F = \langle S, Q_i, R_i^{\mathbf{K}}, R_i^{\mathbf{W}}, A_i, Comm_i, Q_i^{\mathbf{c}}, Q_i^{\mathbf{u}} \rangle_{i \in \mathbf{Ag}},$$

of a non-empty set S of states, families of functions Q_i from the set $\mathbf{AtAc}$ into S^2, families of binary relations $R_i^{\mathbf{W}}$ on S, families of equivalence relations $R_i^{\mathbf{K}}$ on S, families of functions A_i and $Comm_i$ from the set of semi-atomic action into the set of subsets of S, and families of functions $Q_i^{\mathbf{c}}$ and $Q_i^{\mathbf{u}}$ from the set of all actions into the set of binary relations on S.

For any set S, by Id_S we denote the identity relation on S, i.e. the relation $\{(s,s) \mid s \in S\}$. $R \circ R'$ denotes the relational composition of two relations R and R', and R^* denotes the reflexive transitive closure of R.

Definition 1. For some action terms α and β let the relations $Q_i(\alpha)$, $Q_i(\beta)$ and sets $A_i(\alpha)$, $A_i(\beta)$, $Comm_i(\alpha)$, $Comm_i(\beta)$ be given. Define the Q_i, A_i, and $Comm_i$ on complex terms by:

$$
\begin{array}{rlrl}
Q_i(\alpha \cup \beta) & \rightleftharpoons Q_i(\alpha) \cup Q_i(\beta) & Q_i(\alpha;\beta) & \rightleftharpoons Q_i(\alpha) \circ Q_i(\beta) \\
Q_i(\alpha^*) & \rightleftharpoons (Q_i(\alpha))^* & A_i(\alpha \cup \beta) & \rightleftharpoons A_i(\alpha) \cup A_i(\beta) \\
A_i(\alpha;\beta) & \rightleftharpoons A_i(\alpha) \cap \{s \mid \forall t \in S\,(sQ_i(\alpha)t \Rightarrow t \in A_i(\beta))\} & & \\
A_i(\alpha^*) & \rightleftharpoons A_i(\alpha) \cap \{s \mid \forall t,u \in S\,(sQ_i(\alpha^*)t \;\&\; t \in A_i(\alpha) \Rightarrow (tQ_i(\alpha)u \Rightarrow u \in A_i(\alpha))\,)\} & & \\
A_i(\mathbf{c}\alpha) & \rightleftharpoons S & A_i(\mathbf{u}\alpha) & \rightleftharpoons S \\
Comm_i(\alpha \cup \beta) & \rightleftharpoons Comm_i(\alpha) \cup Comm_i(\beta) & & \\
Comm_i(\alpha;\beta) & \rightleftharpoons Comm_i(\alpha) \cap \{s \mid \forall t \in S\,(sQ_i(\alpha)t \Rightarrow t \in Comm_i(\beta))\} & & \\
Comm_i(\alpha^*) & \rightleftharpoons Comm_i(\alpha) \cap \{s \mid \forall t,u \in S\;(\,sQ_i(\alpha^*)t \;\&\; t \in Comm_i(\alpha) \Rightarrow & & \\
 & \qquad (tQ_i(\alpha)u \Rightarrow u \in Comm_i(\alpha))\,)\} & & \\
Comm_i(\mathbf{c}\alpha) & \rightleftharpoons Comm_i(\alpha) & Comm_i(\mathbf{u}\alpha) & \rightleftharpoons S \setminus Comm_i(\alpha) \\
Q_i(\mathbf{u}\alpha) & \rightleftharpoons Q_i^{\mathbf{u}}(\alpha) \circ \mathrm{Id}_{S \setminus Comm_i(\alpha)} & Q_i(\mathbf{c}\alpha) & \rightleftharpoons Q_i^{\mathbf{c}}(\alpha) \circ \mathrm{Id}_{Comm_i(\alpha)}
\end{array}
$$

An example of a frame is a tuple F_0 in which $S = \{s\}$, $Q_i(a) = R_i^{\mathbf{K}} = R_i^{\mathbf{W}} = \{(s,s)\}$, $A_i(a) = Comm_i(a) = S$, and $Q_i^{\mathbf{c}}(\alpha) = Q_i^{\mathbf{u}}(\alpha) = \emptyset$.

Definition 2. An *ADL-model* M is any tuple $\langle F, V \rangle$, where F is a frame and V is a function, mapping each propositional variable p to some subset $V(p)$ of the set of states of F. V extends to the set of all formulae in the expected way, namely, using induction on the structure of a formula ϕ define $V(\phi)$ by:

$$
\begin{array}{rl}
V(\bot) \rightleftharpoons \emptyset & \qquad V(\phi \to \psi) \rightleftharpoons (S \setminus V(\phi)) \cup V(\psi) \\
V(\mathbf{K}_i\phi) & \rightleftharpoons \{s \mid \forall t (sR_i^{\mathbf{K}}t \Rightarrow t \in V(\phi))\} \\
V(\mathbf{W}_i\phi) & \rightleftharpoons \{s \mid \forall t (sR_i^{\mathbf{W}}t \Rightarrow t \in V(\phi))\} \\
V([\alpha]_i\phi) & \rightleftharpoons \{s \mid \forall t (sQ_i(\alpha)t \Rightarrow t \in V(\phi))\} \\
V(\mathbf{A}_i\alpha) \rightleftharpoons A_i(\alpha) & \qquad V(\mathbf{Comm}_i\alpha) \rightleftharpoons Comm_i(\alpha)
\end{array}
$$

$Q_i(\alpha)$ is defined by induction on the structure of α: 1. If $\alpha \in \mathbf{AtAc}$ then $Q_i(\alpha) \rightleftharpoons Q_i(a)$. If $\alpha = \psi?$ for some formula ψ, then $Q_i(\psi?) \rightleftharpoons \{(t,t) \mid t \in V(\psi)\}$. 2. If $\alpha = \beta \cup \gamma$, $\alpha = \beta;\gamma$, $\alpha = \beta^*$, $\alpha = \mathbf{c}\beta$ or $\alpha = \mathbf{u}\beta$ then $Q_i(\beta)$, $Q_i(\gamma)$, $Comm_i(\beta)$, $Comm_i(\gamma)$ are defined, and $Q_i(\alpha)$ is as given in Definition 1.

$A_i(\alpha)$ and $Comm_i(\alpha)$ are defined by induction on the structure of α: For any semi-atomic action α, $A_i(\alpha)$ and $Comm_i(\alpha)$ are defined in the frame, or otherwise, the specification is as in Definition 1.

As usual, we write $M, s \models \phi$ (or just $s \models \phi$) iff $s \in V(\phi)$ in M, and $M \models \phi$ iff $M, s \models \phi$ for all s in M. For any class of models K and any set of formulae $\Gamma \cup \{\phi\}$ we write $\Gamma \models_K \phi$ if $\forall M \in K\ ((\forall \gamma \in \Gamma\ M \models \gamma) \Rightarrow M \models \phi)$. A logic L is *sound* with respect to some class K of models if for any formula ϕ $L \vdash \phi$ implies that $M \models \phi$ for any model M from K. L is *complete* with respect to class K of models if the backward implication holds, i.e. if $M \models \phi$ for any model M from K then $L \vdash \phi$. L is *strongly complete* with respect to K if $\Gamma \models_K \phi$ implies $\Gamma \vdash_L \phi$ for any set of formulae $\Gamma \cup \{\phi\}$.

Consider $M_0 \rightleftharpoons \langle F_0, V_0 \rangle$, where F_0 is the frame defined above, and $V_0(p) = \{s\}$ for any propositional variable p. It is not difficult to check that M_0 is an ADL-model.

From the above definition each model M can be viewed as a tuple

$$\langle S, Q_i, R_i^{\mathbf{K}}, R_i^{\mathbf{W}}, \models \rangle_{i \in \mathbf{Ag}},$$

where S, Q_i, $R_i^{\mathbf{K}}$ and $R_i^{\mathbf{W}}$ are defined as above, and $\models$ is a truth relation on $S \times \mathbf{For}$.

As usual the following soundness theorem can be shown.

Theorem 3 (Soundness). ADL is sound with respect to the class of all ADL-models.

4 Properties of ADL

In this section, we apply standard techniques of modal logic for constructing canonical models and filtrations (cf. e.g. [2, 12]) to prove completeness and decidability results for ADL. We skip all the standard details and state only definitions and results.

4.1 Completeness with respect to non-standard models

To begin with, recall canonical models are built from maximally consistent sets of formulae. A *maximal consistent set (MCS)* is a consistent set Γ of formulae such that Γ includes all theorems of ADL, and any superset of Γ is inconsistent. Any consistent set of ADL-formulae can be extended to a MCS in a standard way.

Let S be the set of all MCSs. We define the appropriate relations on S by the following equivalences:

$$\begin{array}{rlrl} sR_i^{\mathbf{K}}t & \text{iff } \forall \mathbf{K}_i\phi \in s,\ \phi \in t & \quad sR_i^{\mathbf{W}}t & \text{iff } \forall \mathbf{W}_i\phi \in s,\ \phi \in t \\ sQ_i(\alpha)t & \text{iff } \forall [\alpha]_i\phi \in s,\ \phi \in t & s \models \phi & \text{iff } \phi \in s \end{array}$$

The model $M = \langle S, Q_i, R_i^{\mathbf{K}}, R_i^{\mathbf{W}}, \models \rangle$ thus obtained is the *canonical model* of ADL.

Lemma 4 (Existence Lemma). For any state $s \in S$ and any formula ϕ

1. if $\neg \mathbf{K}_i \neg \phi \in s$ then $\exists t \in S\ (sR_i^{\mathbf{K}}t\ \&\ \phi \in t)$
2. if $\neg \mathbf{W}_i \neg \phi \in s$ then $\exists t \in S\ (sR_i^{\mathbf{W}}t\ \&\ \phi \in t)$
3. if $\langle \alpha \rangle_i \phi \in s$ then $\exists t \in S\ (sQ_i(\alpha)t\ \&\ \phi \in t)$

Lemma 5.

1. $R_i^{\mathbf{K}}$ is an equivalence relation
2. $s \models \mathbf{Comm}_i\alpha$ whenever $tQ_i(\mathbf{c}\alpha)s$ for some $t \in S$
3. $s \not\models \mathbf{Comm}_i\alpha$ whenever $tQ_i(\mathbf{u}\alpha)s$ for some $t \in S$
4. $Q_i(\alpha \cup \beta) = Q_i(\alpha) \cup Q_i(\beta)$

5. $Q_i(\alpha;\beta) = Q_i(\alpha) \circ Q_i(\beta)$
6. $s \models [\alpha^*]_i\phi$ iff $s \models \phi \wedge [\alpha]_i[\alpha^*]_i\phi$
7. $s \models [\alpha^*]_i\phi$ iff $s \models \phi \wedge [\alpha^*]_i(\phi \to [\alpha]_i\phi)$,
8. $s \models \mathbf{Comm}_i\alpha \cup \beta$ iff $s \models \mathbf{Comm}_i\alpha \vee \mathbf{Comm}_i\beta$
9. $s \models \mathbf{Comm}_i\alpha;\beta$ iff $s \models \mathbf{Comm}_i\alpha \wedge [\alpha]_i\mathbf{Comm}_i\beta$
10. $s \models \mathbf{Comm}_i\alpha^*$ iff $s \models \mathbf{Comm}_i\alpha \wedge [\alpha^*]_i(\mathbf{Comm}_i\alpha \to [\alpha]_i\mathbf{Comm}_i\alpha)$
11. $s \models \mathbf{Comm}_i\mathbf{c}\alpha$ iff $s \models \mathbf{Comm}_i\alpha$
12. $s \models \mathbf{Comm}_i\mathbf{u}\alpha$ iff $s \not\models \mathbf{Comm}_i\alpha$
13. $s \models \mathbf{A}_i\alpha \cup \beta$ iff $s \models \mathbf{A}_i\alpha \vee \mathbf{A}_i\beta$
14. $s \models \mathbf{A}_i\alpha;\beta$ iff $s \models \mathbf{A}_i\alpha \wedge [\alpha]_i\mathbf{A}_i\beta$
15. $s \models \mathbf{A}_i\alpha^*$ iff $s \models \mathbf{A}_i\alpha \wedge [\alpha^*]_i(\mathbf{A}_i\alpha \to [\alpha]_i\mathbf{A}_i\alpha)$
16. $s \models \mathbf{A}_i\mathbf{c}\alpha$ for all $s \in S$
17. $s \models \mathbf{A}_i\mathbf{u}\alpha$ for all $s \in S$

Unfortunately, the canonical model does not satisfy the property $Q_i(\alpha^*) = (Q_i(\alpha))^*$, and is thus not a standard model for ADL in the sense of Definition 2. Any model which satisfies all the properties of Lemma 5 will be called a *non-standard model*.

Using Lemmas 4 and 5 we can prove that the truth relation in the canonical model is defined correctly in the sense of Definition 2. Thus, as for standard modal logics, the completeness of ADL with respect to M follows from the definition of M.

Theorem 6. ADL is complete with respect to its canonical model.

Theorem 7. ADL is strongly complete with respect to the class of all non-standard ADL-models.

It is worth remarking that strong completeness of ADL without the $*$ operator can be obtained by an analogous argument, because the canonical model M is a standard model in this restricted language.

Theorem 8. The $*$-free fragment of ADL is strongly complete with respect to the class of all ADL-models.

4.2 Decidability and completeness with respect to standard models

For simplicity, we will consider ADL without operators $\mathbf{K}_i$ and $\mathbf{W}_i$, because if we obtain a completeness theorem and the finite model property for this fragment, then the full ADL will possess these properties because of a well-known preservation property of fusions of modal logics [25, Theorem 2.6].

Definition 9. A set X of formulae is *FL-closed* if it satisfies the following properties for any $i \in \mathbf{Ag}$:

(FL1) if $\phi \to \psi \in X$ then $\phi, \psi \in X$
(FL2) if $[\alpha \cup \beta]_i\phi \in X$ then $[\alpha]_i\phi, [\beta]_i\phi \in X$
(FL3) if $[\alpha;\beta]_i\phi \in X$ then $[\alpha]_i[\beta]_i\phi \in X$
(FL4) if $[\alpha^*]_i\phi \in X$ then $[\alpha]_i[\alpha^*]_i\phi \in X$
(FL5) if $[\phi?]_i\psi \in X$ then $\phi \to \psi \in X$
(FL6) if $[\mathbf{c}\alpha]_i\phi \in X$ then $\mathbf{Comm}_i\alpha \in X$
(FL7) if $[\mathbf{u}\alpha]_i\phi \in X$ then $\mathbf{Comm}_i\alpha \in X$

(FL8) if $[\alpha]_i\phi \in X$ then $\phi \in X$
(FL9) if $\mathbf{Comm}_i\alpha \cup \beta \in X$ then $\mathbf{Comm}_i\alpha, \mathbf{Comm}_i\beta \in X$
(FL10) if $\mathbf{Comm}_i\alpha;\beta \in X$ then $\mathbf{Comm}_i\alpha, [\alpha]_i\mathbf{Comm}_i\beta \in X$
(FL11) if $\mathbf{Comm}_i\alpha^* \in X$ then $[\alpha^*]_i\mathbf{Comm}_i\alpha \in X$
(FL12) if $\mathbf{Comm}_i\mathbf{c}\alpha \in X$ then $\mathbf{Comm}_i\alpha \in X$
(FL13) if $\mathbf{Comm}_i\mathbf{u}\alpha \in X$ then $\mathbf{Comm}_i\alpha \in X$
(FL14) if $\mathbf{A}_i\alpha \cup \beta \in X$ then $\mathbf{A}_i\alpha, \mathbf{A}_i\beta \in X$
(FL15) if $\mathbf{A}_i\alpha;\beta \in X$ then $\mathbf{A}_i\alpha, [\alpha]_i\mathbf{A}_i\beta \in X$
(FL16) if $\mathbf{A}_i\alpha^* \in X$ then $[\alpha^*]_i\mathbf{A}_i\alpha \in X$

Let Σ be a set of formulae. The *Fischer-Ladner closure* (or, *FL-closure* for short) of Σ is the smallest FL-closed set which contains Σ. We denote it by $FL(\Sigma)$.

Definition 10. For any ADL-formula ϕ, we define the set $FL(\phi)$ using induction on the structure of ϕ.

1. $FL(\bot) \rightleftharpoons \{\bot\}$ and $FL(p) \rightleftharpoons \{p\}$ for any atomic proposition p
2. $FL(\phi \to \psi) \rightleftharpoons \{\phi \to \psi\} \cup FL(\phi) \cup FL(\psi)$
3. $FL([\alpha]_i\phi) \rightleftharpoons FL^{\Box}([\alpha]_i\phi) \cup FL(\phi)$
4. $FL^{\Box}([a]_i\phi) \rightleftharpoons \{[a]_i\phi\}$ for any atomic action a
5. $FL^{\Box}([\phi?]_i\psi) \rightleftharpoons \{[\phi?]_i\psi\} \cup FL(\phi)$
6. $FL^{\Box}([\alpha \cup \beta]_i\phi) \rightleftharpoons \{[\alpha \cup \beta]_i\phi\} \cup FL^{\Box}([\alpha]_i\phi) \cup FL^{\Box}([\beta]_i\phi)$
7. $FL^{\Box}([\alpha;\beta]_i\phi) \rightleftharpoons \{[\alpha;\beta]_i\phi\} \cup FL^{\Box}([\alpha]_i[\beta]_i\phi) \cup FL^{\Box}([\beta]_i\phi)$
8. $FL^{\Box}([\alpha^*]_i\phi) \rightleftharpoons \{[\alpha^*]_i\phi\} \cup FL^{\Box}([\alpha]_i[\alpha^*]_i\phi)$
9. $FL^{\Box}([\mathbf{c}\alpha]_i\phi) \rightleftharpoons \{[\mathbf{c}\alpha]_i\phi\} \cup FL(\mathbf{Comm}_i\alpha)$
10. $FL^{\Box}([\mathbf{u}\alpha]_i\phi) \rightleftharpoons \{[\mathbf{u}\alpha]_i\phi\} \cup FL(\mathbf{Comm}_i\alpha)$
11. $FL(\mathbf{Comm}_i a) \rightleftharpoons \{\mathbf{Comm}_i a\}$ for any atomic action a
12. $FL(\mathbf{Comm}_i\phi?) \rightleftharpoons \{\mathbf{Comm}_i\phi?\} \cup FL(\phi)$
13. $FL(\mathbf{Comm}_i\alpha \cup \beta) \rightleftharpoons \{\mathbf{Comm}_i\alpha \cup \beta\} \cup FL(\mathbf{Comm}_i\alpha) \cup FL(\mathbf{Comm}_i\beta)$
14. $FL(\mathbf{Comm}_i\alpha;\beta) \rightleftharpoons \{\mathbf{Comm}_i\alpha;\beta\} \cup FL(\mathbf{Comm}_i\alpha) \cup FL([\alpha]_i\mathbf{Comm}_i\beta)$
15. $FL(\mathbf{Comm}_i\alpha^*) \rightleftharpoons \{\mathbf{Comm}_i\alpha^*\} \cup FL([\alpha^*]_i\mathbf{Comm}_i\alpha)$
16. $FL(\mathbf{Comm}_i\mathbf{c}\alpha) \rightleftharpoons \{\mathbf{Comm}_i\mathbf{c}\alpha\} \cup FL(\mathbf{Comm}_i\alpha)$
17. $FL(\mathbf{Comm}_i\mathbf{u}\alpha) \rightleftharpoons \{\mathbf{Comm}_i\mathbf{u}\alpha\} \cup FL(\mathbf{Comm}_i\alpha)$
18. $FL(\mathbf{A}_i a) \rightleftharpoons \{\mathbf{A}_i a\}$ for any atomic action a
19. $FL(\mathbf{A}_i\phi?) \rightleftharpoons \{\mathbf{A}_i\phi?\} \cup FL(\phi)$
20. $FL(\mathbf{A}_i\alpha \cup \beta) \rightleftharpoons \{\mathbf{A}_i\alpha \cup \beta\} \cup FL(\mathbf{A}_i\alpha) \cup FL(\mathbf{A}_i\beta)$
21. $FL(\mathbf{A}_i\alpha;\beta) \rightleftharpoons \{\mathbf{A}_i\alpha;\beta\} \cup FL(\mathbf{A}_i\alpha) \cup FL([\alpha]_i\mathbf{A}_i\beta)$
22. $FL(\mathbf{A}_i\alpha^*) \rightleftharpoons \{\mathbf{A}_i\alpha^*\} \cup FL([\alpha^*]_i\mathbf{A}_i\alpha)$
23. $FL(\mathbf{A}_i\mathbf{c}\alpha) \rightleftharpoons \{\mathbf{A}_i\mathbf{c}\alpha\}$
24. $FL(\mathbf{A}_i\mathbf{u}\alpha) \rightleftharpoons \{\mathbf{A}_i\mathbf{u}\alpha\}$

Lemma 11. $FL(\phi) = FL(\{\phi\})$, i.e. $FL(\phi)$ is the FL-closure of the formula ϕ.

It is easy to see that $FL(\Sigma)$ is finite whenever Σ is finite. The following lemma allows us to find an upper bound for the cardinality of $FL(\Sigma)$ in terms of the lengths of formulae from Σ.

We denote by $|\phi|$ and $|\alpha|$ the length (as number of the symbols excluding parentheses) of ϕ and α, respectively. By $\#\phi$ we denote a maximal number of occurrences of the symbols ; and $*$ below the operators $\mathbf{Comm}_i$, $\mathbf{A}_i$, $\mathbf{c}$ and $\mathbf{u}$ in ϕ.

Lemma 12.

1. $\mathrm{Card}(FL(\phi)) \leq |\phi| 2^{\#\phi}$
2. $\mathrm{Card}(FL^{\Box}([\alpha]_i\phi)) \leq |\alpha| 2^{\#[\alpha]_i\phi}$
3. $\mathrm{Card}(FL(\mathbf{Comm}_i\alpha)) \leq |\alpha| 2^{\#\mathbf{Comm}_i\alpha}$
4. $\mathrm{Card}(FL(\mathbf{A}_i\alpha)) \leq |\alpha| 2^{\#\mathbf{A}_i\alpha}$

Proof. All properties can be proved by simultaneous induction on the length of ϕ and the length of α (cf. [12, Lemma 6.1]). First, we prove (i).

The cases $\phi = \bot, p$ are easy.

$$\begin{aligned}
\mathrm{Card}(FL(\xi \to \eta)) &\leq \mathrm{Card}(FL(\xi)) + \mathrm{Card}(FL(\eta)) \leq \\
&\leq |\xi| 2^{\#\xi} + |\eta| 2^{\#\eta} \leq \\
&\leq (|\xi| + |\eta|) 2^{\#\phi} = |\phi| 2^{\#\phi} \\
\mathrm{Card}(FL([\alpha]_i\psi)) &\leq \mathrm{Card}(FL^{\Box}(\phi)) + \mathrm{Card}(FL(\psi)) \leq \\
&\leq |\alpha| 2^{\#\phi} + |\psi| 2^{\#\psi} \leq \\
&\leq (|\alpha| + |\psi|) 2^{\#\phi} = |\phi| 2^{\#\phi} \\
\mathrm{Card}(FL(\mathbf{Comm}_i\alpha)) &\leq |\alpha| 2^{\#\mathbf{Comm}_i\alpha} \leq |\mathbf{Comm}_i\alpha| 2^{\#\mathbf{Comm}_i\alpha} \\
\mathrm{Card}(FL(\mathbf{A}_i\alpha)) &\leq |\alpha| 2^{\#\mathbf{A}_i\alpha} \leq |\mathbf{A}_i\alpha| 2^{\#\mathbf{A}_i\alpha}
\end{aligned}$$

Using induction on α we will prove (ii), (iii) and (iv) simultaneously. By the induction hypothesis we have (i) for all ϕ' with length strictly less then ϕ, and (ii), (iii), (iv) for all α' with length strictly less then α. The following inequalities make the induction step for different formulae.

$$\begin{aligned}
\mathrm{Card}(FL^{\Box}([a]_i\psi)) &= 1 \leq |a| 2^{\#[a]_i\psi} \\
\mathrm{Card}(FL^{\Box}([\xi?]_i\psi)) &= 1 + \mathrm{Card}(FL(\xi)) \leq \\
&\leq 1 + |\xi| 2^{\#\xi} \leq |\xi?| 2^{\#[\xi?]_i\psi} \\
\mathrm{Card}(FL^{\Box}([\beta \cup \gamma]_i\psi)) &\leq 1 + \mathrm{Card}(FL^{\Box}([\beta]_i\psi)) + \mathrm{Card}(FL^{\Box}([\gamma]_i\psi)) \leq \\
&\leq 1 + |\beta| 2^{\#[\beta]_i\psi} + |\gamma| 2^{\#[\gamma]_i\psi} \leq \\
&\leq (1 + |\beta| + |\gamma|) 2^{\#[\beta\cup\gamma]_i\psi} = |\beta \cup \gamma| 2^{\#[\beta\cup\gamma]_i\psi} \\
\mathrm{Card}(FL^{\Box}([\beta;\gamma]_i\psi)) &\leq 1 + \mathrm{Card}(FL^{\Box}([\beta]_i[\gamma]_i\psi)) + \mathrm{Card}(FL^{\Box}([\gamma]_i\psi)) \leq \\
&\leq 1 + |\beta| 2^{\#[\beta]_i[\gamma]_i\psi} + |\gamma| 2^{\#[\gamma]_i\psi} \leq \\
&\leq (1 + |\beta| + |\gamma|) 2^{\#[\beta;\gamma]_i\psi} = |\beta;\gamma| 2^{\#[\beta;\gamma]_i\psi} \\
\mathrm{Card}(FL^{\Box}([\beta^*]_i\psi)) &\leq 1 + \mathrm{Card}(FL^{\Box}([\beta]_i[\beta^*]_i\psi)) \leq \\
&\leq 1 + |\beta| 2^{\#[\beta]_i[\beta^*]_i\psi} \leq \\
&\leq (1 + |\beta|) 2^{\#[\beta^*]_i\psi} = |\beta^*| 2^{\#[\beta^*]_i\psi} \\
\mathrm{Card}(FL^{\Box}([\mathbf{c}\beta]_i\psi)) &= \mathrm{Card}(FL^{\Box}([\mathbf{u}\beta]_i\psi)) = \\
&\leq 1 + \mathrm{Card}(FL(\mathbf{Comm}_i\beta)) \leq \\
&\leq 1 + |\beta| 2^{\#\mathbf{Comm}_i\beta} \leq \\
&\leq (1 + |\beta|) 2^{\#[\mathbf{c}\beta]_i\psi} = (1 + |\beta|) 2^{\#[\mathbf{u}\beta]_i\psi} = \\
&= |\mathbf{c}\beta| 2^{\#[\mathbf{c}\beta]_i\psi} = |\mathbf{u}\beta| 2^{\#[\mathbf{u}\beta]_i\psi} \\
\mathrm{Card}(FL(\mathbf{Comm}_i a)) &= 1 \leq |a| 2^{\#\mathbf{Comm}_i a}
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Card}(FL(\mathbf{Comm}_i\xi?)) &= 1+\mathrm{Card}(FL(\xi)) \leq \\
&\leq 1+|\xi|2^{\#\xi} \leq |\xi?|2^{\#\mathbf{Comm}_i\xi?} \\
\mathrm{Card}(FL(\mathbf{Comm}_i\beta\cup\gamma)) &\leq 1+\mathrm{Card}(FL(\mathbf{Comm}_i\beta))+ \\
&+\mathrm{Card}(FL(\mathbf{Comm}_i\gamma)) \leq \\
&\leq 1+|\beta|2^{\#\mathbf{Comm}_i\beta}+|\gamma|2^{\#\mathbf{Comm}_i\gamma} \leq \\
&\leq (1+|\beta|+|\gamma|)2^{\#\mathbf{Comm}_i\beta\cup\gamma} = |\beta\cup\gamma|2^{\#\mathbf{Comm}_i\beta\cup\gamma} \\
\mathrm{Card}(FL(\mathbf{Comm}_i\beta;\gamma)) &\leq 1+\mathrm{Card}(FL([\beta]_i\mathbf{Comm}_i\gamma))+ \\
&+\mathrm{Card}(FL(\mathbf{Comm}_i\beta)) \leq \\
&\leq 1+\mathrm{Card}(FL^{\Box}([\beta]_i\mathbf{Comm}_i\gamma))+ \\
&+\mathrm{Card}(FL(\mathbf{Comm}_i\gamma))+\mathrm{Card}(FL(\mathbf{Comm}_i\beta)) \leq \\
&\leq 1+|\beta|2^{\#[\beta]_i\mathbf{Comm}_i\gamma}+ \\
&+|\gamma|2^{\#\mathbf{Comm}_i\gamma}+|\beta|2^{\#\mathbf{Comm}_i\beta} \leq \\
&\leq (1+|\beta|+|\gamma|)2^{\#\mathbf{Comm}_i\beta;\gamma} = |\beta;\gamma|2^{\#\mathbf{Comm}_i\beta;\gamma} \\
\mathrm{Card}(FL(\mathbf{Comm}_i\beta^*)) &\leq 1+\mathrm{Card}(FL([\beta^*]_i\mathbf{Comm}_i\beta)) \leq \\
&\leq 1+\mathrm{Card}(FL^{\Box}([\beta^*]_i\mathbf{Comm}_i\beta))+ \\
&+\mathrm{Card}(FL(\mathbf{Comm}_i\beta)) \leq \\
&\leq 1+|\beta^*|2^{\#[\beta^*]_i\mathbf{Comm}_i\beta}+ \\
&+|\beta|2^{\#\mathbf{Comm}_i\beta} \leq \\
&\leq (1+|\beta|)2^{\#\mathbf{Comm}_i\beta^*} = |\beta^*|2^{\#\mathbf{Comm}_i\beta^*} \\
\mathrm{Card}(FL(\mathbf{Comm}_i\mathbf{c}\beta)) &= \mathrm{Card}(FL(\mathbf{Comm}_i\mathbf{u}\beta)) = \\
&\leq 1+\mathrm{Card}(FL(\mathbf{Comm}_i\beta)) \leq \\
&\leq 1+|\beta|2^{\#\mathbf{Comm}_i\beta} \leq \\
&\leq (1+|\beta|)2^{\#\mathbf{Comm}_i\mathbf{c}\beta} = (1+|\beta|)2^{\#\mathbf{Comm}_i\mathbf{u}\beta}
\end{aligned}
$$

The last inequation can be proved similarly.

Definition 13. Let $M = \langle S, Q, \models\rangle$ be a (non-standard) model for ADL and Σ a set of formulae. Define a relation $\sim_\Sigma$ on S by:

$$s \sim_\Sigma t \;\text{ iff }\; \forall\phi \in FL(\Sigma),\; s \models \phi \text{ iff } t \models \phi.$$

It is in fact an equivalence relation and is used to define a *filtration* $M^\Sigma \rightleftharpoons \langle S^\Sigma, Q^\Sigma, \models\rangle$ of M through Σ, where

$$\|s\| \rightleftharpoons \{t \mid s \sim_\Sigma t\} \text{ and } S^\Sigma \rightleftharpoons \{\|s\| \mid s \in S\}.$$

Furthermore, for any atomic action a,

$$\|s\| Q_i^\Sigma(a) \|t\| \;\text{ iff }\; \forall[a]_i\phi \in FL(\Sigma),\; \text{if } s \models [a]_i\phi \text{ then } t \models \phi,$$

for any propositional variable p,

$$\|s\| \models p \;\text{ iff }\; s \models p \text{ and } p \in FL(\Sigma),$$

for any semi-atomic action α,

$$\|s\| \models \mathbf{Comm}_i\alpha \ \text{ iff } \ s \models \mathbf{Comm}_i\alpha \text{ and } \mathbf{Comm}_i\alpha \in FL(\Sigma)$$
$$\|s\| \models \mathbf{A}_i\alpha \ \text{ iff } \ s \models \mathbf{A}_i\alpha \text{ and } \mathbf{A}_i\alpha \in FL(\Sigma),$$

for any action α,

$$\|s\|Q_i^{\mathbf{c}}(\alpha)\|t\| \ \text{ iff } \ \forall[\mathbf{c}\alpha]_i\phi \in FL(\Sigma), \text{ if } s \models [\mathbf{c}\alpha]_i\phi \text{ then } t \models \phi$$
$$\|s\|Q_i^{\mathbf{u}}(\alpha)\|t\| \ \text{ iff } \ \forall[\mathbf{u}\alpha]_i\phi \in FL(\Sigma), \text{ if } s \models [\mathbf{u}\alpha]_i\phi \text{ then } t \models \phi.$$

The definition of the truth relation is extended to all formulae and all relations Q_i^{Σ} using Definitions 1 and 2. This completes the definition of M^{Σ}.

Lemma 14. Let M be a (non-standard) model. Then:
1. $M \models \mathbf{Comm}_i\alpha^* \leftrightarrow [\alpha^*]_i\mathbf{Comm}_i\alpha$
2. $M \models \mathbf{A}_i\alpha^* \leftrightarrow [\alpha^*]_i\mathbf{A}_i\alpha$

We need Lemma 14 to prove the following Filtration Lemma.

Lemma 15 (Filtration Lemma). Let Σ be a finite set of formulae.
1. For all $\phi \in FL(\Sigma)$, $\|s\| \models \phi$ iff $s \models \phi$
2. For all $[\alpha]_i\eta \in FL(\Sigma)$,
   (a) if $sQ_i(\alpha)t$ then $\|s\|Q_i^{\Sigma}(\alpha)\|t\|$
   (b) if $\|s\|Q_i^{\Sigma}(\alpha)\|t\|$ and $s \models [\alpha]_i\eta$ then $t \models \eta$

Proof. We use simultaneous induction on the structure of formulae and actions [12].

$\phi = \bot$. By the definition of models $s \not\models \bot$ and $\|s\| \not\models \bot$.
$\phi = p$. $\|s\| \models p$ iff $s \models p$ by the construction of M^{Σ}.
$\phi = \xi \to \eta$. ϕ is in $FL(\Sigma)$, hence, by the properties of $FL(\Sigma)$, ξ and η belong $FL(\Sigma)$ too. Therefore, by the definition of M^{Σ} and the induction hypothesis we have the following equivalences.

$$\|s\| \models \xi \to \eta \text{ iff}$$
$$(\|s\| \models \xi \Rightarrow \|s\| \models \eta) \text{ iff}$$
$$(s \models \xi \Rightarrow s \models \eta) \text{ iff } s \models \xi \to \eta$$

$\phi = \mathbf{Comm}_i\alpha$. In this case we use induction on the structure of α.
$\alpha = a$. By the definition of M^{Σ}, $\|s\| \models \mathbf{Comm}_i a$ iff $s \models \mathbf{Comm}_i a$.
$\alpha = \xi?$. Again, the statement follows from the definition of M^{Σ}.
$\alpha = \beta;\gamma$. By the properties of $FL(\Sigma)$ we have that $\mathbf{Comm}_i\beta \in FL(\Sigma)$ and $[\beta]_i\mathbf{Comm}_i\gamma \in FL(\Sigma)$. By the induction hypothesis, 1 holds for $\mathbf{Comm}_i\beta$ and 2 holds for $[\beta]_i\mathbf{Comm}_i\gamma$. Let $\|s\| \models \mathbf{Comm}_i\alpha$. By the definition $\|s\| \models \mathbf{Comm}_i\beta$ and $\|s\| \models [\beta]_i\mathbf{Comm}_i\gamma$. Hence, by 1, $s \models \mathbf{Comm}_i\beta$. By 2b

$$s \models [\beta]_i\mathbf{Comm}_i\gamma \Rightarrow \forall t(\|s\|Q_i^{\Sigma}\|t\| \Rightarrow t \models \mathbf{Comm}_i\gamma)$$

Let $sQ_i(\beta)t$. Then, by 2a $\|s\|Q_i^{\Sigma}\|t\|$. Therefore, $\|t\| \models \mathbf{Comm}_i\gamma$ and, by the induction, $t \models \mathbf{Comm}_i\gamma$. Hence, $s \models [\beta]_i\mathbf{Comm}_i\gamma$ and, therefore, $s \models \mathbf{Comm}_i\alpha$.
Conversely, let $s \models \mathbf{Comm}_i\alpha$. Therefore, $s \models \mathbf{Comm}_i\beta$ and $s \models [\beta]_i\mathbf{Comm}_i\gamma$. By 1 we have $\|s\| \models \mathbf{Comm}_i\beta$. Let $\|s\|Q_i^{\Sigma}\|t\|$. By 2b we have $t \models \mathbf{Comm}_i\gamma$ and by 1 $\|t\| \models \mathbf{Comm}_i\gamma$, that is $\|s\| \models [\beta]_i\mathbf{Comm}_i\gamma$.

$\alpha = \beta \cup \gamma$**.** By the properties of $FL(\Sigma)$ we have that $\mathbf{Comm}_i\beta \in FL(\Sigma)$ and $\mathbf{Comm}_i\gamma \in FL(\Sigma)$. By the induction hypothesis, 1 holds for $\mathbf{Comm}_i\beta$ and for $\mathbf{Comm}_i\gamma$ and, in this case, we get what we want quite easily:

$$\|s\| \models \mathbf{Comm}_i\alpha \text{ iff}$$
$$(\|s\| \models \mathbf{Comm}_i\beta \text{ or } \|s\| \models \mathbf{Comm}_i\gamma) \text{ iff}$$
$$(s \models \mathbf{Comm}_i\beta \text{ or } s \models \mathbf{Comm}_i\gamma) \text{ iff } s \models \mathbf{Comm}_i\alpha$$

$\alpha = \beta^*$**.** By the properties of $FL(\Sigma)$ we have that $[\beta^*]_i\mathbf{Comm}_i\beta \in FL(\Sigma)$. By the induction hypothesis, clause 1 holds for $\mathbf{Comm}_i\beta$ and clause 2 holds for $[\beta^*]_i\mathbf{Comm}_i\beta$. Let $\|s\| \models \mathbf{Comm}_i\alpha$. Then by Lemma 14 $\|s\| \models [\beta^*]_i\mathbf{Comm}_i\beta$. Assume $sQ_i(\beta^*)t$, where t is arbitrary. By clause 2a we have $\|s\|Q_i^\Sigma(\beta^*)\|t\|$. So, $\|t\| \models \mathbf{Comm}_i\beta$. By clause 1, $t \models \mathbf{Comm}_i\beta$. Thus, $s \models [\beta^*]_i\mathbf{Comm}_i\beta$ which is equivalent to $s \models \mathbf{Comm}_i\beta^*$ by Lemma 14.
Conversely, suppose $s \models \mathbf{Comm}_i\alpha$. By Lemma 14 we have $s \models [\beta^*]\mathbf{Comm}_i\beta$. Let $\|s\|Q_i^\Sigma(\beta^*)\|t\|$. By clause 2b, $t \models \mathbf{Comm}_i\beta$. Finally, by clause 2a $\|t\| \models \mathbf{Comm}_i\beta$. Hence $\|s\| \models [\beta^*]_i\mathbf{Comm}_i\beta$ which is equivalent to $\|s\| \models \mathbf{Comm}_i\beta^*$ by Lemma 14.

$\alpha = \mathbf{c}\beta$**.** By the construction of M^Σ (see Definition 2), $\|s\| \models \mathbf{Comm}_i\mathbf{c}\beta$ iff $\|s\| \models \mathbf{Comm}_i\beta$ which is equivalent to $s \models \mathbf{Comm}_i\beta$ by the induction hypothesis. Finally, $s \models \mathbf{Comm}_i\mathbf{c}\beta$ iff $s \models \mathbf{Comm}_i\beta$ by the properties of nonstandard models.

$\alpha = \mathbf{u}\beta$**.** Again, by the construction of M^Σ (see Definition 2), $\|s\| \models \mathbf{Comm}_i\mathbf{u}\beta$ iff $\|s\| \not\models \mathbf{Comm}_i\beta$ which is equivalent to $s \not\models \mathbf{Comm}_i\beta$ by the induction hypothesis. The latter is equivalent to $s \models \mathbf{Comm}_i\mathbf{u}\beta$.

$\phi = \mathbf{A}_i\alpha$**.** In this case we again use induction on the length of action α. Cases $\alpha = a$, $\alpha = \xi?$, $\alpha = \beta \cup \gamma$, $\alpha = \beta;\gamma$, $\alpha = \beta^*$ can be proved as in the case $\phi = \mathbf{Comm}_i\alpha$.
The rest cases $\alpha = \mathbf{c}\beta$ and $\alpha = \mathbf{u}\beta$ are obvious because $M \models \mathbf{A}_i\mathbf{c}\beta$ and $M \models \mathbf{A}_i\mathbf{u}\beta$ in any (nonstandard) model M.

$\phi = [\alpha]_i\eta$**.** For the proof of 1 refer to e.g. [8, 12]. We use the induction hypothesis for α and η. By the properties of $FL(\Sigma)$, $\eta \in FL(\Sigma)$. By the induction hypothesis, 1 holds for η and 2 holds for $[\alpha]_i\eta$. Thus, we have

$$s \models [\alpha]_i\eta \Rightarrow \forall t(\|s\|Q_i^\Sigma(\alpha)\|t\| \Rightarrow t \models \eta)$$

Conversely,

$$\forall t(\|s\|Q_i^\Sigma(\alpha)\|t\| \Rightarrow t \models \eta) \Rightarrow$$
$$\forall t(sQ_i(\alpha)t \Rightarrow t \models \eta) \Rightarrow s \models [\alpha]_i\eta$$

Thus, using induction hypothesis $\|t\| \models \eta$ iff $t \models \eta$, we get

$$\|s\| \models [\alpha]_i \text{ iff}$$
$$\forall t(\|s\|Q_i^\Sigma(\alpha)\|t\| \Rightarrow \|t\| \models \eta) \text{ iff}$$
$$\forall t(\|s\|Q_i^\Sigma(\alpha)\|t\| \Rightarrow t \models \eta) \text{ iff } s \models [\alpha]_i\eta$$

To prove 2 we use the induction on the structure of α.

$\alpha = a$**.** First, we prove 2a. Let $sQ_i(a)t$. Fix arbitrary $[a]_i\xi$ from $FL(\Sigma)$ such that $s \models [a]_i\xi$. By the properties of (nonstandard) models, we have $t \models \xi$. Hence, by the definition of $Q_i^\Sigma(a)$, $\|s\|Q_i^\Sigma(a)\|t\|$.
To prove 2b suppose that $\|s\|Q_i^\Sigma\|t\|$. By the definition of Q_i^σ, it means that for all $[a]_i\xi$ from $FL(\Sigma)$ if $s \models [a]_i\xi$ then $t \models \xi$. Therefore, this also holds for $[a]_i\eta$.

$\alpha = \xi?$**.** As we have $\xi \in FL(\Sigma)$, 1 holds for ξ. We obtain:

$$sQ_i(\xi?)s \text{ iff } s \models \xi \text{ iff } \|s\| \models \xi \text{ iff } \|s\|Q_i^{\Sigma}(\xi?)\|s\|$$

Thus, 2 trivially holds for $[\xi?]_i\eta$.

$\alpha = \beta;\gamma$**.** By the properties of $FL(\Sigma)$ we have $[\beta]_i[\gamma]_i\eta \in FL(\Sigma)$ and $[\gamma]_i\eta \in FL(\Sigma)$. Thus, 2a holds for β and γ. Then 2a follows immediately:

$$\begin{gathered} sQ_i(\beta;\gamma)t \text{ iff} \\ \exists u\ sQ_i(\beta)uQ_i(\gamma)t \Rightarrow \\ \exists\|u\|\ \|s\|Q_i^{\Sigma}(\beta)\|u\|Q_i^{\Sigma}(\gamma)\|t\| \text{ iff } \|s\|Q_i^{\Sigma}(\beta;\gamma)\|t\| \end{gathered}$$

Further, by the induction hypothesis 2b holds for $[\beta]_i[\gamma]_i\eta$ and $[\gamma]_i\eta$. Suppose that $\|s\|Q_i^{\Sigma}(\beta;\gamma)\|t\|$ and $s \models [\beta;\gamma]_i\eta$. Then there exists $\|u\|$ from S^{Σ} such that $\|s\|Q_i^{\Sigma}(\beta)\|u\|Q_i^{\Sigma}(\gamma)\|t\|$ and $s \models [\beta]_i[\gamma]_i\eta$. Therefore, by 2b for β we have $\|u\|Q_i^{\Sigma}(\gamma)\|t\|$ and $u \models [\gamma]_i\eta$, and, finally, by 2b for γ, $t \models \eta$.

$\alpha = \beta \cup \gamma$**.** By the properties of $FL(\Sigma)$ we have $[\beta]_i\eta \in FL(\Sigma)$ and $[\gamma]_i\eta \in FL(\Sigma)$. Thus, 2a holds for β and γ. Then

$$\begin{gathered} sQ_i(\beta \cup \gamma)t \text{ iff} \\ sQ_i(\beta)t \text{ or } sQ_i(\gamma)t \Rightarrow \\ \|s\|Q_i^{\Sigma}(\beta)\|t\| \text{ or } \|s\|Q_i^{\Sigma}(\gamma)\|t\| \text{ iff } \|s\|Q_i^{\Sigma}(\beta \cup \gamma)\|t\| \end{gathered}$$

By the induction hypothesis 2b holds for $[\beta]_i\eta$ and $[\gamma]_i\eta$. Suppose that $\|s\|Q_i^{\Sigma}(\beta\cup\gamma)\|t\|$ and $s \models [\beta \cup \gamma]_i\eta$. From the first assumption we have $\|s\|Q_i^{\Sigma}(\beta)\|t\|$ or $\|s\|Q_i^{\Sigma}(\gamma)\|t\|$, and from the second one $s \models [\beta]_i\eta$ and $s \models [\gamma]_i\eta$. Therefore, by 2b we have $t \models \eta$.

$\alpha = \beta^*$**.** Suppose $sQ(\beta^*)t$. Let

$$E \rightleftharpoons \{u \in S \mid \|s\|Q_i^{\Sigma}(\beta)\|u\|\}.$$

Then $\|s\|Q_i^{\Sigma}(\beta)\|t\|$ is equivalent to $t \in E$.

Each equivalence class $\|u\|$ can be completely defined by a set $\Psi(\|u\|)$ of formulae from $FL(\Sigma)$ which are true in each state from this class. Thus, because $FL(\Sigma)$ is finite each $\|u\|$ can be defined by a formula $\psi(\|u\|)$ which is the conjunction of formulae from $\Psi(\|u\|)$ and negotiations of formulae from $FL(\Sigma) \setminus \Psi(\|u\|)$. That is, $\psi(\|u\|)$ is true in all states from $\|u\|$ and false in all other states of M. Therefore, there is a formula $\psi(E)$ defining E in M because E is a union of equivalence classes. The formula $\psi(E)$ is a disjunction of formulae $\psi(\|u\|)$ for each $\|u\| \subseteq E$.

By the definition of $Q_i(\beta^*)$, $\|s\|Q_i^{\Sigma}(\beta^*)\|s\|$, and, therefore, $s \in E$.

Let $u \in E$ and $uQ_i(\beta)v$. Then, we have $\|s\|Q_i^{\Sigma}(\beta)\|u\|$ by the definition of E and $\|u\|Q_i^{\Sigma}(\beta)\|v\|$ by the induction hypothesis. Therefore, $\|s\|Q_i^{\Sigma}(\beta)\|v\|$, that is $v \in E$. Thus, $u \in E$ and $uQ_i(\beta)v$ implies $v \in E$. Because E is formula definable this is equivalent to

$$M \models \psi(E) \to [\beta]_i\psi(E).$$

By the loop invariance rule (which is easily derivable in ADL using the axiom A2.6)

$$M \models \psi(E) \to [\beta^*]_i\psi(E).$$

By assumption $sQ(\beta^*)t$, and $s \in E$, therefore, $t \in E$, that is $\|s\|Q_i^{\Sigma}(\beta)\|t\|$. This completes the induction step for 2a in this case.

For 2b suppose $\|s\|Q_i^{\Sigma}(\beta^*)\|t\|$ and $s \models [\beta^*]_i\eta$. Then there exists $t_0, \dots, t_n$ such that $s = t_0$, $t = t_n$ and $\|t_i\|Q_i^{\Sigma}(\beta)\|t_{i+1}\|$ for $0 \leq i < n$. We have $t_0 = s \models [\beta^*]_i\eta$ by assumption. Then, $t_0 \models [\beta]_i[\beta^*]_i\eta$. The formula $[\beta]_i[\beta^*]_i\eta$ is in $FL(\Sigma)$, so by the induction for it we have $t_1 \models [\beta^*]_i\eta$. Continuing for n steps, we get $t = t_n \models [\beta^*]_i\eta$. Thus, $t \models \eta$.

$\alpha = \mathbf{c}\beta$. Taking into account $M \models [\mathbf{c}\beta]_i\mathbf{Comm}_i\beta$ we have

$$\begin{aligned}
&sQ_i(\mathbf{c}\beta)t \Rightarrow\\
\forall[\beta]_i\eta(s \models [\mathbf{c}\beta]_i\eta \Rightarrow t \models \eta)\ \&\ t \models \mathbf{Comm}_i\beta \Rightarrow&\\
\forall[\beta]_i\eta \in FL(\Sigma)\ (s \models [\mathbf{c}\beta]_i\eta \Rightarrow t \models \eta)\ \&\ t \models \mathbf{Comm}_i\beta \text{ iff}&\\
\|s\|Q_i^{\mathbf{c}}(\beta)\|t\|\ \&\ t \models \mathbf{Comm}_i\beta \text{ iff}&\\
\|s\|Q_i^{\mathbf{c}}(\beta)\|t\|\ \&\ \|t\| \models \mathbf{Comm}_i\beta \text{ iff}&\ \|s\|Q_i^{\Sigma}(\mathbf{c}\beta)\|t\|
\end{aligned}$$

Now we prove 2b. Let $s \models [\mathbf{c}\beta]_i\eta$ and $\|s\|Q_i^{\Sigma}(\mathbf{c}\beta)\|t\|$.

$$\begin{aligned}
&\|s\|Q_i^{\Sigma}(\mathbf{c}\beta)\|t\| \text{ iff}\\
\|s\|Q_i^{\mathbf{c}}(\beta)\|t\|\ \&\ \|t\| \models \mathbf{Comm}_i\beta \Rightarrow&\\
\forall[\mathbf{c}\beta]_i\eta \in FL(\Sigma)(s \models [\mathbf{c}\beta]_i\eta \Rightarrow t \models \eta)&
\end{aligned}$$

Thus, $t \models \eta$.

$\alpha = \mathbf{u}\beta$. The proof is similar to that of the previous item.

As in the case of standard modal logics the Filtration Lemma has a few useful consequences.

Theorem 16 (Small Model Theorem). Let ϕ be a formula, $n = |\phi|2^{\#\phi}$, and m be a number of different knowledge modalities in ϕ. If ϕ is satisfiable in some ADL-model then it is satisfiable in some finite ADL-model with no more than $2^n \cdot (2^{2^n})^m$ states.

Proof. To proof the theorem we modify the definition of the filtrated model.

Let Σ be a finite set of formulae and $\mathbf{K}_{i_1}, \dots, \mathbf{K}_{i_m}$ be all knowledge modalities occurred in formulae from $FL(\Sigma)$. For any s in M we denote by $\Sigma(s)$ [9] the set $\{\phi \in FL(\Sigma) \mid s \models \phi\}$.

First, we add the following conditions to the definition of FL-closed set X:

(FL17) if $\mathbf{K}_i\phi \in X$ then $\phi \in X$;
(FL18) if $\mathbf{W}_i\phi \in X$ then $\phi \in X$.

Second, we restrict equivalence relation $\sim_\Sigma$ in the following way:

$$s \sim_\Sigma t \quad \text{iff} \quad \Sigma(s) = \Sigma(t) \text{ and } \forall j = 1, \dots, m\ \{\Sigma(u) \mid sR^{\mathbf{K}}_{i_j}u\} = \{\Sigma(u) \mid tR^{\mathbf{K}}_{i_j}u\}.$$

It is easy to see that any equivalence class $\|s\|$ is uniquely determined by the tuple of sets

$$\langle \Sigma(s), \{\Sigma(u) \mid sR^{\mathbf{K}}_{i_1}u\}, \dots, \{\Sigma(u) \mid sR^{\mathbf{K}}_{i_m}u\}\rangle.$$

If $n = \operatorname{Card} FL(\Sigma)$, then we have 2^n possibilities for $\Sigma(s)$ and 2^{2^n} possibilities for $\{\Sigma(u) \mid sR^{\mathbf{K}}_{i_j}u\}$. Thus, we obtain $\operatorname{Card} S^{\Sigma} \leq 2^n \cdot (2^{2^n})^m$. Therefore, if $\Sigma = \{\phi\}$, then $\operatorname{Card} S^{\Sigma} \leq 2^N \cdot (2^{2^N})^m$ by Lemma 12.

Third, we use *the least* filtration of M that is the relations on the M^{Σ} are defined by the following.

For any atomic action a,

$$\begin{aligned} Q_i^{\Sigma}(a) &\rightleftharpoons \{(\|s\|,\|t\|) \mid \exists s' \in \|s\| \; \exists t' \in \|t\| \; (s' Q_i t')\}, \\ (R_i^{\mathbf{K}})^{\Sigma} &\rightleftharpoons \begin{cases} \{(\|s\|,\|t\|) \mid \exists s' \in \|s\| \; \exists t' \in \|t\| \; (s' R_i^{\mathbf{K}} t')\}, \text{ if } i \in \{i_1,\ldots,i_m\} \\ \mathrm{Id}_{S^{\Sigma}}, \text{ otherwise}, \end{cases} \\ (R_i^{\mathbf{W}})^{\Sigma} &\rightleftharpoons \{(\|s\|,\|t\|) \mid \exists s' \in \|s\| \; \exists t' \in \|t\| \; (s' R_i^{\mathbf{W}} t')\}, \end{aligned}$$

for any action α,

$$\begin{aligned} Q_i^{\mathbf{c}}(\alpha) &\rightleftharpoons \{(\|s\|,\|t\|) \mid \exists s' \in \|s\| \; \exists t' \in \|t\| \; (s' Q_i(\mathbf{c}\alpha) t')\}, \\ Q_i^{\mathbf{u}}(\alpha) &\rightleftharpoons \{(\|s\|,\|t\|) \mid \exists s' \in \|s\| \; \exists t' \in \|t\| \; (s' Q_i(\mathbf{u}\alpha) t')\}. \end{aligned}$$

Next, it is now possible to prove Filtration Lemma within these definitions using the induction on the length of formulae as usual. The new is the case $\phi = \mathbf{K}_i\psi$ which can be proved as follows.

Let $\mathbf{K}_i\psi \in FL(\Sigma)$. Therefore, $\mathbf{K}_i$ is one of the $\mathbf{K}_{i_1},\ldots,\mathbf{K}_{i_m}$ and $\psi \in FL(\Sigma)$. Let $\|s\| \models \mathbf{K}_i\psi$ and $sR_i^{\mathbf{K}}t$. By the construction of the model, $\|s\|(R_i^{\mathbf{K}})^{\Sigma}\|t\|$ and, thus, $\|t\| \models \psi$. By the induction hypothesis, $t \models \psi$ that means $s \models \mathbf{K}_i\psi$.

For the converse assume that $s \models \mathbf{K}_i\psi$ and $\|s\|(R_i^{\mathbf{K}})^{\Sigma}\|t\|$. Hence, $s' R_i^{\mathbf{K}} t'$ for some $s' \in \|s\|$ and $t' \in \|t\|$. By the definition of $\sim_{\Sigma}$, there exists t'' such that $s\mathbf{K}_i t''$ and

$$\forall \phi \in FL(\Sigma) \; t'' \models \phi \text{ iff } t' \models \phi.$$

Therefore, $t'' \models \psi$ and, consequently, $t' \models \psi$. By the induction hypothesis we obtain $\|t\| \models \psi$ and, therefore, $\|s\| \models \mathbf{K}_i\psi$ by the definition of the model.

The following lemma completes the proof of the theorem.

Lemma 1. *All $(R_i^{\mathbf{K}})^{\Sigma}$ are equivalence relations on S^{Σ}.*

Proof. The trivial case is when $\mathbf{K}_i$ does not occur in any formula of $FL(\Sigma)$. For the otherwise we refer to the proof of Proposition 12.5 in [9].

Theorem 17 (Completeness). ADL is complete with respect to the class of all ADL-models.

Proof. Suppose ϕ is not derivable in ADL. Then the set $\{\neg\phi\}$ is ADL-consistent. Thus, $\neg\phi$ is satisfiable in some non-standard model by Theorem 7. By the Filtration Lemma (Lemma 15), $\neg\phi$ is satisfiable in the model filtrated through $FL(\neg\phi)$. This filtrated model is standard by construction.

Theorem 18 (Decidability). ADL is decidable.

Proof. Applying the Small Model Theorem, we can test the satisfiability of a formula ϕ by enumerating all ADL-models with no more than $2^{|\phi|2^{\#\phi}}$ states (there are only finitely many) and test the satisfiability of ϕ in each of these.

4.3 Embedding into CPDL

In order to establish an upper bound for the complexity of ADL we will now exhibit an embedding of ADL into converse propositional dynamic logic (CPDL).

The key to reducing ADL-(un)satisfiability to CPDL-(un)satisfiability is to find a suitable renaming of certain subformula occurrences. In our reduction there is a unique association between each pair $\langle i, a\rangle$ (where a is an atomic action) and an atomic action a_i. Now, associate with any i from $\mathbf{Ag}$ unique atomic actions $b_i^{\mathbf{K}}$ and $b_i^{\mathbf{W}}$ which differ from all the a_i above. Also, associate with any pair $\langle i, \alpha\rangle$, where α is an arbitrary action, unique atomic actions c_i^{α} and d_i^{α}.

Now a suitable reduction is a mapping σ of formulae, which uses an auxiliary mapping σ_i of actions ($i \in \mathbf{Ag}$), and is defined to satisfy the following conditions (using simultaneous induction on the structure of formulae and actions).

$$\begin{array}{rlrl}
\sigma\bot \rightleftharpoons \bot \qquad \sigma p \rightleftharpoons p & & \sigma(\phi \to \psi) \rightleftharpoons \sigma\phi \to \sigma\psi & \\
\sigma\mathbf{W}_i\phi \rightleftharpoons [b_i^{\mathbf{W}}]\sigma\phi & & \sigma\mathbf{K}_i\phi \rightleftharpoons [(b_i^{\mathbf{K}} \cup (b_i^{\mathbf{K}})^{-1})^*]\sigma\phi & \\
\sigma[\alpha]_i\phi \rightleftharpoons [\sigma_i\alpha]\sigma\phi & & \sigma_i a \rightleftharpoons a_i & \\
\sigma_i\alpha \cup \beta \rightleftharpoons \sigma_i\alpha \cup \sigma_i\beta & & \sigma_i\alpha;\beta \rightleftharpoons \sigma_i\alpha;\sigma_i\beta & \\
\sigma_i(\alpha^*) \rightleftharpoons (\sigma_i\alpha)^* & & \sigma_i(\phi?) \rightleftharpoons (\sigma\phi)? & \\
\sigma_i(\mathbf{c}\alpha) \rightleftharpoons c_i^{\alpha};(\sigma\mathbf{Comm}_i\alpha)? & & \sigma_i(\mathbf{u}\alpha) \rightleftharpoons d_i^{\alpha};(\neg\sigma\mathbf{Comm}_i\alpha)? & \\
\sigma\mathbf{A}_i a \rightleftharpoons p_i^a & & \sigma\mathbf{Comm}_i a \rightleftharpoons q_i^a & \\
\sigma\mathbf{A}_i\phi? \rightleftharpoons p_i^{\phi?} & & \sigma\mathbf{Comm}_i\phi? \rightleftharpoons q_i^{\phi?} & \\
\sigma\mathbf{A}_i\mathbf{c}\alpha \rightleftharpoons \top & & \sigma\mathbf{Comm}_i\mathbf{c}\alpha \rightleftharpoons \sigma\mathbf{Comm}_i\alpha & \\
\sigma\mathbf{A}_i\mathbf{u}\alpha \rightleftharpoons \top & & \sigma\mathbf{Comm}_i\mathbf{u}\alpha \rightleftharpoons \neg\sigma\mathbf{Comm}_i\alpha & \\
\sigma\mathbf{A}_i\alpha^* \rightleftharpoons [\sigma_i\alpha^*]\sigma\mathbf{A}_i\alpha & & \sigma\mathbf{Comm}_i\alpha^* \rightleftharpoons [\sigma_i\alpha^*]\sigma\mathbf{Comm}_i\alpha & \\
\sigma\mathbf{A}_i\alpha \cup \beta \rightleftharpoons \sigma\mathbf{A}_i\alpha \vee \sigma\mathbf{A}_i\beta & & \sigma\mathbf{Comm}_i\alpha \cup \beta \rightleftharpoons \sigma\mathbf{Comm}_i\alpha \vee \sigma\mathbf{Comm}_i\beta & \\
\sigma\mathbf{A}_i\alpha;\beta \rightleftharpoons \sigma\mathbf{A}_i\alpha \wedge \sigma[\alpha]_i\mathbf{A}_i\beta & & \sigma\mathbf{Comm}_i\alpha;\beta \rightleftharpoons \sigma\mathbf{Comm}_i\alpha \wedge \sigma[\alpha]_i\mathbf{Comm}_i\beta &
\end{array}$$

Taking into account the completeness theorems for CPDL and ADL, and the model correspondence for these logics we obtain the following result.

Theorem 19. Let ϕ be any ADL-formula. Then: 1. ADL $\vdash \phi$ iff CPDL $\vdash \sigma\phi$, and 2. $\sigma\phi$ can be computed in exponential time.

It is well-known that the satisfiability problem for CPDL and PDL are EXPTIME-complete [12]. Because ADL contains PDL as a sublogic, we have:

Theorem 20. ADL is EXPTIME-hard.

By Theorem 19:

Theorem 21. The satisfiability problem for ADL-formulae is in 2EXPTIME.

It is interesting to note that the above translation of ADL into CPDL can also be used to give a syntactic proof of the soundness and completeness of ADL.

5 Conclusion

In this paper we applied logical tools from modal logic to develop a sound, complete and decidable formalisation of the informational and motivational attitudes of dynamic agents. Despite the decidability result for ADL, the standard decision procedure which follows from small model theorem is very expensive and thus useless as a practical reasoning method for this logic. Therefore, a primary task of future research is the development of effective proof methods for ADL.

As we have a translation ADL into CPDL one could exploit existing decision procedures for CPDL. Recently, a tableaux calculus was proposed for CPDL [5], on the basis of which a decision procedure can be developed. Unfortunately, as yet no implementation is available for this calculus. Until an implementation is available, it is possible to use existing PDL provers by using an encoding [4] of CPDL-satisfiability in PDL. Although this translation has only polynomial complexity, the increase in complexity is too high with respect to the length of the given formula so that this method is impractical for large formulae. If we find an improved method for encoding CPDL in PDL then it seems possible to use for ADL the decision methods developed for PDL by [18, 19]. However we do not know of any implementation of a prover for PDL with the test operator. In an application where we could make do without the test operator it is possible to apply available implementations of decision procedures developed for expressive description logics [1], for example, the DLP prover [17] would be a suitable candidate. It may be possible to extend the language and implementations of description logic provers that already include operators corresponding to the action forming operators $\cup$, ; and $*$ with the test operator.

The $*$-free fragment of ADL can also be embedded into CPDL, which means decision procedures for PDL are applicable in this case, too. More interestingly, the $*$-free fragment of ADL can be translated into first order logic (along the lines as discussed in [6], for example) and, hence, powerful automated proof methods developed for first order logic can be applied to the satisfiability problem in this fragment.

This shows there are many ways of going about the future development of an implemented reasoning system for ADL. First of all, it will be useful to provide an implementation of the CPDL tableaux method of [5]. Second, it might be useful to develop inference calculi for ADL directly. For instance, it may be more practical to extend decision procedures for PDL with rules for the additional operators of ADL, thereby avoiding the overhead of the translation via CPDL into PDL. In this respect it seems also useful to develop tableaux methods and pure Gentzen-type cut-free calculi for ADL.

A question left unanswered in this paper is the exact complexity of the satisfiability problem for ADL. Our lower and upper bounds for the complexity leave a wide gap. In particular, the question is, does ADL have the same complexity as PDL and CPDL, is it NEXPTIME-complete, is it EXPSPACE-complete, or does it belong to 2EXPTIME?

Further, it is of interest how far ADL can be enhanced as a modal logic (or otherwise) without loosing the properties of soundness, completeness and decidability. Of particular interest is what happens if interaction axioms like the following are added.

$$\mathbf{Comm}_i\alpha \to \mathbf{K}_i\mathbf{Comm}_i\alpha \qquad [\alpha]_i\mathbf{K}_i\phi \to \mathbf{K}_i[\alpha]_i\phi \qquad \mathbf{K}_i[\alpha]_i\phi \to [\alpha]_i\mathbf{K}_i\phi$$

These axioms are natural and important in applications. The first axiom expresses that an agent always knows her commitments, while the second axiom says that an agent knows the result of her action in advance if it is contained in her knowledge base, in other words, there is no learning. The last axiom expresses persistence of knowledge after the execution of an action, that is, perfect recall. Ultimately one would like to accommodate more features of real agents, for example, the notions of goals and the implementability of actions from the KARO framework [15, 24], or also notions like belief and intentions from other formalisations of motivational attitudes [3, 21]. It should be noted that adding beliefs, which are commonly modelled by KD45 modalities, poses no technical problems. It is not difficult to extend the results of this paper to hold also for ADL with KD45 modalities.

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