DAY7

64) An electric potential ϕ is given by

$$\phi(x, y, z) = xy\sin z + x^2 y + y^2 z + z^2 x$$

Find the directional derivative of the electric potential ϕ at the point $P(1, -1, \pi)$ in the direction of the vector $\boldsymbol{n} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$.

- First approach The gradient $\nabla\phi$ is

$$\nabla\phi = \frac{d\{\phi\}}{dx}\boldsymbol{i} + \frac{d\{\phi\}}{dy}\boldsymbol{j} + \frac{d\{\phi\}}{dy}\boldsymbol{k}$$
$$= (y\sin z + 2xy + z^2)\boldsymbol{i} + (x\sin z + x^2 + 2yz)\boldsymbol{j} + (xy\cos z + y^2 + 2zx)\boldsymbol{k}$$

The magnitude of $\boldsymbol{n}$ is $|\boldsymbol{n}| = \sqrt{1+1+1} = \sqrt{3}$. Therefore the unit vector of $\boldsymbol{n}$ is $\frac{\boldsymbol{n}}{|\boldsymbol{n}|} = \frac{\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}}{\sqrt{3}}$. So the directional derivative at $P(1, -1, \pi)$ is

$$\nabla\phi \cdot \frac{\boldsymbol{n}}{|\boldsymbol{n}|}\bigg|_{(x,y,z)=(1,-1,\pi)} = \frac{y\sin z + 2xy + z^2 - (x\sin z + x^2 + 2yz) - (xy\cos z + y^2 + 2zx)}{\sqrt{3}}\bigg|_{(x,y,z)=(1,-1,\pi)}$$
$$= \frac{-2 + \pi^2 - 1 + 2\pi - 2 - 2\pi}{\sqrt{3}} = \frac{-5 + \pi^2}{\sqrt{3}}$$

- Second approach The gradient $\nabla\phi$ is

$$\nabla\phi = \frac{d\{\phi\}}{dx}\boldsymbol{i} + \frac{d\{\phi\}}{dy}\boldsymbol{j} + \frac{d\{\phi\}}{dy}\boldsymbol{k}$$
$$= (y\sin z + 2xy + z^2)\boldsymbol{i} + (x\sin z + x^2 + 2yz)\boldsymbol{j} + (xy\cos z + y^2 + 2zx)\boldsymbol{k}$$

At $P(1, -1, \pi)$, the gradient is $(-2+\pi^2)\boldsymbol{i} + (1-2\pi)\boldsymbol{j} + (2+2\pi)\boldsymbol{k} \triangleq \boldsymbol{v}$. Now we need to find the magnitude of $\boldsymbol{n}$-directional component of $\boldsymbol{v}$. When the angle between $\boldsymbol{n}$ and $\boldsymbol{v}$ is θ, the magnitude of $\boldsymbol{n}$-directional component of $\boldsymbol{v}$ can be written as $|\boldsymbol{v}|\cos\theta$. As $\boldsymbol{n} \cdot \boldsymbol{v} = |\boldsymbol{n}||\boldsymbol{v}|\cos\theta$, we can obtain the magnitude as

$$|\boldsymbol{v}|\cos\theta = |\boldsymbol{v}|\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}||\boldsymbol{v}|} = \frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}|}$$

The magnitude of $\boldsymbol{n}$ is $|\boldsymbol{n}| = \sqrt{1+1+1} = \sqrt{3}$. Therefore

$$\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}|} = \frac{-2 + \pi^2 - (1 - 2\pi) - (2 + 2\pi)}{\sqrt{3}} = \frac{-2 + \pi^2 - 1 + 2\pi - 2 - 2\pi}{\sqrt{3}} = \frac{-5 + \pi^2}{\sqrt{3}}$$

65) A total resistance Z is given by the formula

$$\frac{1}{Z} = \jmath\omega L + \frac{1}{\jmath\omega C} + \frac{1}{R}$$

Find the derivative $\frac{dZ}{dC}$. Using the chain rule, we take the partial derivative of both sides with respect to C. Note that L, C and R are all independent of each other because Z can take any values. Thus $\frac{dL}{dC} = \frac{dR}{dC} = 0$. On the other hand Z changes depending on C. Therefore Z is the function of C and $\frac{dZ}{dC}$ does exist.

$$\frac{1}{Z} = \jmath\omega L + \frac{1}{\jmath\omega C} + \frac{1}{R}$$
$$\therefore \frac{d}{dC}\frac{1}{Z} = \frac{d\left(\jmath\omega L + \frac{1}{\jmath\omega C} + \frac{1}{R}\right)}{dC}$$
$$\therefore \frac{dZ}{dC}\frac{d}{dZ}\frac{1}{Z} = \frac{d(\jmath\omega L)}{dC} + \frac{d\left(\frac{1}{\jmath\omega C}\right)}{dC} + \frac{d\left(\frac{1}{R}\right)}{dC}$$
$$\therefore \frac{dZ}{dC}(-\frac{1}{Z^2}) = \frac{d\left(\frac{1}{\jmath\omega C}\right)}{dC} = \jmath\omega\frac{d\left(\frac{1}{C}\right)}{dC} = \jmath\omega(-\frac{1}{C^2})$$
$$\therefore \frac{dZ}{dC} = \jmath\omega(\frac{Z^2}{C^2})$$

66) Let $P = P(x, y)$, and $x = \mathfrak{e}^t$ and $y = \mathfrak{e}^{-t}$. Find the total derivative $\frac{dP}{dt}$ in terms of partial derivatives $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$. Hence find the second total derivative $\frac{d^2P}{dt^2}$ in terms of partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial^2 P}{\partial x^2}$, $\frac{\partial^2 P}{\partial y^2}$, and $\frac{\partial^2 P}{\partial x \partial y}$. You may assume that the two mixed partial derivatives are equal. From the chain rule, we can say that :

$$\begin{aligned}
\frac{d\{P\}}{dt} &= \frac{d\{P\}}{dx}\cdot\frac{d\{x\}}{dt} + \frac{d\{P\}}{dy}\cdot\frac{d\{y\}}{dt}\\
&= \frac{d\{P\}}{dx}\cdot\frac{d\{\mathfrak{e}^t\}}{dt} + \frac{d\{P\}}{dy}\cdot\frac{d\{\mathfrak{e}^{-t}\}}{dt}\\
&= \frac{d\{P\}}{dx}\cdot\mathfrak{e}^t + \frac{d\{P\}}{dy}\cdot(-\mathfrak{e}^{-t})\\
&= \frac{d\{P\}}{dx}\cdot x - \frac{d\{P\}}{dy}\cdot y
\end{aligned}$$

Now we differentiate again with respect to t as

$$\begin{aligned}
\frac{\partial^2 P}{\partial t^2} &= \frac{d\left\{\frac{d\{P\}}{dt}\right\}}{dt} = \frac{d\left\{x\frac{d\{P\}}{dx} - y\frac{d\{P\}}{dy}\right\}}{dt}\\
&= \frac{d\{x\}}{dt}\frac{d\{P\}}{dx} + x\frac{d\left\{\frac{d\{P\}}{dx}\right\}}{dt} - \frac{d\{y\}}{dt}\frac{d\{P\}}{dy} - y\frac{d\left\{\frac{d\{P\}}{dy}\right\}}{dt}\\
&= \frac{d\{x\}}{dt}\frac{d\{P\}}{dx} + x(\frac{d\{x\}}{dt}\frac{d\left\{\frac{d\{P\}}{dx}\right\}}{dx} + \frac{d\{y\}}{dt}\frac{d\left\{\frac{d\{P\}}{dx}\right\}}{dy})\\
&\quad -\frac{d\{y\}}{dt}\frac{d\{P\}}{dy} - y(\frac{d\{x\}}{dt}\frac{d\left\{\frac{d\{P\}}{dy}\right\}}{dx} + \frac{d\{y\}}{dt}\frac{d\left\{\frac{d\{P\}}{dy}\right\}}{dy})\\
&= \frac{d\{\mathfrak{e}^t\}}{dt}\frac{d\{P\}}{dx} + x(\frac{d\{\mathfrak{e}^t\}}{dt}\frac{d\left\{\frac{d\{P\}}{dx}\right\}}{dx} + \frac{d\{\mathfrak{e}^{-t}\}}{dt}\frac{d\left\{\frac{d\{P\}}{dx}\right\}}{dy})\\
&\quad -\frac{d\{\mathfrak{e}^{-t}\}}{dt}\frac{d\{P\}}{dy} - y(\frac{d\{\mathfrak{e}^t\}}{dt}\frac{d\left\{\frac{d\{P\}}{dy}\right\}}{dx} + \frac{d\{\mathfrak{e}^{-t}\}}{dt}\frac{d\left\{\frac{d\{P\}}{dy}\right\}}{dy})\\
&= \mathfrak{e}^t\frac{d\{P\}}{dx} + x(\mathfrak{e}^t\frac{d\left\{\frac{d\{P\}}{dx}\right\}}{dx} - \mathfrak{e}^{-t}\frac{d\left\{\frac{d\{P\}}{dx}\right\}}{dy})\\
&\quad +\mathfrak{e}^{-t}\frac{d\{P\}}{dy} - y(\mathfrak{e}^t\frac{d\left\{\frac{d\{P\}}{dy}\right\}}{dx} - \mathfrak{e}^{-t}\frac{d\left\{\frac{d\{P\}}{dy}\right\}}{dy})\\
&= x\frac{d\{P\}}{dx} + x(x\frac{d^2P}{dx^2} - y\frac{\partial^2 P}{\partial x\partial y})\\
&\quad +y\frac{d\{P\}}{dy} - y(x\frac{\partial^2 P}{\partial y\partial x} - y\frac{\partial^2 P}{\partial y^2})\\
&= x\frac{d\{P\}}{dx} + x^2\frac{d^2P}{dx^2} - xy\frac{\partial^2 P}{\partial x\partial y} + y\frac{d\{P\}}{dy} - yx\frac{\partial^2 P}{\partial y\partial x} + y^2\frac{\partial^2 P}{\partial y^2}\\
&= x\frac{d\{P\}}{dx} + x^2\frac{d^2P}{dx^2} - 2xy\frac{\partial^2 P}{\partial y\partial x} + y\frac{d\{P\}}{dy} + y^2\frac{\partial^2 P}{\partial y^2}(\because \frac{\partial^2 P}{\partial y\partial x} = \frac{\partial^2 P}{\partial x\partial y})
\end{aligned}$$

DAY8

67) Locate all stationary points for the function $f(x,y) = 2x^3 + 3x^2y + 2y^3 - 144y + 7$. How many stationary points are there ?

At the stationary point, $\frac{d\{f(x,y)\}}{dx} = \frac{d\{f(x,y)\}}{dy} = 0$. Therefore we find (x,y) which satisfies $\frac{d\{f(x,y)\}}{dx} = \frac{d\{f(x,y)\}}{dy} = 0$ as follows:

$$\frac{d\{f(x,y)\}}{dx} = 6x^2 + 6xy = 6x(x+y) = 0 \qquad ① \; ; \quad \frac{d\{f(x,y)\}}{dy} = 3x^2 + 6y^2 - 144 = 0 \qquad ②$$

① gives $x = 0, -y$. When $x = 0$, ② gives $6y^2 = 144$ *i.e.*, $y = \pm 2\sqrt{6}$. When $x = -y$, ② gives $9y^2 = 144$ *i.e.*, $y = \pm 4$. Therefore the stationary points are 4 points of $(x,y) = (0, \pm 2\sqrt{6}), (\mp 4, \pm 4)$.

68) A point $(x,y) = (-4,-8)$ is one of the stationary points of the function $f(x,y) = 12xy - 3y^2 + 2x^3$. Find the nature of this stationary point.

We need $\frac{d^2 f(x,y)}{dx^2}, \frac{\partial^2 f(x,y)}{\partial y \partial x}, \frac{\partial^2 f(x,y)}{\partial y^2}$ to find out the nature of the stationary point.

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{12y + 6x^2\right\}}{dx} = 12x \qquad ③ \; ; \quad \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{d\{12x - 6y\}}{dx} = 12 \qquad ④$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\{12x - 6y\}}{dy} = -6 \qquad ⑤$$

The value of the discriminant at $(x,y) = (-4,-8)$ is

$$\left.\frac{d^2 f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2\right|_{(x,y)=(-4,-8)} = 12 \cdot (-4) \cdot (-6) - 12^2 > 0$$

$$\left.\frac{d^2 f(x,y)}{dx^2}\right|_{(x,y)=(-4,-8)} = \left.\frac{d\left\{12y + 6x^2\right\}}{dx}\right|_{(x,y)=(-4,-8)} = 12x|_{(x,y)=(-4,-8)} = 12 \cdot (-4) < 0$$

Therefore the the stationary point at $(x,y) = (-4,-8)$ corresponds to a local maximum point.

69) Explain why, for the function $f(x,y) = (x+y)\mathfrak{e}^{-xy}$, the stationary point at $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$ is a saddle point despite both $\frac{d^2 f(x,y)}{dx^2}$ and $\frac{\partial^2 f(x,y)}{\partial y^2}$ being negative.

We need $\frac{d^2 f(x,y)}{dx^2}, \frac{\partial^2 f(x,y)}{\partial y \partial x}, \frac{\partial^2 f(x,y)}{\partial y^2}$ to find out the nature of the stationary points.

$$\frac{d^2 f(x,y)}{dx^2} = \mathfrak{e}^{-xy}(-2y + xy^2 + y^3) \qquad ③ \; ; \quad \frac{\partial^2 f(x,y)}{\partial x \partial y} = \mathfrak{e}^{-xy}(-2x - 2y + xy^2 + x^2 y) \qquad ④$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \mathfrak{e}^{-xy}(-2x + yx^2 + x^3) \qquad ⑤$$

When $(x,y) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$$\frac{d^2 f(x,y)}{dx^2} = \mathfrak{e}^{-\frac{1}{2}}(\frac{1}{\sqrt{2}}) \; ; \quad \frac{\partial^2 f(x,y)}{\partial x \partial y} = \mathfrak{e}^{-\frac{1}{2}}(\frac{3}{\sqrt{2}}) \; ; \quad \frac{\partial^2 f(x,y)}{\partial y^2} = \mathfrak{e}^{-\frac{1}{2}}(\frac{1}{\sqrt{2}})$$

The discriminant D is

$$D = \left.\frac{d^2 f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2\right|_{(x,y)=(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}$$

$$= \left.\frac{d^2 f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2\right|_{(x,y)=(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})} = \mathfrak{e}^{-\frac{1}{2}-\frac{1}{2}}(\frac{1}{2} - \frac{9}{2}) < 0$$

Thus the stationary point $(x,y) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ is a saddle point.

DAY4

46) The current, I, is given by

$$I(V) = I_s \sinh(V)$$

where V is the applied voltage and I_s is a constant. If the operating voltage is given by $V_a = \pi$ (measured in Volts), find a second order Taylor approximation for $I(V)$ about this operating voltage.

A second order Taylor expansion for $I(V)$ about $V = \pi$ is

$$I(V) = I(\pi) + \underline{(V-\pi)}\, \frac{dI}{dV}\bigg|_{V=\pi} + \underline{\frac{(V-\pi)^2}{2!}}\, \frac{d^2I}{dV^2}\bigg|_{V=\pi}$$

We now need $\frac{dI}{dV}$ and $\frac{d^2I}{dV^2}$.

$$I(V) = I_s \sinh(V) = I_s \frac{\mathfrak{e}^V - \mathfrak{e}^{-V}}{2}$$
$$\frac{dI}{dV} = I_s \frac{\mathfrak{e}^V + \mathfrak{e}^{-V}}{2}$$
$$\frac{d^2I}{dV^2} = I_s \frac{\mathfrak{e}^V - \mathfrak{e}^{-V}}{2}$$

Therefore

$$\begin{aligned} I(V) &= I_s \sinh(\pi) + \underline{(V-\pi)}\, I_s \frac{\mathfrak{e}^V + \mathfrak{e}^{-V}}{2}\bigg|_{V=\pi} + \underline{\frac{(V-\pi)^2}{2!}}\, I_s \frac{\mathfrak{e}^V - \mathfrak{e}^{-V}}{2}\bigg|_{V=\pi} \\ &= I_s \sinh(\pi) + I_s \frac{\mathfrak{e}^\pi + \mathfrak{e}^{-\pi}}{2}(V-\pi) + I_s \frac{\mathfrak{e}^\pi - \mathfrak{e}^{-\pi}}{2}\frac{(V-\pi)^2}{2!} \\ &= I_s \sinh(\pi) + I_s \cosh(\pi)(V-\pi) + I_s \sinh(\pi)\frac{(V-\pi)^2}{2} \end{aligned}$$

47) The current, I, is given by

$$I(V,t) = \mathfrak{e}^{-V}\cos(\omega t)$$

where V is the applied voltage and t is time. Find the term in t^2V^3 in the Taylor series expansion around $t = 0, V = 0$.

When $t = 0, V = 0$, the term that has t^2V^3 in it must be the term $\frac{1}{5!}\, {}_5C_2 \frac{\partial^5 I}{\partial t^2 \partial V^3}\Big|_{t=0,V=0} t^2V^3$ as the overall order is equal to 5. Therefore

$$I(V,t) = \mathfrak{e}^{-V}\cos(\omega t)$$
$$\frac{\partial I}{\partial t} = -\omega \mathfrak{e}^{-V}\sin(\omega t)$$
$$\frac{\partial^2 I}{\partial t^2} = -\omega^2 \mathfrak{e}^{-V}\cos(\omega t)$$
$$\frac{\partial^3 I}{\partial V \partial t^2} = -(-1)\omega^2 \mathfrak{e}^{-V}\cos(\omega t) = \omega^2 \mathfrak{e}^{-V}\cos(\omega t)$$
$$\frac{\partial^4 I}{\partial V^2 \partial t^2} = -\omega^2 \mathfrak{e}^{-V}\cos(\omega t)$$
$$\frac{\partial^5 I}{\partial V^3 \partial t^2} = \omega^2 \mathfrak{e}^{-V}\cos(\omega t)$$

When we put this into $\frac{1}{5!}\, {}_5C_2 \frac{\partial^5 I}{\partial t^2 \partial V^3}\Big|_{t=0,V=0} t^2V^3$

$$\frac{1}{5!}\, {}_5C_2 \omega^2 \mathfrak{e}^{-V}\cos(\omega t)\Big|_{t=0,V=0} t^2V^3 = \frac{1}{5!}\, {}_5C_2 \omega^2 \mathfrak{e}^{-0}\cos(\omega 0) t^2V^3 = \frac{1}{5\cdot 4\cdot 3\cdot 2\cdot 1}\frac{5\cdot 4}{2\cdot 1}\omega^2 t^2 V^3 = \frac{1}{12}\omega^2 t^2 V^3$$

48) The current, I, is given by

$$I(V,R) = \frac{\mathfrak{e}^V}{R}$$

where V is the applied voltage and R is a variable.

a) Find the second order Taylor series for I around $V = 0, R = 1$

The first derivatives of I with respect to V and R are

$$\frac{\partial I}{\partial V} = \frac{\mathfrak{e}^V}{R}$$

$$\frac{\partial I}{\partial R} = -\mathfrak{e}^V R^{-2}$$

The second derivatives of I with respect to V and R are

$$\frac{\partial^2 I}{\partial V^2} = \frac{\mathfrak{e}^V}{R}$$

$$\frac{\partial^2 I}{\partial R^2} = 2\mathfrak{e}^V R^{-3}$$

$$\frac{\partial^2 I}{\partial R \partial V} = -\mathfrak{e}^V R^{-2}$$

We evaluate these derivatives at $(V,R) = (0,1)$ as follows.

$$\left.\frac{\partial I}{\partial V}\right|_{(V,R)=(0,1)} = 1$$

$$\left.\frac{\partial I}{\partial R}\right|_{(V,R)=(0,1)} = -1$$

$$\left.\frac{\partial^2 I}{\partial V^2}\right|_{(V,R)=(0,1)} = 1$$

$$\left.\frac{\partial^2 I}{\partial R^2}\right|_{(V,R)=(0,1)} = 2$$

$$\left.\frac{\partial^2 I}{\partial R \partial V}\right|_{(V,R)=(0,1)} = -1$$

Therefore

$$I(V,R) = I(0,1) + (V-0)\left.\frac{\partial I}{\partial V}\right|_{(V,R)=(0,1)} + (R-1)\left.\frac{\partial I}{\partial R}\right|_{(V,R)=(0,1)}$$
$$+\frac{1}{2!}\left[(V-0)^2\left.\frac{\partial^2 I}{\partial V^2}\right|_{(V,R)=(0,1)} + 2(V-0)(R-1)\left.\frac{\partial^2 I}{\partial R \partial V}\right|_{(V,R)=(0,1)} + (R-1)^2\left.\frac{\partial^2 I}{\partial R^2}\right|_{(V,R)=(0,1)}\right]$$
$$= 1 + V - R + 1 + \frac{1}{2}\left[V^2 - 2V(R-1) - (R-1)^2\right] = 2 + V - R + \frac{1}{2}\left[V^2 - 2V(R-1) - (R-1)^2\right]$$

b) Using the series estimate $I(0.1, 0.9)$ and compare it with the exact value of $I(0.1, 0.9)$

We substitute $V = 0.1, R = 0.9$ into the second-order Taylor series we found in question 48a.

$$I(0.1, 0.9) = 2 + 0.1 - 0.9 + \frac{1}{2}\left[0.1^2 - 2\cdot 0.1(0.9-1) - (0.9-1)^2\right] = 1.21$$

If we work it out manually using $I(V,R) = \frac{\mathfrak{e}^V}{R}$ it becomes $I(0.1, 0.9) = \frac{\mathfrak{e}^{0.1}}{0.9} = 1.22797$ Therefore the two results differ by 0.0179677.

DAY7

71) A loudspeaker cone is generated by rotating the curve $y = \cosh x - 1$ about the $x-$ axis through 2π radians from $x = 0$ to $x = 1$. Calculate the surface area of the cone excluding the two ends.

[5 marks]

Since $\cosh x = \frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}$, $\frac{d\{\cosh x\}}{dx} = \frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2}$. Surface area is

$$\int_0^1 (2\pi y)\sqrt{dx^2 + dy^2} = \int_0^1 (2\pi y)\sqrt{1 + \left(\frac{d\{y\}}{dx}\right)^2}dx = \int_0^1 (2\pi y)\sqrt{1 + \left(\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2}\right)^2}dx$$

$$= \int_0^1 (2\pi y)\sqrt{1 + \frac{\mathfrak{e}^{2x} + \mathfrak{e}^{-2x} - 2}{4}}dx = \int_0^1 (2\pi y)\sqrt{\frac{4 + \mathfrak{e}^{2x} + \mathfrak{e}^{-2x} - 2}{4}}dx = \int_0^1 (2\pi y)\sqrt{\frac{\mathfrak{e}^{2x} + \mathfrak{e}^{-2x} + 2}{4}}dx$$

$$= \int_0^1 (2\pi y)\sqrt{\left(\frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}\right)^2}dx = \int_0^1 (2\pi y) \times \frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}dx$$

$$= 0.5\pi\int_0^1 (\mathfrak{e}^x + \mathfrak{e}^{-x} - 2) \times (\mathfrak{e}^x + \mathfrak{e}^{-x})dx = 0.5\pi\int_0^1 (\mathfrak{e}^{2x} + \mathfrak{e}^{-2x} + 2 - 2\mathfrak{e}^x - 2\mathfrak{e}^{-x})dx$$

$$= 0.5\pi[\frac{1}{2}\mathfrak{e}^{2x} - \frac{1}{2}\mathfrak{e}^{-2x} + 2x - 2\mathfrak{e}^x + 2\mathfrak{e}^{-x}]_0^1 = 0.5\pi(\frac{1}{2}\mathfrak{e}^2 - \frac{1}{2}\mathfrak{e}^{-2} + 2 - 2\mathfrak{e} + 2\mathfrak{e}^{-1})$$

72) For the force

$$\mathbf{F} = (y + 3x^2z^2)\mathbf{i} + (x - z)\mathbf{j} + (2x^3z - y)\mathbf{k}$$

find the potential ϕ such that $\mathbf{F} = \nabla\phi$. Hence evaluate

$$\int_{(0,0,0)}^{(1,2,3)} (y + 3x^2z^2)dx + (x - z)dy + (2x^3z - y)dz$$

$$\mathbf{F} = \nabla\phi = \frac{d\{\phi\}}{dx}\mathbf{i} + \frac{d\{\phi\}}{dy}\mathbf{j} + \frac{d\{\phi\}}{dz}\mathbf{k}$$
$$\equiv (y + 3x^2z^2)\mathbf{i} + (x - z)\mathbf{j} + (2x^3z - y)\mathbf{k}$$

Therefore

$$\frac{d\{\phi\}}{dx} = y + 3x^2z^2 \;\; ; \;\; \frac{d\{\phi\}}{dy} = x - z \;\; ; \;\; \frac{d\{\phi\}}{dz} = 2x^3z - y$$

This is written as

$$\partial\phi = (y + 3x^2z^2)\partial x \;\; ; \;\; \partial\phi = (x - z)\partial y \;\; ; \;\; \partial\phi = (2x^3z - y)\partial z$$

Thus

$$\int \partial\phi = \int (y + 3x^2z^2)\partial x \;\; ; \;\; \therefore \phi = xy + x^3z^2 + c_\alpha(y, z)$$
$$\int \partial\phi = \int (x - z)\partial y \;\; ; \;\; \therefore \phi = xy - yz + c_\beta(x, z)$$
$$\int \partial\phi = \int (2x^3z - y)\partial z \;\; ; \;\; \therefore \phi = x^3z^2 - yz + c_\gamma(x, y)$$

Thus we can tell that $\phi = xy - yz + x^3z^2$ When we define $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$, $(y + 3x^2z^2)dx + (x - z)dy + (2x^3z - y)dz = F \cdot d\mathbf{r}$. Since $\mathbf{F} = \nabla\phi$, the integral in question is manipulated as

$$\int_{(0,0,0)}^{(1,2,3)} \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0,0)}^{(1,2,3)} \nabla\phi \cdot d\mathbf{r} = \int_{(0,0,0)}^{(1,2,3)} \frac{\partial \{\phi\}}{\partial \{\mathbf{r}\}} \cdot d\mathbf{r} = \int_{(0,0,0)}^{(1,2,3)} d\phi$$
$$= [\phi]_{(0,0,0)}^{(1,2,3)} = \left[xy - yz + x^3z^2\right]_{(0,0,0)}^{(1,2,3)} = 1 \cdot 2 - 2 \cdot 3 + 1^3 \cdot 3^2 = 5$$

Alternatively

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ 3t \end{pmatrix}$$

where $0 \leq t \leq 1$.

$$\boldsymbol{F} = \begin{pmatrix} y + 3x^2z^2 \\ x - z \\ 2x^3z - y \end{pmatrix} = \begin{pmatrix} 2t + 3t^2(3t)^2 \\ t - 3t \\ 2t^3(3t) - 2t \end{pmatrix} = \begin{pmatrix} 2t + 27t^4 \\ -2t \\ 6t^4 - 2t \end{pmatrix}$$

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Thus the integration in question can be re-written as

$$\int_0^1 \boldsymbol{F} \cdot \frac{d\{\boldsymbol{r}\}}{dt} dt$$
$$= \int_0^1 2t + 27t^4 - 2t \cdot 2 + (6t^4 - 2t) \cdot 3dt = \int_0^1 2t + 27t^4 - 4t + 18t^4 - 6tdt$$
$$= \int_0^1 45t^4 - 8tdt = [9t^5 - 4t^2]_0^1 = 9 - 4 = 5$$

73) For the double integral

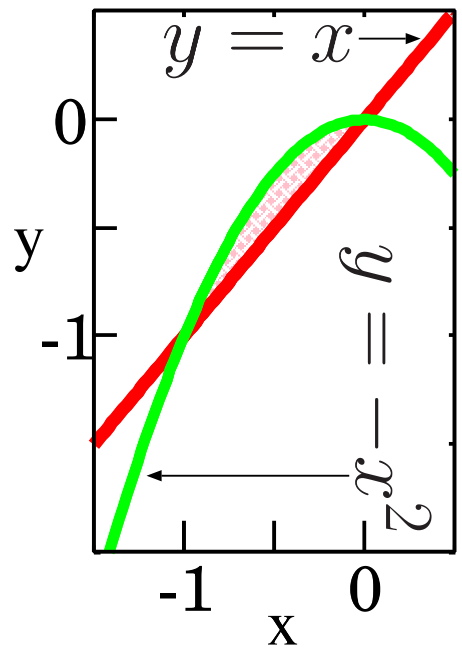
$$\int_{-1}^{0} \int_{x}^{-x^2} 12xydydx$$

draw a clear, labelled sketch of the region of integration and evaluate the integral using any suitable method.

From the given equation we get the range of x and y as follows

$$x \leq y \leq -x^2 \qquad -1 \leq x \leq 0$$

From these four conditions, we obtain



$$\int_{-1}^{0}\int_{x}^{-x^2} 12xy\,dy\,dx = \int_{-1}^{0}\left[12xy^2\right]_{x}^{-x^2} dx = \int_{-1}^{0} 12x(x^4 - x^2)dx$$

$$= \int_{-1}^{0} 12(x^5 - x^3)dx = 12\left[\frac{x^6}{6} - \frac{x^4}{4}\right]_{-1}^{0} = \left[2x^6 - 3x^4\right]_{-1}^{0} = -(2-3) = 1$$

74) Find the work

$$W = \int_C \mathbf{F}\cdot d\boldsymbol{r}$$

done by the force $\mathbf{F} = x^2\boldsymbol{i} + xy\boldsymbol{j}$ in moving a particle along the curve given parametrically by

$$x(t) = 1 - t$$

and

$$y(t) = t$$

where $0 \le t \le 1$.

a) Express x, y, z on the curve C using t and set the range of t

$$x(t) = 1 - t$$
$$y(t) = t$$
$$0 \le t \le 1$$

b) Express $\mathbf{F}$ as the function of t

$$\mathbf{F} = \begin{pmatrix} x^2 \\ xy \end{pmatrix} = \begin{pmatrix} (1-t)^2 \\ (1-t)t \end{pmatrix} = \begin{pmatrix} 1 + t^2 - 2t \\ t - t^2 \end{pmatrix}$$

c) Express $\dfrac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \dfrac{d\{x\}}{dt} \\ \dfrac{d\{y\}}{dt} \\ \dfrac{d\{z\}}{dt} \end{pmatrix}$ using t

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \dfrac{d\{x\}}{dt} \\ \dfrac{d\{y\}}{dt} \end{pmatrix} = \begin{pmatrix} \dfrac{d\{1-t\}}{dt} \\ \dfrac{d\{t\}}{dt} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

d) Put all of them into $\int \mathbf{F} \cdot \dfrac{d\{\boldsymbol{r}\}}{dt} dt$

$$\int \mathbf{F} \cdot \frac{d\{\boldsymbol{r}\}}{dt} dt = \int_0^1 \begin{pmatrix} 1+t^2-2t \\ t-t^2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} dt = \int_0^1 -1-t^2+2t+t-t^2 dt$$

$$= \int_0^1 -1-2t^2+3t dt = \left[-t-\frac{2}{3}t^3+\frac{3}{2}t^2\right]_{t=0}^{t=1} = -1-\frac{2}{3}+\frac{3}{2} = \frac{-1}{6}$$

DAY5

65) Consider the differential equation

$$\frac{d\{I\}}{dt} - I = r(t)$$

a) For the homogeneous equation with $r(t) = 0$, find the general solution.

$$\frac{d\{I\}}{dt} - I = 0 \;\; ; \quad \therefore \frac{d\{I\}}{dt} = I \;\; ; \quad \therefore \frac{1}{I}dI = dt \;\; ; \quad \therefore \int \frac{1}{I}dI = \int dt \;\; ; \quad \therefore \ln I = t + c = \ln \mathfrak{e}^{t+c}$$

$$\therefore I = \mathfrak{e}^{t+c} = D\mathfrak{e}^{t}$$

b) Find the general solution to the homogeneous equation charaterised by $r(t) = \mathfrak{e}^{-t}$ and the particular solution for $I = 0.5$ at $t = 0$ and describe the behavior for large t

i) **Allocate** $P(t)$ **and** $Q(t)$
When we compare the equation with Equation (85), we obtain $P(t) = -1$ and $Q(t) = \mathfrak{e}^{-t}$

ii) **Calculate** $A = \int P(t)dt$

$$A = \int P(t)dt = \int -1dt = -t$$

iii) **Obtain** $\Phi(t) = \mathfrak{e}^{A}$
From Equation (86),

$$\Phi(t) = \mathfrak{e}^{A} = \mathfrak{e}^{-t}$$

iv) **Calculate** $B = \int \Phi(t)Q(t)dt$

$$B = \int \Phi(t)Q(t)dt = \int \mathfrak{e}^{-t}\mathfrak{e}^{-t}dt = \int \mathfrak{e}^{-2t}dt = -\frac{1}{2}\mathfrak{e}^{-2t} \qquad ①$$

v) **Obtain the general solution** $I = \frac{1}{\Phi(t)}[B + c]$

$$I = \frac{1}{\Phi(t)}[B + c] = \frac{1}{\mathfrak{e}^{-t}}\left[-\frac{1}{2}\mathfrak{e}^{-2t} + c\right] = -\frac{1}{2}\mathfrak{e}^{-t} + c\mathfrak{e}^{t}$$

vi) **Apply the condition to** $I = \frac{1}{\Phi(t)}[B + c]$ **in order to find out** c **and thus the particular solution**
As the condition $(t, I) = (0, 0.5)$

$$0.5 = -\frac{1}{2}\mathfrak{e}^{-0} + c\mathfrak{e}^{0} \;\; ; \quad \therefore c = 1$$

Thus the particular solution is $I = -\frac{1}{2}\mathfrak{e}^{-t} + \mathfrak{e}^{t}$. For large t, the term of $-\frac{1}{2}\mathfrak{e}^{-t}$ goes to zero and the term of $\mathfrak{e}^{t}$ diverges and in the end I goes to infinite.

66) Solve the differential equation

$$\frac{\partial^2 I}{\partial t^2} - I = 2\mathfrak{e}^{-t} - 1$$

subject to the conditions that I remains finite for large t and that $I = 2$ when $t = 0$
To solve the following we must first find the complementary function $Y_1(t)$ and and then the particular integral $Y_2(t)$. The final answer will be the sum of both:

$$I(t) = Y_1(t) + Y_2(t)$$

In order to find out $Y_1(t)$, substituting Equation (103) into $\frac{\partial^2 I}{\partial t^2} - I = 0$, we produce an auxiliary equation:

$$\lambda^2 - 1 = 0 \;\;;\;\; \therefore \lambda = -1, 1 \equiv \alpha, \beta$$

Since α and β are real and $\alpha \neq \beta$, the complementary function $Y_1(t)$ is

$$Y_1(t) = a\mathfrak{e}^{-t} + b\mathfrak{e}^{t}$$

Now in order to find the particular solution, you need to know the correct substitution. One of two terms in $r(t)$ is $2\mathfrak{e}^{-t}$. This means the coefficient of t, *i.e.*, $c = -1$. Since $c = \alpha$, the particular integral is Equation (109). By taking into account of the fact that the other term in $r(t)$ is -1, we set $Y_2(t)$ as

$$Y_2(t) = gt\mathfrak{e}^{-t} + h \;\;;\;\; \therefore \frac{d\left\{Y_2(t)\right\}}{dt} = g\mathfrak{e}^{-t} - gt\mathfrak{e}^{-t}$$

$$\therefore \frac{\partial^2 Y_2(t)}{\partial t^2} = -g\mathfrak{e}^{-t} - (g\mathfrak{e}^{-t} - gt\mathfrak{e}^{-t}) - g\mathfrak{e}^{-t} - g\mathfrak{e}^{-t} + gt\mathfrak{e}^{-t} = -2g\mathfrak{e}^{-t} + gt\mathfrak{e}^{-t}$$

Substituting these into the original ODE

$$\frac{\partial^2 I}{\partial t^2} - I = 2\mathfrak{e}^{-t} - 1$$

$$\therefore -2g\mathfrak{e}^{-t} + gt\mathfrak{e}^{-t} - gt\mathfrak{e}^{-t} - h = 2\mathfrak{e}^{-t} - 1$$

$$\therefore -2g\mathfrak{e}^{-t} - h = 2\mathfrak{e}^{-t} - 1$$

$$\therefore -2g = 2 \;\;;\;\; \therefore g = -1 \;\;;\;\; -h = -1 \;\;;\;\; \therefore h = 1$$

Thus the particular integral is

$$Y_2(t) = -t\mathfrak{e}^{-t} + 1$$

Therefore the general solution for this ODE is

$$I(t) = Y_1(t) + Y_2(t) = a\mathfrak{e}^{-t} + b\mathfrak{e}^{t} - t\mathfrak{e}^{-t} + 1$$

For the value of I to be finite for large t, the value of b must be zero. Thus

$$I(t) = a\mathfrak{e}^{-t} - t\mathfrak{e}^{-t} + 1$$

To find a we need to substitute $I = 2$ and $t = 0$.

$$2 = a\mathfrak{e}^{0} + 1 \;\;;\;\; \therefore a = 1$$

Therefore the particular solution to the ODE is

$$I(t) = \mathfrak{e}^{-t} - t\mathfrak{e}^{-t} + 1$$

67) The current $I(t)$ at time t in an electric circuit with total resistance R and self-inductance L satisfies

$$L\frac{d\left\{I\right\}}{dt} + RI = \mathfrak{e}^{-Dt}$$

where $0 < D < \frac{R}{L}$.

a) Find the general solution to the problem.

i) **Allocate** $P(t)$ **and** $Q(t)$

When we compare the equation with Equation (85), we obtain $P(t) = \frac{R}{L}$ and $Q(t) = \frac{\mathfrak{e}^{-Dt}}{L}$

ii) **Calculate** $A = \int P(t)dt$

$$A = \int P(t)dt = \int \frac{R}{L}dt = \frac{R}{L}t$$

iii) **Obtain** $\Phi(t) = \mathfrak{e}^{A}$

From Equation (86),

$$\Phi(t) = \mathfrak{e}^{A} = \mathfrak{e}^{\frac{R}{L}t}$$

iv) **Calculate** $B = \int \Phi(t)Q(t)dt$

$$B = \int \Phi(t)Q(t)dt = \int \mathfrak{e}^{\frac{R}{L}t}\frac{\mathfrak{e}^{-Dt}}{L}dt = \frac{1}{L}\int \mathfrak{e}^{\frac{R}{L}t-Dt}dt$$

$$= \frac{1}{L}\int \mathfrak{e}^{(\frac{R}{L}-D)t}dt = \frac{1}{L(\frac{R}{L}-D)}\mathfrak{e}^{(\frac{R}{L}-D)t} = \frac{1}{R-LD}\mathfrak{e}^{(\frac{R}{L}-D)t} \qquad ①$$

v) **Obtain the general solution** $I = \frac{1}{\Phi(t)}\left[B + c\right]$

$$I = \frac{1}{\Phi(t)}\left[B+c\right] = \frac{1}{\mathfrak{e}^{\frac{R}{L}t}}\left[\frac{1}{R-LD}\mathfrak{e}^{(\frac{R}{L}-D)t} + c\right] = \mathfrak{e}^{-\frac{R}{L}t}\left[\frac{1}{R-LD}\mathfrak{e}^{(\frac{R}{L}-D)t} + c\right] = \frac{1}{R-LD}\mathfrak{e}^{-Dt} + c\mathfrak{e}^{-\frac{R}{L}t}$$

b) The switch is closed at $t = 0$ and the initial value of the current is $I(0) = 2$. When $R = 6, L = 1, D = 5$, find the solution to this initial value problem.

i) **Obtain the general solution** $I = \frac{1}{\Phi(t)}\left[B + c\right]$ **when**$R = 6, L = 1, D = 5$

$$I = \frac{1}{R-LD}\mathfrak{e}^{-Dt} + c\mathfrak{e}^{-\frac{R}{L}t} = \mathfrak{e}^{-5t} + c\mathfrak{e}^{-6t}$$

ii) **Apply the condition to** $I = \frac{1}{\Phi(t)}\left[B + c\right]$ **in order to find out** c **and thus the particular solution**

As the condition $(t, I) = (0, 2)$

$$2 = \mathfrak{e}^{0} + c\mathfrak{e}^{0} \;\; ; \quad \therefore c = 2 - 1 = 1$$

Thus the particular solution is $I = \mathfrak{e}^{-5t} + \mathfrak{e}^{-6t}$

c) Write down the steady-state value I_s of the current

The particular solution is $I = \mathfrak{e}^{-5t} + \mathfrak{e}^{-6t}$. When t is infinite, both $\mathfrak{e}^{-5t}$ and $\mathfrak{e}^{-6t}$ approach zero. Thus $I = 0$ is the steady-state value.