Engineering Maths

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Contents

I. Prerequisites

In order to successfully complete this Engineering Mathematics course you must be competent with the following material. If you are unfamiliar with the any of the following material it is recommended that you attempt some practice questions before undertaking the main course material.

1) **Logarithms**

$$\begin{aligned}
\log_a(x) = m &\equiv a^m = x \\
\log(x) &\equiv \log_{10}(x) \\
\ln(x) &\equiv \log_e(x) \\
\log_a(a) &= 1 \\
\log_a(m \cdot n) &= \log_a(m) + \log_a(n) \\
\log_a\left(\frac{m}{n}\right) &= \log_a(m) - \log_a(n) \\
\log_a(m^n) &= n \cdot \log_a(m) \\
\log_a b &= \frac{\log_c b}{\log_c a}
\end{aligned}$$

2) **Indices**

$$\begin{aligned}
a^m \cdot a^n &= a^{(m+n)} \\
\frac{a^m}{a^n} &= a^{(m-n)} \\
(a^m)^n &= a^{(m \cdot n)} \\
a^{-m} &= \frac{1}{a^m} \\
a^{(m/n)} &= \sqrt[n]{a^m} \\
a^0 &= 1 \\
a^1 &= a
\end{aligned}$$

3) **Trigonometric Identities**

$$\begin{aligned}
y = \sin^{-1} x = \arcsin x &\iff x = \sin y \\
y = \cos^{-1} x = \arccos x &\iff x = \cos y \\
y = \tan^{-1} x = \arctan x &\iff x = \tan y \\
\text{cosec } x &= \frac{1}{\sin x} \\
\sec x &= \frac{1}{\cos x} \\
\cot x &= \frac{1}{\tan x} \\
y = \text{cosec }^{-1} x \iff x = \text{ cosec } y &= \frac{1}{\sin y} \\
y = \sec^{-1} x \iff x = \sec y &= \frac{1}{\cos y} \\
y = \cot^{-1} x \iff x = \cot y &= \frac{1}{\tan y} \\
\tan(x) &= \frac{\sin(x)}{\cos(x)} \\
\sin^2(x) + \cos^2(x) &= 1 \\
\sec^2(x) &= 1 + \tan^2(x) \\
\sin(A \pm B) &= \sin(A)\cos(B) \pm \cos(A)\sin(B) \\
\cos(A \pm B) &= \cos(A)\cos(B) \mp \sin(A)\sin(B) \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \\
\sin(2A) &= 2\sin(A)\cos(A) \\
\cos(2A) &= \cos^2(A) - \sin^2(A) \\
&= 2\cos^2(A) - 1 \\
&= 1 - 2\sin^2(A) \\
\tan(2A) &= \frac{2\tan(A)}{1 - \tan^2(A)} \\
2\sin(A)\cos(B) &= \sin(A+B) + \sin(A-B) \\
2\cos(A)\sin(B) &= \sin(A+B) - \sin(A-B) \\
2\cos(A)\cos(B) &= \cos(A+B) + \cos(A-B) \\
-2\sin(A)\sin(B) &= \cos(A+B) - \cos(A-B)
\end{aligned}$$

$$\cos(x) = \frac{\mathfrak{e}^{\jmath x} + \mathfrak{e}^{-\jmath x}}{2}$$
$$\sin(x) = \frac{\mathfrak{e}^{\jmath x} - \mathfrak{e}^{-\jmath x}}{2\jmath}$$

4) **Hyperbolic Identities**

$$\begin{aligned}
\cosh(x) &= (\mathfrak{e}^{x} + \mathfrak{e}^{-x})/2 \\
x &= \cosh^{-1}\left(\frac{\mathfrak{e}^{x} + \mathfrak{e}^{-x}}{2}\right) \\
\sinh(x) &= (\mathfrak{e}^{x} - \mathfrak{e}^{-x})/2 \\
x &= \sinh^{-1}\left(\frac{\mathfrak{e}^{x} - \mathfrak{e}^{-x}}{2}\right) \\
\tanh(x) &= (\mathfrak{e}^{x} - \mathfrak{e}^{-x})/(\mathfrak{e}^{x} + \mathfrak{e}^{-x}) \\
\cosh^2(A) - \sinh^2(A) &= 1
\end{aligned}$$

When you need x which satisfies $\cosh(x) = \alpha$ where α is a real number,

$$\begin{aligned}
\frac{\mathfrak{e}^{x} + \mathfrak{e}^{-x}}{2} &= \alpha \\
\therefore \mathfrak{e}^{x} + \mathfrak{e}^{-x} &= 2\alpha \\
\therefore \mathfrak{e}^{2x} + 1 &= 2\alpha\mathfrak{e}^{x} \\
\therefore \mathfrak{e}^{2x} - 2\alpha\mathfrak{e}^{x} + 1 &= 0 \\
\therefore \mathfrak{e}^{x} &= \alpha \pm \sqrt{\alpha^2 - 1} \\
\therefore x &= \ln(\alpha \pm \sqrt{\alpha^2 - 1})
\end{aligned}$$

When you need x which satisfies $\sinh(x) = \alpha$ where α is a real number,

$$\begin{aligned}
\frac{\mathfrak{e}^{x} - \mathfrak{e}^{-x}}{2} &= \alpha \\
\therefore \mathfrak{e}^{x} - \mathfrak{e}^{-x} &= 2\alpha \\
\therefore \mathfrak{e}^{2x} - 1 &= 2\alpha\mathfrak{e}^{x} \\
\therefore \mathfrak{e}^{2x} - 2\alpha\mathfrak{e}^{x} - 1 &= 0 \\
\therefore \mathfrak{e}^{x} &= \alpha \pm \sqrt{\alpha^2 + 1} \\
\therefore x &= \ln(\alpha \pm \sqrt{\alpha^2 + 1}) \\
\therefore x &= \ln(\alpha + \sqrt{\alpha^2 + 1})(\because A > 0 \text{ for } \ln A)
\end{aligned}$$

5) **Completing the Square**

$$4x^2 - 2x - 5 = 0$$

We can solve the above equation by completing the square as follows

$$\begin{aligned}
4x^2 - 2x - 5 &= 0 \\
4x^2 - 2x &= 5 \\
x^2 - \frac{1}{2}x &= \frac{5}{4} \\
\left(x - \frac{1}{4}\right)^2 - \frac{1}{16} &= \frac{5}{4} \\
\left(x - \frac{1}{4}\right)^2 &= \frac{5}{4} + \frac{1}{16} \\
\left(x - \frac{1}{4}\right)^2 &= \frac{21}{16} \\
\therefore x &= \frac{1}{4} \pm \sqrt{\frac{21}{16}}
\end{aligned}$$

6) **Quadratic Equation**
We can use completing the square to derive the quadratic equation.

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
ax^2 + bx &= -c \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} &= -\frac{c}{a}
\end{aligned}$$

$$
\begin{aligned}
\left(x+\frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2}-\frac{c}{a}\\
\left(x+\frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2}-\frac{4ac}{4a^2}\\
\left(x+\frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2}\\
x+\frac{b}{2a} &= \pm\sqrt{\frac{b^2-4ac}{4a^2}}\\
x+\frac{b}{2a} &= \frac{\pm\sqrt{b^2-4ac}}{2a}\\
x &= -\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}\\
x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}
$$

7) **Polynomial Long Division**
If we know one factor of a polynomial equation, in order to find out the other factor we perform a division. In this example we know that $x^2-9x-10$ has a factor of $x+1$. Therefore

$$
\begin{array}{r}
x-10 \\
x+1\,\overline{)x^2\;\;-9x-10} \\
-)x^2\;\;\;+x \\
\hline
-10x-10 \\
-)\;\;\;-10x-10 \\
\hline
\\
0\;\;\;\;0
\end{array}
$$

Thus, we find the other factor to be

$$x-10$$

In order to confirm this is correct we can multiply this factor by the known factor to find the original polynomial.

$$
\begin{aligned}
(x-10)(x+1) &= x^2+x-10x-10\\
&= x^2-9x-10
\end{aligned}
$$

8) **Area of a Triangle in Vector Form**
When a triangle is defined with two sides $|\boldsymbol{p}|$ and $|\boldsymbol{q}|$ and the angle between these two sides is θ, the area of triangle is

$$\frac{1}{2}|\boldsymbol{p}|\cdot|\boldsymbol{q}|\cdot\sin\theta$$

9) **Inequalities**

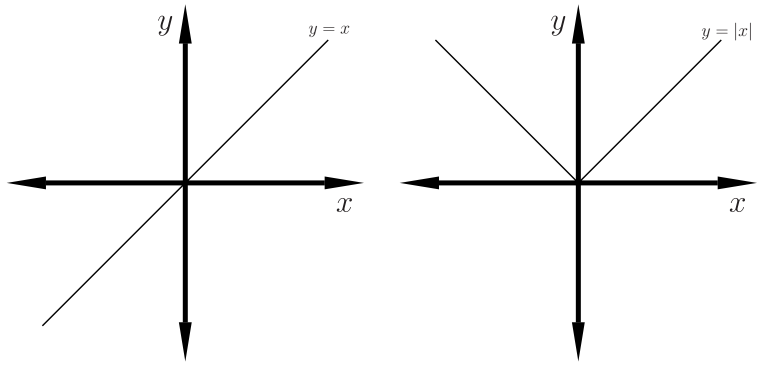
Symbol	Meaning
$<$	is less than
$>$	is greater than
$\leq$	is less than or equal to
$\geq$	is greater than or equal to

The one rule for inequalities is if you multiply or divide by a negative number the inequality sign is reversed as follows

$$
\begin{aligned}
-ax+c &\leq d\\
-ax &\leq d-c\\
x &\geq -\frac{(d-c)}{a}
\end{aligned}
$$

$$
\begin{aligned}
\frac{x}{-e}-f &> g\\
\frac{x}{-e} &> g+f\\
x &< -e(g+f)
\end{aligned}
$$

10) **Modulus**
The modulus symbol is $||$. Anything that is enclosed within this can not evaluate to a negative number. For example $|-4+2|=2$.



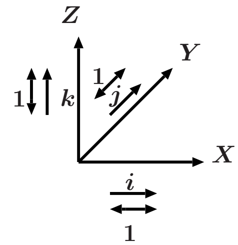
y
y = x
x
y
y = |x|
x

II. Key points on vectors

Key Points

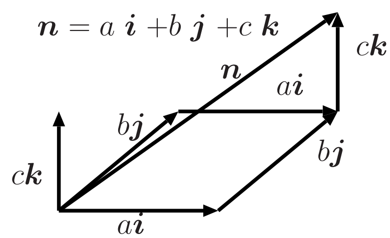
$\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are unit vector in x, y, and z directions respectively. $\jmath$ is $\sqrt{-1}$.

1) A vector has a x component, y component, and z component
   - A vector is expressed as $\boldsymbol{i}$ when it has only a x component and its modulus is 1.
   - A vector is expressed as $\boldsymbol{j}$ when it has only a y component and its modulus is 1.
   - A vector is expressed as $\boldsymbol{k}$ when it has only a z component and its modulus is 1.



2) When a vector has an amount of a in x component, an amount of b in y component, and an amount of c in z component, the vector can be expressed as

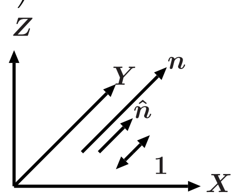
$$\boldsymbol{n} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} \equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1}$$



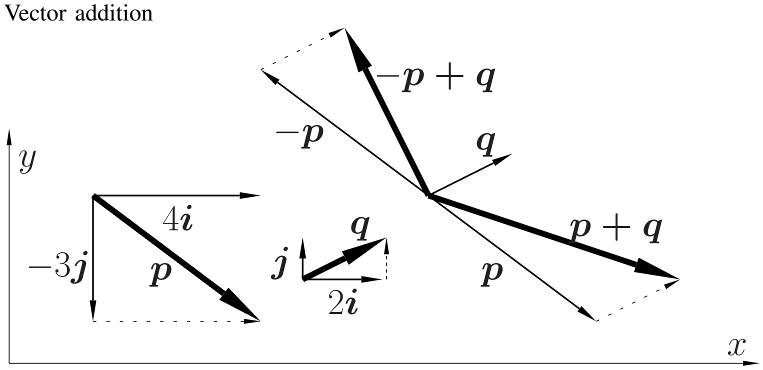
3) A unit vector can be found by dividing a vector by its modulus.

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{|\boldsymbol{n}|} \tag{2}$$

   where $|\boldsymbol{n}|$ is $\sqrt{a^2 + b^2 + c^2}$ when $\boldsymbol{n} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} \equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.



4) Vector addition



   When there are two vectors

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and

$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

the addition of the vectors is

$$\begin{aligned} \boldsymbol{a} + \boldsymbol{b} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \end{aligned} \tag{3}$$

5) The position vector of P with coordinates (a, b, c) is

$$\overrightarrow{OP} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} \tag{4}$$

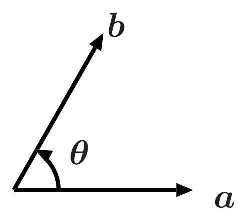
6) When there are two vectors

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and

$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

and these two vectors subtend an angle θ,



the scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3 = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta \tag{5}$$

7) When there are two vectors

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and

$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

and these two vectors subtend an angle θ, the vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} = & \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \boldsymbol{i} \\ & + \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} \boldsymbol{j} \\ & + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \boldsymbol{k} \\ = & (a_2 b_3 - a_3 b_2)\boldsymbol{i} \\ & + (a_3 b_1 - a_1 b_3)\boldsymbol{j} \\ & + (a_1 b_2 - a_2 b_1)\boldsymbol{k} \\ = & |\boldsymbol{a}||\boldsymbol{b}| \sin\theta \hat{\boldsymbol{n}} \end{aligned} \tag{6}$$

where $\hat{\boldsymbol{n}}$ is a unit vector and the direction of $\hat{\boldsymbol{n}}$ is the same as $\boldsymbol{a} \times \boldsymbol{b}$ in Fig. 1.

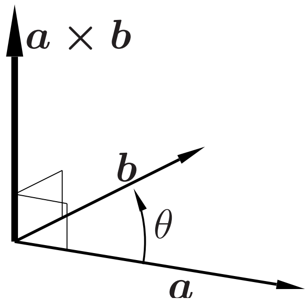


Fig. 1. $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$

8) The vector equation of the line which goes through a point A and is parallel to a vector c is

$$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{c} \tag{7}$$

where t is the real number. Please note that 'x','y','z' are not involved in the vector equation. The cartesian form of Equation (7) is obtained as follows:

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} &= t \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ \therefore \begin{pmatrix} x - a_1 \\ y - a_2 \\ z - a_3 \end{pmatrix} &= t \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \end{aligned}$$

This can be expressed in the scalar manner as

$$\begin{aligned} x - a_1 &= tc_1 \\ \therefore \frac{x - a_1}{c_1} &= t \\ y - a_2 &= tc_2 \\ \therefore \frac{y - a_2}{c_2} &= t \\ z - a_3 &= tc_3 \\ \therefore \frac{z - a_3}{c_3} &= t \end{aligned}$$

By getting rid of t in these three equations, we get the cartesian equation:

$$\frac{x - a_1}{c_1} = \frac{y - a_2}{c_2} = \frac{z - a_3}{c_3} \tag{8}$$

9) The vector equation of the line through points A and B with position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ is

$$\boldsymbol{r} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}) \tag{9}$$

where t is the real number. Please note that 'x','y','z' are not involved in the vector equation. When $0 \leq t \leq 1$, then $\boldsymbol{r}$ is in-between A and B. The cartesian form of Equation (9) is obtained as follows:

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \left(\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right) \\ \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix} \\ \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} &= t \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix} \\ \therefore \begin{pmatrix} x - a_1 \\ y - a_2 \\ z - a_3 \end{pmatrix} &= t \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix} \end{aligned}$$

This can be expressed in the scalar manner as

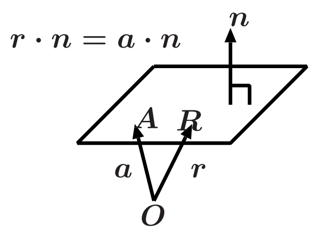
$$x - a_1 = t(b_1 - a_1)$$
$$\therefore \frac{x - a_1}{b_1 - a_1} = t$$
$$y - a_2 = t(b_2 - a_2)$$
$$\therefore \frac{y - a_2}{b_2 - a_2} = t$$
$$z - a_3 = t(b_3 - a_3)$$
$$\therefore \frac{z - a_3}{b_3 - a_3} = t$$

By getting rid of t in these three equations, we get the cartesian equation:

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3} \tag{10}$$

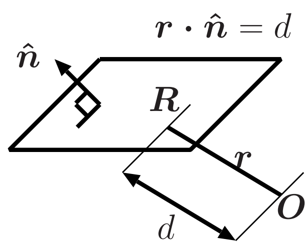
10) A plane perpendicular to the vector $\boldsymbol{n}$ and passing through the point with position vector $\boldsymbol{a}$, has equation

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n} \tag{11}$$



11) A plane with unit normal $\hat{\boldsymbol{n}}$, which has a perpendicular distance d from the origin is given by

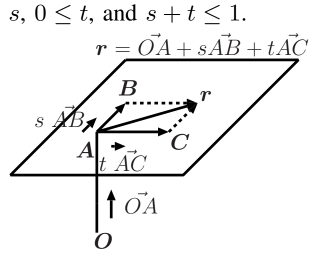
$$\boldsymbol{r} \cdot \hat{\boldsymbol{n}} = d \tag{12}$$



12) A plane which goes through $A(\boldsymbol{a})$, $B(\boldsymbol{b})$ and $C(\boldsymbol{c})$ is given by

$$\boldsymbol{r} = \overrightarrow{OA} + s\overrightarrow{AB} + t\overrightarrow{AC} \tag{13}$$

If the point $R(\boldsymbol{r})$ is inside of the triangle ABC then $0 \leq s$, $0 \leq t$, and $s + t \leq 1$.

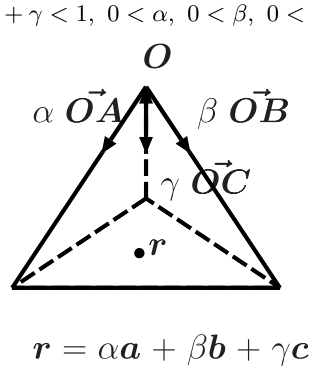


13) A point $R(\boldsymbol{r})$ which is inside the tetrahedron O, $A(\boldsymbol{a})$, $B(\boldsymbol{b})$ and $C(\boldsymbol{c})$ is given by

$$\boldsymbol{r} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \tag{14}$$

where α, β, γ are real numbers and satisfy

$$\alpha + \beta + \gamma < 1,\ 0 < \alpha,\ 0 < \beta,\ 0 < \gamma \tag{15}$$



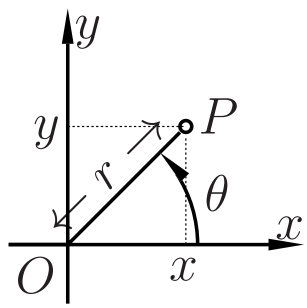


Fig. 2. The relashionship between polar and Cartesian coordinates

III. Key points on coordinates

Key Points

1) If the Cartesian coordinates of a point P are (x, y) then P can be located on a Cartesian plane as indicated in Fig. 2. r is the distance of P from the origin and θ is the angle, measured anti-clockwise, which the line OP makes when measured from the positive $x-$direction. If (x, y) are the Cartesian coordinates and $[r, \theta]$ the polar coordinates of a point P, then

$$x = r\cos\theta, \quad y = r\sin\theta \tag{16}$$

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = y/x \tag{17}$$

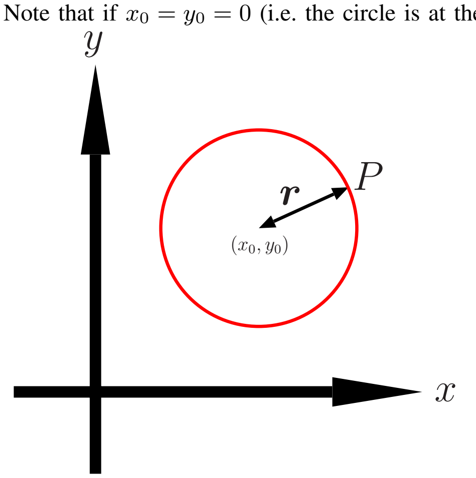
2) If the Cartesian coordinates (x, y) are any point P on a circle of radius r whose centre is at the origin. Then since $\sqrt{x^2 + y^2}$ is the distance of P from the origin, the equation of the circle is,

$$r = \sqrt{x^2 + y^2}, \quad x^2 + y^2 = r^2 \tag{18}$$

3) If the Cartesian coordinates (x, y) are any point P on a circle of radius r whose centre is (x_0, y_0). Then since $\sqrt{(x - x_o)^2 + (y - y_0)^2}$ is the distance of P from the origin, the equation of the circle is,

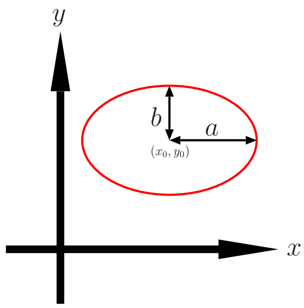
$$r = \sqrt{(x - x_o)^2 + (y - y_0)^2}, \quad (x - x_o)^2 + (y - y_0)^2 = r^2 \tag{19}$$

Note that if $x_0 = y_0 = 0$ (i.e. the circle is at the origin) then Equation (19) reduces to Equation (18).



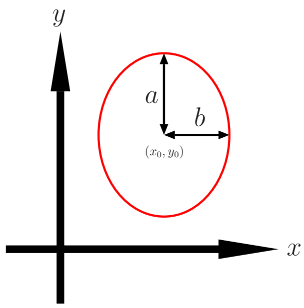
4) An ellipse with centre (x_0, y_0) satifies the equation

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \tag{20}$$



or

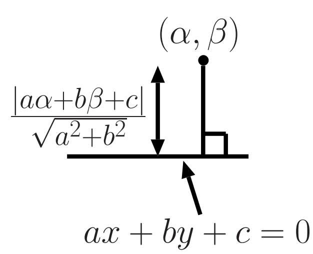
$$\frac{(x-x_0)^2}{b^2} + \frac{(y-y_0)^2}{a^2} = 1 \tag{21}$$



The parameter b is called the semiminor axis by analogy with the parameter a, which is called the semimajor axis (assuming $a > b$). When the major axis is horizontal use Equation (20). If on the other hand the major axis is vertical use Equation (21).

5) The minimum distance between a point $Q(\alpha, \beta)$ and a line $ax + by + c = 0$ is expressed as

$$\frac{|a\alpha + b\beta + c|}{\sqrt{a^2 + b^2}} \tag{22}$$



Proof: The line $ax + by + c = 0$ goes through the point $R(\boldsymbol{r})$ where

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -\frac{c}{b} \end{pmatrix}$$

and it is parallel to

$$\boldsymbol{l} = \begin{pmatrix} b \\ -a \end{pmatrix}$$

A point $P(\boldsymbol{p})$ on the line can be written as

$$\boldsymbol{p} = \boldsymbol{r} + t\boldsymbol{l}$$

where t is a real value. Since

$$\overrightarrow{QP} \perp \boldsymbol{l}$$

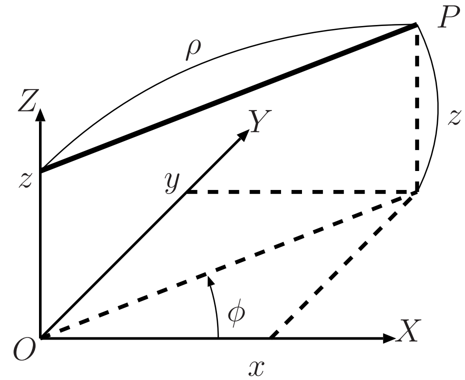


Fig. 3. The relashionship between Cylindrical and Cartesian coordinates

we can express this as the following equation:

$$\begin{aligned} &\overrightarrow{QP}\cdot\boldsymbol{l} \\ &= (\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{l} \\ &= (\boldsymbol{r}+t\boldsymbol{l}-qv)\cdot\boldsymbol{l} \\ &= (\boldsymbol{r}-\boldsymbol{q})\cdot\boldsymbol{l}+t|\boldsymbol{l}|^2 = 0 \\ &\therefore t = \frac{(\boldsymbol{q}-\boldsymbol{r})\cdot\boldsymbol{l}}{|\boldsymbol{l}|^2} \end{aligned}$$

Now we need to get $\overrightarrow{QP}$ as follows:

$$\begin{aligned} |\overrightarrow{QP}|^2 &= |\boldsymbol{p}-\boldsymbol{q}|^2 \\ &= |\boldsymbol{r}+t\boldsymbol{l}-\boldsymbol{q}|^2 \\ &= |\boldsymbol{r}|^2+|\boldsymbol{q}|^2+t^2|\boldsymbol{l}|^2+2t\boldsymbol{r}\boldsymbol{l}-2t\boldsymbol{l}\boldsymbol{q}-2\boldsymbol{r}\boldsymbol{q} \\ &= |\boldsymbol{r}|^2+|\boldsymbol{q}|^2+\frac{((\boldsymbol{q}-\boldsymbol{r})\cdot\boldsymbol{l})^2}{|\boldsymbol{l}|^4}\cdot|\boldsymbol{l}|^2+2\frac{(\boldsymbol{q}-\boldsymbol{r})\cdot\boldsymbol{l}}{|\boldsymbol{l}|^2}(\boldsymbol{r}\boldsymbol{l}-\boldsymbol{l}\boldsymbol{q})-2\boldsymbol{r}\boldsymbol{q} \\ &= |\boldsymbol{r}|^2+|\boldsymbol{q}|^2+\frac{((\boldsymbol{q}-\boldsymbol{r})\cdot\boldsymbol{l})^2}{|\boldsymbol{l}|^2}-2\frac{(\boldsymbol{q}-\boldsymbol{r})\cdot\boldsymbol{l}}{|\boldsymbol{l}|^2}(\boldsymbol{q}-\boldsymbol{r})\boldsymbol{l}-2\boldsymbol{r}\boldsymbol{q} \\ &= |\boldsymbol{r}|^2+|\boldsymbol{q}|^2+\frac{((\boldsymbol{q}-\boldsymbol{r})\cdot\boldsymbol{l})^2}{|\boldsymbol{l}|^2}-2\frac{((\boldsymbol{q}-\boldsymbol{r})\boldsymbol{l})^2}{|\boldsymbol{l}|^2}-2\boldsymbol{r}\boldsymbol{q} \\ &= |\boldsymbol{r}|^2+|\boldsymbol{q}|^2-\frac{((\boldsymbol{q}-\boldsymbol{r})\cdot\boldsymbol{l})^2}{|\boldsymbol{l}|^2}-2\boldsymbol{r}\boldsymbol{q} \\ &= \frac{|a\alpha+b\beta+c|^2}{a^2+b^2} \\ \therefore |\overrightarrow{QP}| &= \frac{|a\alpha+b\beta+c|}{\sqrt{a^2+b^2}} \end{aligned}$$

6) 3D Cylindrical polar coordinate (ρ,ϕ,z) in Fig. 3 can be obtained from

$$\rho = \sqrt{x^2+y^2}; \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{23}$$

$$\because x = \rho\cos\phi, y = \rho\sin\phi \tag{24}$$

You need to draw a diagram to determine the correct ϕ

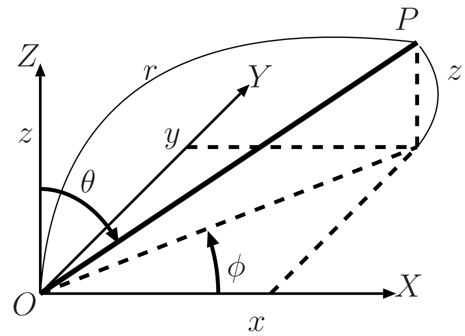


Fig. 4. The relashionship between Spherical and Cartesian coordinates

7) 3D Spherical polar coordinate (r, θ, ϕ) in Fig. 4 can be obtained from

$$r = \sqrt{x^2 + y^2 + z^2}; \theta = \cos^{-1}\left(\frac{z}{r}\right); \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{25}$$

$$\because x = r\sin\theta\cos\phi; y = r\sin\theta\sin\phi; z = r\cos\theta \tag{26}$$

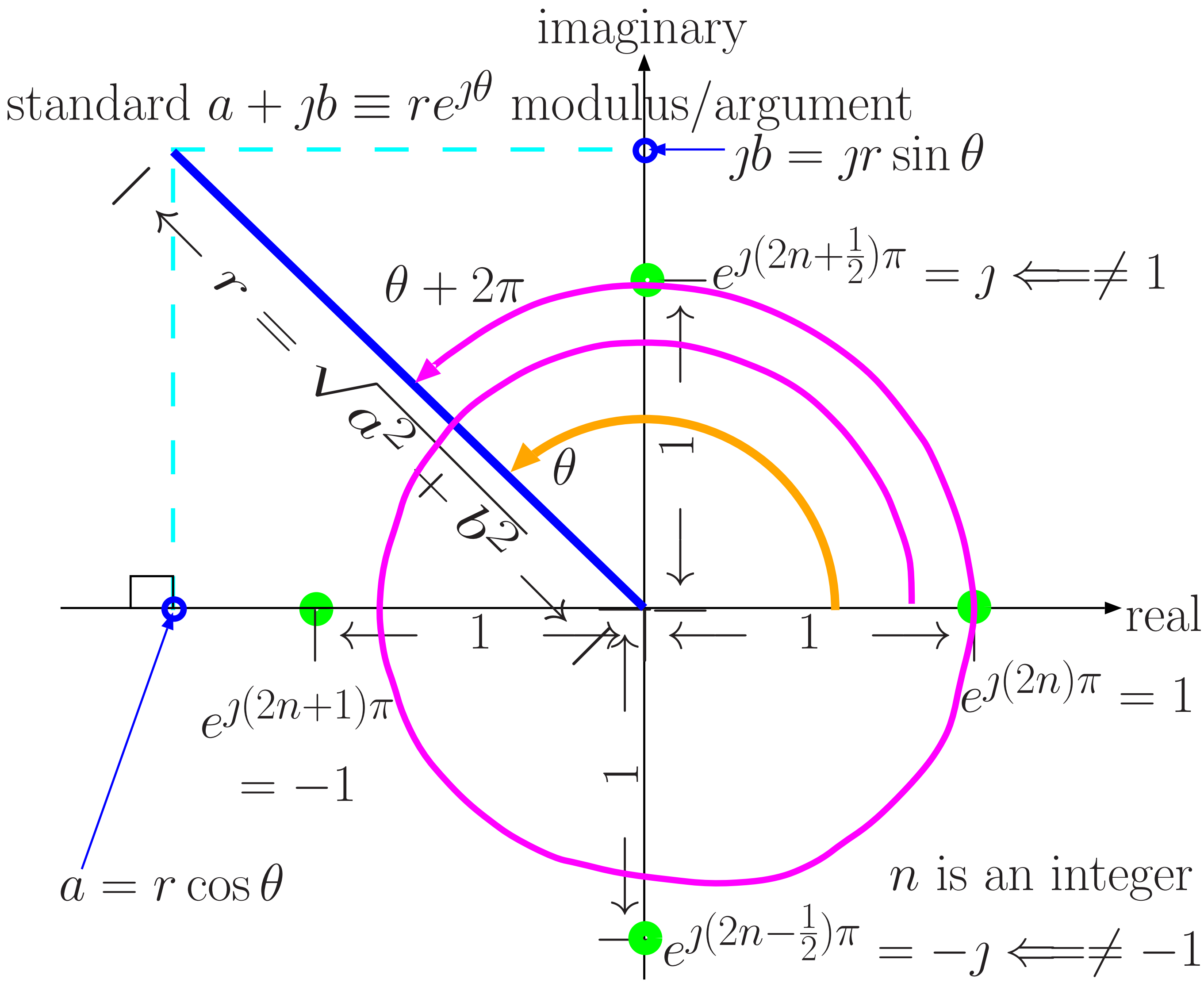
You need to draw a diagram to determine the correct ϕ. θ should satisfy $0 \leq \theta \leq \pi$ without a diagram

IV. Key points on complex numbers

Key Points

1) The symbol $\jmath$ is such that

$$\jmath^2 = -1 \qquad \jmath = \sqrt{-1} \tag{27}$$



2) In Argand diagram, the complex number $a + \jmath b$ (the standard form) can be expressed as

$$a + \jmath b = r\mathfrak{e}^{\jmath\theta} = r(\cos\theta + \jmath\sin\theta) \tag{28}$$

,which is the modulus/argument form,where

$$r = |a + \jmath b| = \sqrt{a^2 + b^2} \qquad \tan\theta = \frac{b}{a} \tag{29}$$

$$a = r\cos\theta \qquad b = r\sin\theta \tag{30}$$

Be careful: $a^2 - b^2 + 2ab\jmath = (a + \jmath b)^2 \neq |a + \jmath b|^2 = a^2 + b^2$.
$\frac{a\jmath}{b\jmath} = \frac{a}{b}$, *i.e.*, $\frac{a\jmath}{b\jmath} \neq a - b$.
$\frac{\mathfrak{e}^{a\jmath}}{\mathfrak{e}^{b\jmath}} = \mathfrak{e}^{a\jmath - b\jmath} = \mathfrak{e}^{(a-b)\jmath}$

3) From the figure, $\pm\jmath$ can be expressed as

$$\jmath = \mathfrak{e}^{\frac{\pi}{2}\jmath}, -\jmath = \mathfrak{e}^{-\frac{\pi}{2}\jmath} \tag{31}$$

4) If $a + \jmath b$ is any complex number then its complex conjugate is

$$a - \jmath b \tag{32}$$

5) In the Argand diagram, the argument can be $2\pi n$ rotated to have an identical value:

$$\mathfrak{e}^{\jmath\theta} = \mathfrak{e}^{\jmath(\theta + 2\pi n)} \tag{33}$$

where n is an integer.

6) De Moivre's theorem

$$(r\mathfrak{e}^{\jmath\theta})^n = [r(\cos\theta + \jmath\sin\theta)]^n = r^n(\cos n\theta + \jmath\sin n\theta) = r^n\mathfrak{e}^{\jmath n\theta} \tag{34}$$

7) nth roots of complex numbers

If

$$z^n = r\mathfrak{e}^{\jmath\theta} = r(\cos\theta + \jmath\sin\theta)$$

then

$$z = \sqrt[n]{r}\mathfrak{e}^{\jmath(\theta+2k\pi)/n} \qquad k = 0, \pm 1, \pm 2, \dots \tag{35}$$

In other words, if

$$a\mathfrak{e}^{\jmath b} = c\mathfrak{e}^{\jmath d}$$

then

$$a = c$$
$$b = d + 2n\pi$$

8) If $a + \jmath b = c + \jmath d$, where a, b, c, and d, are real, then we can say

$$a = c, b = d \tag{36}$$

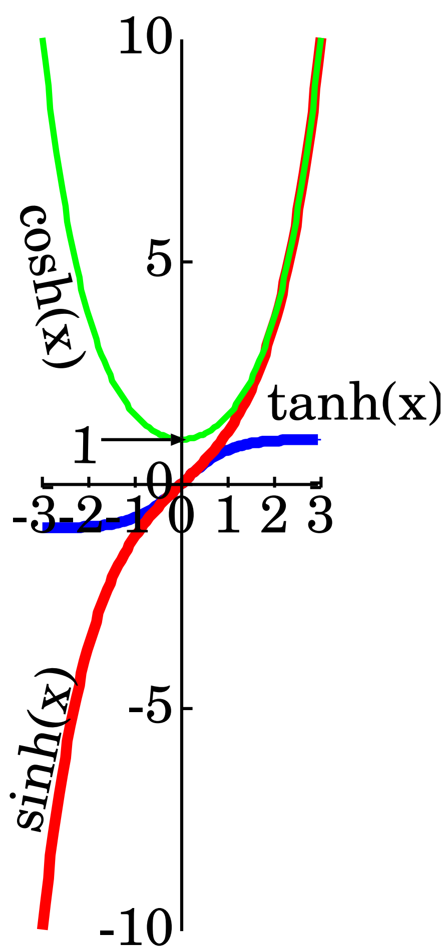
If $a + \jmath b = 0$, then $a = b = 0$

9) $\cosh x$ and $\sinh x$ are defined as

$$\cosh x = \frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}, \sinh x = \frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2} \tag{37}$$

$$x = \cosh^{-1}\left(\frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}\right), x = \sinh^{-1}\left(\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2}\right)$$

$$\tanh(x) = (\mathfrak{e}^x - \mathfrak{e}^{-x})/(\mathfrak{e}^x + \mathfrak{e}^{-x})$$
$$\cosh^2(A) - \sinh^2(A) = 1$$



When you need x which satisfies $\cosh(x) = \alpha$ where α is a real number, using $x = \cosh^{-1}\left(\frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}\right)$ we get

$$\begin{aligned}\frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2} &= \alpha \\ \therefore \mathfrak{e}^x + \mathfrak{e}^{-x} &= 2\alpha \\ \therefore \mathfrak{e}^{2x} + 1 &= 2\alpha\mathfrak{e}^x \\ \therefore \mathfrak{e}^{2x} - 2\alpha\mathfrak{e}^x + 1 &= 0 \\ \therefore \mathfrak{e}^x &= \alpha \pm \sqrt{\alpha^2 - 1} \\ \therefore x = \cosh^{-1}(\alpha) &= \ln(\alpha \pm \sqrt{\alpha^2 - 1})\end{aligned}$$

When you need x which satisfies $\sinh(x) = \alpha$ where α is a real number, using $x = \sinh^{-1}\left(\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2}\right)$ we get

$$\begin{aligned}\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2} &= \alpha \\ \therefore \mathfrak{e}^x - \mathfrak{e}^{-x} &= 2\alpha \\ \therefore \mathfrak{e}^{2x} - 1 &= 2\alpha\mathfrak{e}^x \\ \therefore \mathfrak{e}^{2x} - 2\alpha\mathfrak{e}^x - 1 &= 0 \\ \therefore \mathfrak{e}^x &= \alpha \pm \sqrt{\alpha^2 + 1} \\ \therefore x &= \ln(\alpha \pm \sqrt{\alpha^2 + 1}) \\ \therefore x = \sinh^{-1}(\alpha) &= \ln(\alpha + \sqrt{\alpha^2 + 1}) (\because A > 0 \text{ for } \ln A)\end{aligned}$$

10) $\cos\theta$ and $\sin\theta$ are defined as

$$\cos\theta = \frac{\mathfrak{e}^{\jmath\theta} + \mathfrak{e}^{-\jmath\theta}}{2}, \sin\theta = \frac{\mathfrak{e}^{\jmath\theta} - \mathfrak{e}^{-\jmath\theta}}{2\jmath} \tag{38}$$

Proof: We know that

$$\mathfrak{e}^{\jmath\theta} = \cos\theta + \jmath\sin\theta \qquad ①$$

By replacing $\jmath$ in ① with $-\jmath$ we get

$$\mathfrak{e}^{-\jmath\theta} = \cos\theta - \jmath\sin\theta \qquad ②$$

① + ② gives us

$$\begin{aligned}\mathfrak{e}^{\jmath\theta} + \mathfrak{e}^{-\jmath\theta} &= 2\cos\theta \\ \therefore \frac{\mathfrak{e}^{\jmath\theta} + \mathfrak{e}^{-\jmath\theta}}{2} &= \cos\theta\end{aligned}$$

① - ② gives us

$$\begin{aligned}\mathfrak{e}^{\jmath\theta} - \mathfrak{e}^{-\jmath\theta} &= 2\jmath\sin\theta \\ \therefore \frac{\mathfrak{e}^{\jmath\theta} - \mathfrak{e}^{-\jmath\theta}}{2\jmath} &= \sin\theta\end{aligned}$$

11) The expression of $\sinh(z), \cosh(z), \sin(z), \cos(z)$ in in standard form.

$$\begin{aligned}\sinh(a + \jmath b) &= \sinh(a)\cos(b) + \jmath\cosh(a)\sin(b) \qquad ① \\ \cosh(a + \jmath b) &= \cosh(a)\cos(b) + \jmath\sinh(a)\sin(b) \qquad ② \\ \sin(a \pm \jmath b) &= \sin(a)\cosh(b) \pm \jmath\cos(a)\sinh(b) \qquad ③ \\ \cos(a \pm \jmath b) &= \cos(a)\cosh(b) \mp \jmath\sin(a)\sinh(b) \qquad ④\end{aligned}$$

They are useful but in most cases you need to prove these before you can use them. Therefore you should go through the following proof and practice the proof.

- The proof of ①

$$\begin{aligned}\sinh(a + \jmath b) &= \frac{\mathfrak{e}^{a+\jmath b} - \mathfrak{e}^{-(a+\jmath b)}}{2} = \frac{\mathfrak{e}^{a+\jmath b} - \mathfrak{e}^{-a-\jmath b}}{2} = \frac{\mathfrak{e}^a\mathfrak{e}^{\jmath b} - \mathfrak{e}^{-a}\mathfrak{e}^{-\jmath b}}{2} \\ &= \frac{\mathfrak{e}^a(\cos b + \jmath\sin b) - \mathfrak{e}^{-a}(\cos(-b) + \jmath\sin(-b))}{2} = \frac{\mathfrak{e}^a(\cos b + \jmath\sin b) - \mathfrak{e}^{-a}(\cos b - \jmath\sin b)}{2} \\ &= \frac{\cos b\mathfrak{e}^a + \jmath\sin b\mathfrak{e}^a}{2} - \frac{\cos b\mathfrak{e}^{-a} - \jmath\sin b\mathfrak{e}^{-a}}{2} = \frac{\cos b\mathfrak{e}^a + \jmath\sin b\mathfrak{e}^a}{2} + \frac{-\cos b\mathfrak{e}^{-a} + \jmath\sin b\mathfrak{e}^{-a}}{2} \\ &= \frac{\cos b\mathfrak{e}^a - \cos b\mathfrak{e}^{-a}}{2} + \jmath\frac{\sin b\mathfrak{e}^a + \sin b\mathfrak{e}^{-a}}{2} = \cos(b)\frac{\mathfrak{e}^a - \mathfrak{e}^{-a}}{2} + \jmath\sin(b)\frac{\mathfrak{e}^a + \mathfrak{e}^{-a}}{2} \\ &= \sinh(a)\cos(b) + \jmath\cosh(a)\sin(b)\end{aligned}$$

- The proof of ②

$$
\begin{aligned}
\cosh(a+\jmath b) &= \frac{\mathfrak{e}^{a+\jmath b}+\mathfrak{e}^{-(a+\jmath b)}}{2} = \frac{\mathfrak{e}^{a+\jmath b}+\mathfrak{e}^{-a-\jmath b}}{2} = \frac{\mathfrak{e}^{a}\mathfrak{e}^{\jmath b}+\mathfrak{e}^{-a}\mathfrak{e}^{-\jmath b}}{2} \\
&= \frac{\mathfrak{e}^{a}(\cos(b)+\jmath\sin(b))+\mathfrak{e}^{-a}(\cos(-b)+\jmath\sin(-b))}{2} = \frac{\mathfrak{e}^{a}(\cos(b)+\jmath\sin(b))+\mathfrak{e}^{-a}(\cos(b)-\jmath\sin(b))}{2} \\
&= \frac{\mathfrak{e}^{a}\cos(b)+\jmath\mathfrak{e}^{a}\sin(b)+\mathfrak{e}^{-a}\cos(b)-\jmath\mathfrak{e}^{-a}\sin(b)}{2} = \frac{\mathfrak{e}^{a}\cos(b)+\mathfrak{e}^{-a}\cos(b)-\jmath\mathfrak{e}^{-a}\sin(b)+\jmath\mathfrak{e}^{a}\sin(b)}{2} \\
&= \frac{\mathfrak{e}^{a}\cos(b)+\mathfrak{e}^{-a}\cos(b)+\jmath(-\mathfrak{e}^{-a}\sin(b)+\mathfrak{e}^{a}\sin(b))}{2} = \frac{\mathfrak{e}^{a}\cos(b)+\mathfrak{e}^{-a}\cos(b)}{2} + \frac{\jmath(-\mathfrak{e}^{-a}\sin(b))+\mathfrak{e}^{a}\sin(b))}{2} \\
&= \cos(b)\frac{\mathfrak{e}^{a}+\mathfrak{e}^{-a}}{2} + \jmath\sin(b)\frac{-\mathfrak{e}^{-a}+\mathfrak{e}^{a}}{2} = \cosh(a)\cos(b)+\jmath\sinh(a)\sin(b)
\end{aligned}
$$

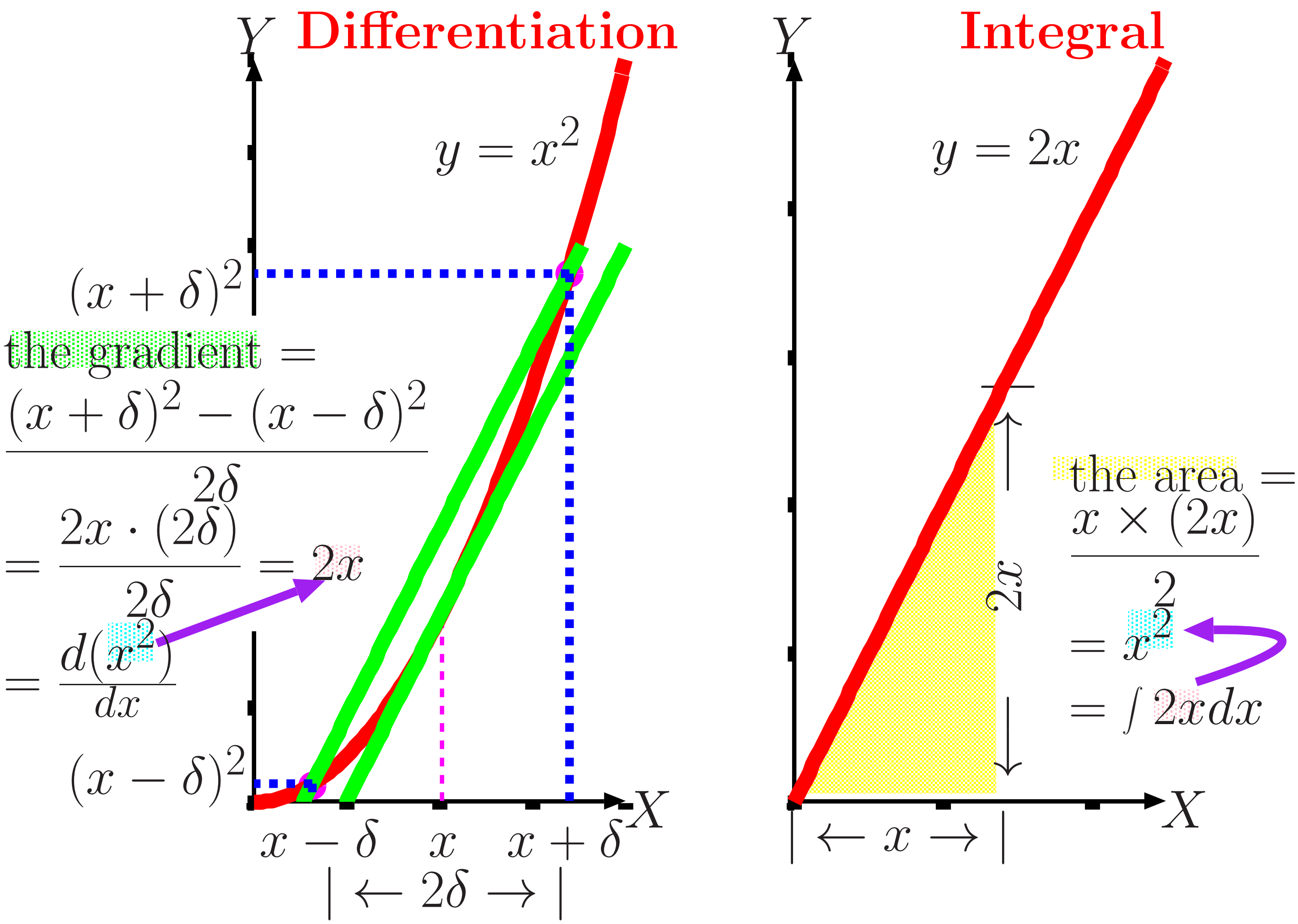
- The proof of ③

$$
\begin{aligned}
\sin(a\pm\jmath b) &= \frac{\mathfrak{e}^{\jmath(a\pm\jmath b)}-\mathfrak{e}^{-\jmath(a\pm\jmath b)}}{2\jmath} = \frac{\mathfrak{e}^{a\jmath\pm\jmath^2 b}-\mathfrak{e}^{-a\jmath\mp\jmath^2 b}}{2\jmath} = \frac{\mathfrak{e}^{a\jmath\mp b}-\mathfrak{e}^{-a\jmath\pm b}}{2\jmath} = \frac{\mathfrak{e}^{\jmath a}\mathfrak{e}^{\mp b}-\mathfrak{e}^{-\jmath a}\mathfrak{e}^{\pm b}}{2\jmath} \\
&= \frac{(\cos(a)+\jmath\sin(a))\mathfrak{e}^{\mp b}-(\cos(a)-\jmath\sin(a))\mathfrak{e}^{\pm b}}{2\jmath} = \frac{\cos(a)\mathfrak{e}^{\mp b}+\jmath\sin(a)\mathfrak{e}^{\mp b}-\cos(a)\mathfrak{e}^{\pm b}+\jmath\sin(a)\mathfrak{e}^{\pm b}}{2\jmath} \\
&= \frac{\cos(a)(\mathfrak{e}^{\mp b}-\mathfrak{e}^{\pm b})+\jmath\sin(a)(\mathfrak{e}^{\mp b}+\mathfrak{e}^{\pm b})}{2\jmath} = -\jmath\frac{\cos(a)(\mathfrak{e}^{\mp b}-\mathfrak{e}^{\pm b})+\jmath\sin(a)(\mathfrak{e}^{\mp b}+\mathfrak{e}^{\pm b})}{2} \\
&= \frac{-\jmath\cos(a)(\mathfrak{e}^{\mp b}-\mathfrak{e}^{\pm b})-\jmath^2\sin(a)(\mathfrak{e}^{\mp b}+\mathfrak{e}^{\pm b})}{2} = \frac{-\jmath\cos(a)(\mathfrak{e}^{\mp b}-\mathfrak{e}^{\pm b})+\sin(a)(\mathfrak{e}^{\mp b}+\mathfrak{e}^{\pm b})}{2} \\
&= \frac{-\jmath\cos(a)(\mathfrak{e}^{\mp b}-\mathfrak{e}^{\pm b})}{2} + \frac{\sin(a)(\mathfrak{e}^{\mp b}+\mathfrak{e}^{\pm b})}{2} = \frac{\jmath\cos(a)(-\mathfrak{e}^{\mp b}+\mathfrak{e}^{\pm b})}{2} + \frac{\sin(a)(\mathfrak{e}^{\mp b}+\mathfrak{e}^{\pm b})}{2} \\
&= \jmath\cos(a)\frac{-\mathfrak{e}^{\mp b}+\mathfrak{e}^{\pm b}}{2} + \sin(a)\frac{\mathfrak{e}^{\mp b}+\mathfrak{e}^{\pm b}}{2} = \pm\jmath\cos(a)\sinh(b)+\sin(a)\cosh(b)
\end{aligned}
$$

- The proof of ④

$$
\begin{aligned}
\cos(a\pm\jmath b) &= \frac{\mathfrak{e}^{\jmath(a\pm\jmath b)}+\mathfrak{e}^{-\jmath(a\pm\jmath b)}}{2} = \frac{\mathfrak{e}^{a\jmath\pm\jmath^2 b}+\mathfrak{e}^{-a\jmath\mp\jmath^2 b}}{2} = \frac{\mathfrak{e}^{a\jmath\mp b}+\mathfrak{e}^{-a\jmath\pm b}}{2\jmath} = \frac{\mathfrak{e}^{\jmath a}\mathfrak{e}^{\mp b}+\mathfrak{e}^{-\jmath a}\mathfrak{e}^{\pm b}}{2} \\
&= \frac{(\cos(a)+\jmath\sin(a))\mathfrak{e}^{\mp b}+(\cos(a)-\jmath\sin(a))\mathfrak{e}^{\pm b}}{2} = \frac{\cos(a)\mathfrak{e}^{\mp b}+\jmath\sin(a)\mathfrak{e}^{\mp b}+\cos(a)\mathfrak{e}^{\pm b}-\jmath\sin(a)\mathfrak{e}^{\pm b}}{2} \\
&= \frac{\cos(a)(\mathfrak{e}^{\mp b}+\mathfrak{e}^{\pm b})+\jmath\sin(a)(\mathfrak{e}^{\mp b}-\mathfrak{e}^{\pm b})}{2} = \frac{\cos(a)(\mathfrak{e}^{\mp b}+\mathfrak{e}^{\pm b})}{2} + \frac{\jmath\sin(a)(\mathfrak{e}^{\mp b}-\mathfrak{e}^{\pm b})}{2} \\
&= \cos(a)\frac{\mathfrak{e}^{\mp b}+\mathfrak{e}^{\pm b}}{2} + \jmath\sin(a)\frac{\mathfrak{e}^{\mp b}-\mathfrak{e}^{\pm b}}{2} = \cos(a)\cosh(b)-\jmath\sin(a)\frac{-\mathfrak{e}^{\mp b}+\mathfrak{e}^{\pm b}}{2} \\
&= \cos(a)\cosh(b)-\jmath\sin(a)(\pm\sinh(b)) = \cos(a)\cosh(b)\mp\jmath\sin(a)\sinh(b)
\end{aligned}
$$

Key points



1) Product rule

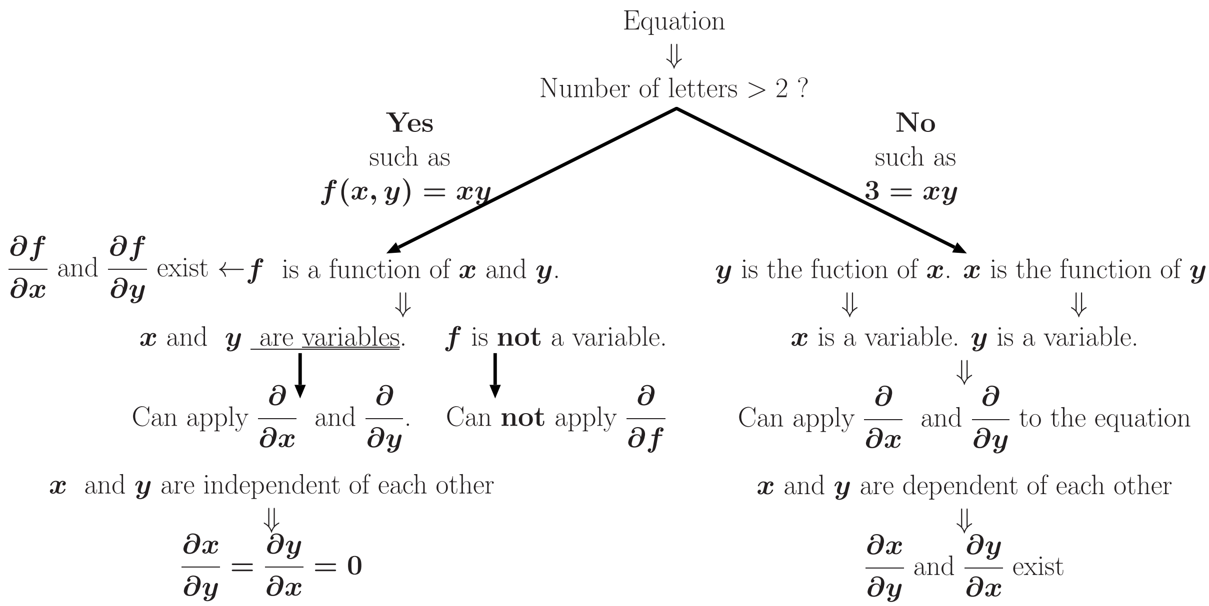
$$\frac{d\{f(x)g(x)\}}{dx} = f(x)\frac{d\{g(x)\}}{dx} + \frac{d\{f(x)\}}{dx}g(x) \tag{39}$$

2) Chain rule
   a) When $y = f(u)$ and $u = g(x)$,

$$\frac{d\{y\}}{dx} = \frac{d\{u\}}{dx} \cdot \frac{\partial\{y\}}{\partial u} \tag{40}$$

It is important that you know the fundamental differentiable functions of Equation (46) $\sim$ Equation (54) so that a complicated function can be simplified to one of the fundamental functions of Equation (46) $\sim$ Equation (54). For example, if you know that 5^x can be differentiable, you can change $\frac{d\left\{5^{x^4-2}\right\}}{dx}$ to $\frac{d\{5^X\}}{dx}$ where $X = x^4 - 2$.

   b) Function and variables



c) When W is a function of x, y and z and x, y, z are the function of s and t, $\frac{d\{W\}}{dt}$ and $\frac{d\{W\}}{ds}$ can not be directly calculated but can be calculated as follows:

$$\frac{d\{W\}}{dt} = \frac{d\{W\}}{dx} \cdot \frac{d\{x\}}{dt} + \frac{d\{W\}}{dy} \cdot \frac{d\{y\}}{dt} + \frac{d\{W\}}{dz} \cdot \frac{d\{z\}}{dt}$$

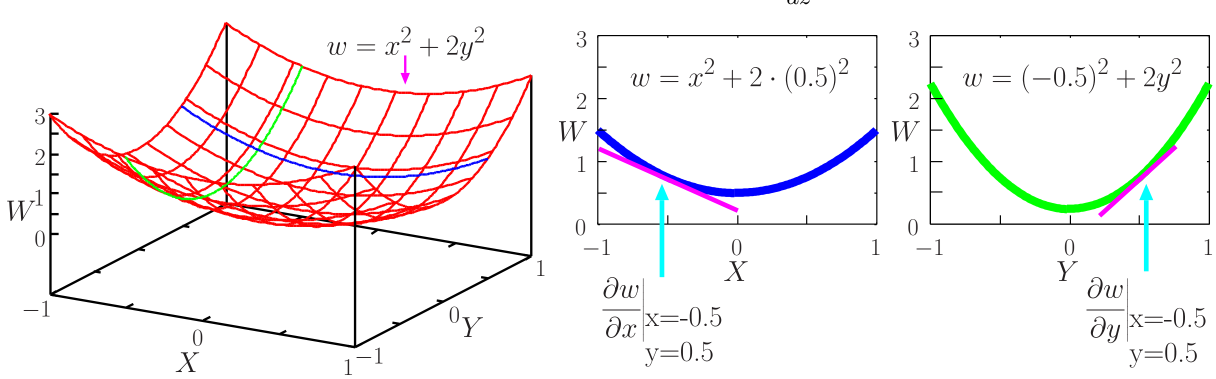
$$\frac{d\{W\}}{ds} = \frac{d\{W\}}{dx} \cdot \frac{d\{x\}}{ds} + \frac{d\{W\}}{dy} \cdot \frac{d\{y\}}{ds} + \frac{d\{W\}}{dz} \cdot \frac{d\{z\}}{ds}$$

d) When W is a function of x, y and z, the total differential dW can be obtained by

$$dW = \frac{d\{W\}}{dx}dx + \frac{d\{W\}}{dy}dy + \frac{d\{W\}}{dz}dz$$

e) When W is a function of x, y and z, the gradient ∇W is defined as

$$\nabla W = \frac{d\{W\}}{dx}\boldsymbol{i} + \frac{d\{W\}}{dy}\boldsymbol{j} + \frac{d\{W\}}{dy}\boldsymbol{k} = \begin{pmatrix} \frac{d\{W\}}{dx} \\ \frac{d\{W\}}{dy} \\ \frac{d\{W\}}{dz} \end{pmatrix}$$



3) Quotient rule

$$\frac{d\left\{\frac{f(x)}{g(x)}\right\}}{dx} = \frac{\frac{d\{f(x)\}}{dx}g(x) - f(x)\frac{d\{g(x)\}}{dx}}{(g(x))^2} \tag{41}$$

Check if $g(x)$ is really a function. If $g(x)$ is a constant, you do not have to use the quatient rule. If $f(x)$ and $g(x)$ are polynomial, check the order of $f(x)$ and $g(x)$. If the order of $f(x)$ is higher than that of $g(x)$ then modify $\frac{f(x)}{g(x)}$ so that the order of the numerator of the resultant function is always lower than the order of denominator.

4) When x and y are the function of t,

$$\frac{d\left\{y\right\}}{dx} = \frac{d\left\{y\right\}}{dt} \cdot \frac{d\left\{t\right\}}{dx} = \frac{d\left\{y\right\}}{dt} \cdot \left(\frac{d\left\{x\right\}}{dt}\right)^{-1}$$

and

$$\frac{d^2 y}{dx^2} = \frac{d\left\{\frac{d\left\{y\right\}}{dx}\right\}}{dx} = \frac{d\left\{t\right\}}{dx}\frac{d\left\{\frac{d\left\{y\right\}}{dx}\right\}}{dt} = \left(\frac{d\left\{x\right\}}{dt}\right)^{-1}\frac{d\left\{\frac{d\left\{y\right\}}{dx}\right\}}{dt}$$

5) Let $F(x)$ and $G(y)$ the function of x and y, respectively.
   a) $\frac{d\left\{y\right\}}{dx}$ for $F(x) + G(y) = 0$ is obtained as

$$\begin{aligned} F(x) + G(y) &= 0 \\ \therefore \frac{d\left\{F(x) + G(y)\right\}}{dx} &= \frac{d\left\{0\right\}}{dx} \\ \therefore \frac{d\left\{F(x)\right\}}{dx} + \frac{d\left\{G(y)\right\}}{dx} &= 0 \\ \therefore \frac{d\left\{F(x)\right\}}{dx} + \frac{d\left\{y\right\}}{dx}\frac{d\left\{G(y)\right\}}{dy} &= 0 \\ \therefore \frac{d\left\{y\right\}}{dx} &= -\frac{\frac{d\left\{F(x)\right\}}{dx}}{\frac{d\left\{G(y)\right\}}{dy}} \end{aligned}$$

   b) $\frac{d\left\{y\right\}}{dx}$ for $F(x) \cdot G(y) = 0$ is obtained as

$$\begin{aligned} F(x) \cdot G(y) &= 0 \\ \therefore \frac{d\left\{F(x) \cdot G(y)\right\}}{dx} &= \frac{d\left\{0\right\}}{dx} \\ \therefore \frac{d\left\{F(x)\right\}}{dx}G(y) + F(x)\frac{d\left\{G(y)\right\}}{dx} &= 0 \\ \therefore \frac{d\left\{F(x)\right\}}{dx}G(y) + F(x) \cdot \frac{d\left\{y\right\}}{dx}\frac{d\left\{G(y)\right\}}{dy} &= 0 \\ \therefore \frac{d\left\{y\right\}}{dx} &= -\frac{\frac{d\left\{F(x)\right\}}{dx}G(y)}{F(x)\frac{d\left\{G(y)\right\}}{dy}} \end{aligned}$$

6) When a graph has a local minimum and local maximum at (x_m, y_m), $\frac{d\left\{y\right\}}{dx}|_{(x,y)=(x_m,y_m)} = 0$. Furthermore, if $\frac{d^2 y}{dx^2}|_{(x,y)=(x_m,y_m)} > 0$, then (x_m, y_m) is the local minimum point. If $\frac{d^2 y}{dx^2}|_{(x,y)=(x_m,y_m)} < 0$, then (x_m, y_m) is the local maximum point.
7) L'Hôpital's Rule
   Let's assume we have a function of

$$y = f(x) = \frac{P(x)}{Q(x)}.$$

   If we want $\lim_{x \to a} f(x)$ but we find out $P(a) = Q(a) = 0$ then we can still find $f(a)$ by

$$\lim_{x \to a} f(x) = \frac{P'(a)}{Q'(a)}.$$

   Please do not mix up with $\frac{df(x)}{dx}$ here. $\frac{\left.\frac{d\frac{P(x)}{Q(x)}}{dx}\right|_{x=a} = \left.\frac{P'(x)Q(x) - P(x)'Q(x)}{Q^2(x)}\right|_{x=a} \neq \frac{P'(a)}{Q'(a)}}{}$.
   You are NOT finding a gradient but you are trying to obtain the value of $f(a) = \frac{P(a)}{Q(a)}$
   Proof: When we use Equation (82) we can write

$$P(a+h) = P(a) + h\left.\frac{d\left\{P\right\}}{dx}\right|_{x=a} + \cdots$$

   and

$$Q(a+h) = Q(a) + h\left.\frac{d\left\{Q\right\}}{dx}\right|_{x=a} + \cdots$$

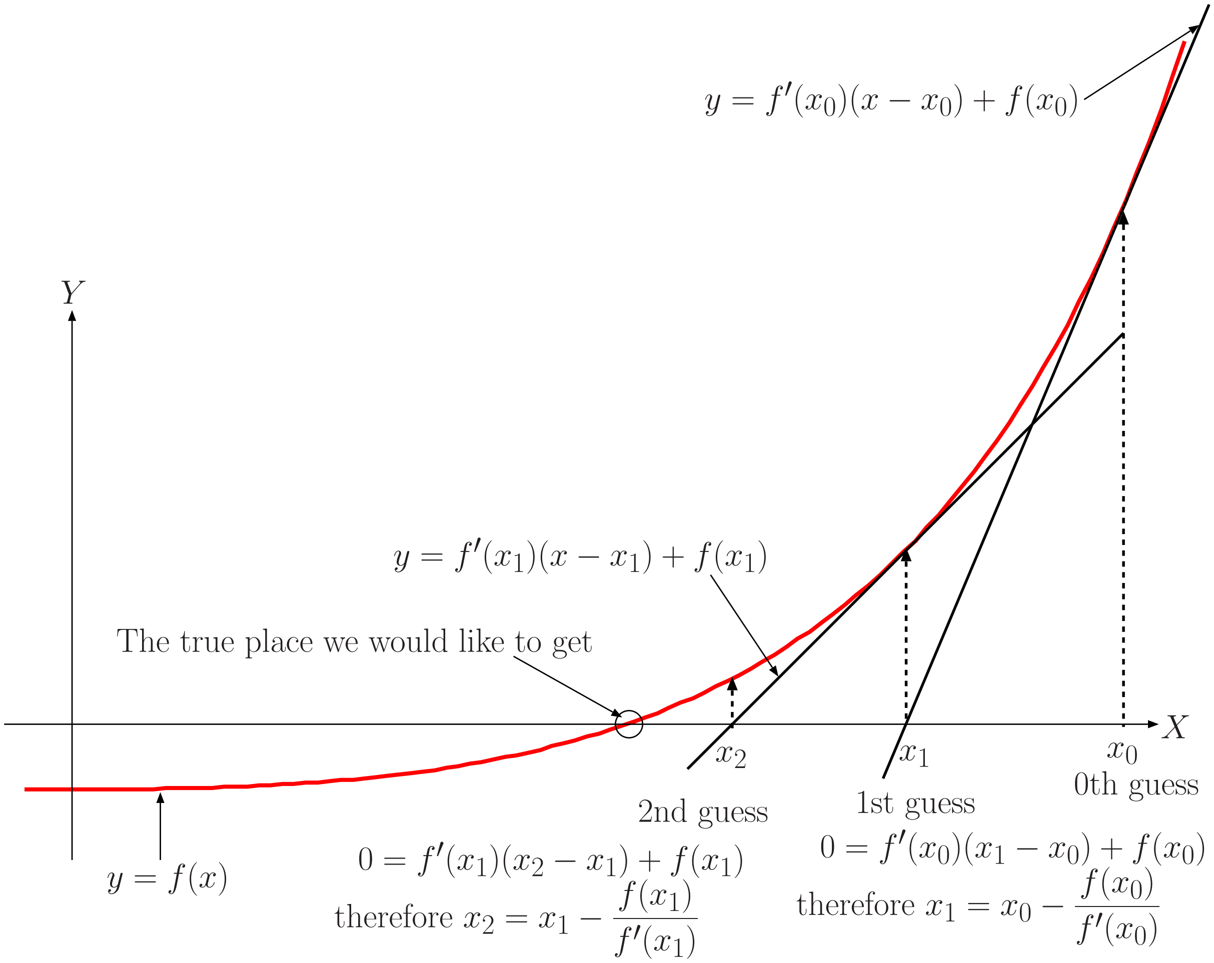


Fig. 5. Estimation of the crossing point between $y = f(x)$ and X axis.

Then we can get the limit as

$$\lim_{x\to a}\frac{P(x)}{Q(x)} = \lim_{h\to 0}\frac{P(a+h)}{Q(a+h)} = \lim_{h\to 0}\frac{P(a) + h\left.\frac{d\{P\}}{dx}\right|_{x=a}}{Q(a) + h\left.\frac{d\{Q\}}{dx}\right|_{x=a}}$$
$$= \lim_{h\to 0}\frac{0 + h\left.\frac{d\{P\}}{dx}\right|_{x=a}}{0 + h\left.\frac{d\{Q\}}{dx}\right|_{x=a}} = \lim_{h\to 0}\frac{h\left.\frac{d\{P\}}{dx}\right|_{x=a}}{h\left.\frac{d\{Q\}}{dx}\right|_{x=a}} = \frac{\left.\frac{d\{P\}}{dx}\right|_{x=a}}{\left.\frac{d\{Q\}}{dx}\right|_{x=a}}$$

8) Newton-Raphson method The crossing point between $y = f(x)$ and X axis can be estimated in an iterative manner as is shown in Fig. 5. The $(n+1)$th guess of the crossing point is obtained using nth guess as in Equation (42).

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{42}$$

9) Multivariable higher order differentiation

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx} \tag{43}$$
$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dy} \tag{44}$$

Please pay attention

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} \neq \frac{d\{f(x,y\}}{dy}\cdot\frac{d\{f(x,y)\}}{dx}.$$

Please also be aware the following difference: Let

$$f(x,y) = axy + bx + cy.$$

When we need $\frac{d\{f(x,y\}}{dx}$, then you assume x and y are independent and we obtain

$$\frac{d\{f(x,y)\}}{dx} = ay + b$$

but if we need $\frac{d\{y\}}{dx}$ for $f(x,y) = 0$, then $f(x,y) = 0$ tells you that x and y are dependent of each other and xy can be regarded as the multiplication of two function x and y and then we obtain

$$\begin{aligned}
\frac{d\{f(x,y)\}}{dx} &= \frac{d\{0\}}{dx} \\
\therefore \frac{d\{axy + bx + cy\}}{dx} &= 0 \\
\therefore a\frac{d\{x\}}{dx}y + ax\frac{d\{y\}}{dx} + b\frac{d\{x\}}{dx} + c\frac{d\{y\}}{dx} &= 0 \\
\therefore ay + ax\frac{d\{y\}}{dx} + b + c\frac{d\{y\}}{dx} &= 0 \\
\therefore (ax + c)\frac{d\{y\}}{dx} &= -ay - b \\
\therefore \frac{d\{y\}}{dx} &= \frac{-ay - b}{ax + c}
\end{aligned}$$

10) Local minimum and local maximum
When $f(x,y)$ has a local minimum or a local maximum at $x = a$ and $y = b$, then $f(x,y)$ satisfies:

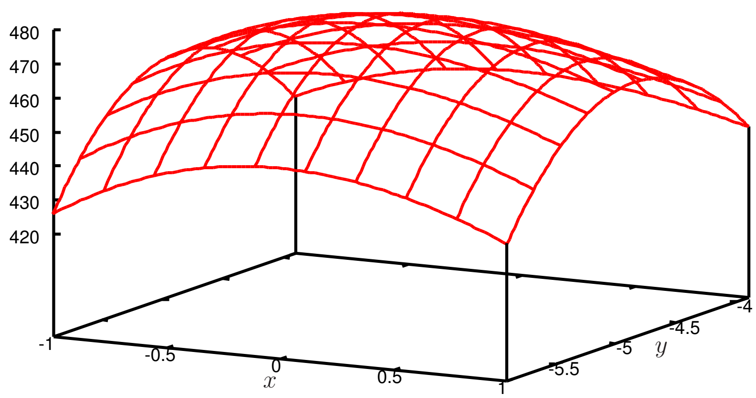
$$\left.\frac{d\{f(x,y)\}}{dx}\right|_{x=a,y=b} = 0, \qquad \left.\frac{d\{f(x,y)\}}{dy}\right|_{x=a,y=b} = 0 \tag{45}$$

This does NOT mean that if $\frac{d\{f(a,b)\}}{dx} = 0$,$\frac{d\{f(a,b)\}}{dy} = 0$, then $f(a,b)$ is a local minimum or a local maximum.
When $\frac{d\{f(a,b)\}}{dx} = 0$,$\frac{d\{f(a,b)\}}{dy} = 0$ is satisfied;

a) $f(a,b)$ is the local maximum when

$$\frac{d^2 f(a,b)}{dx^2}\frac{\partial^2 f(a,b)}{\partial y^2} - \left(\frac{\partial^2 f(a,b)}{\partial y \partial x}\right)^2 > 0$$

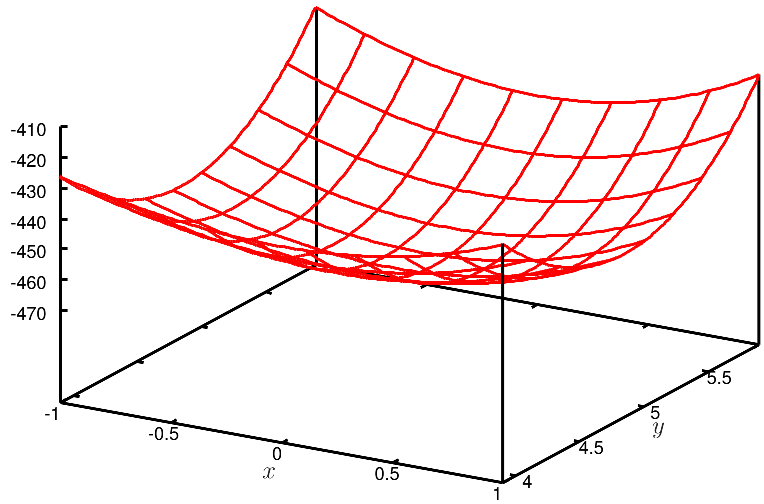
and $\frac{d^2 f(a,b)}{dx^2} < 0$



b) $f(a,b)$ is the local minimum when

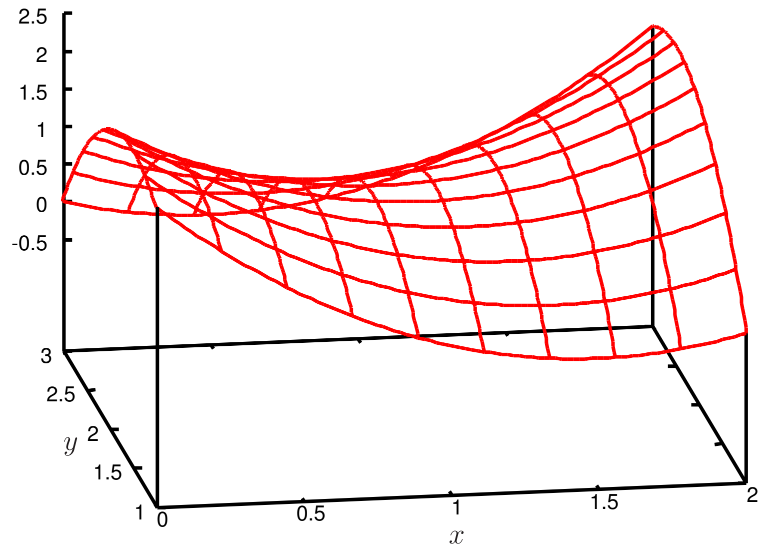
$$\frac{d^2 f(a,b)}{dx^2}\frac{\partial^2 f(a,b)}{\partial y^2} - \left(\frac{\partial^2 f(a,b)}{\partial y \partial x}\right)^2 > 0$$

and $\frac{d^2 f(a,b)}{dx^2} > 0$



c) $f(a,b)$ is a saddle point when

$$\frac{d^2 f(a,b)}{dx^2}\frac{\partial^2 f(a,b)}{\partial y^2} - \left(\frac{\partial^2 f(a,b)}{\partial y \partial x}\right)^2 < 0$$



d) We do not know whether or not $f(a,b)$ is a local maximum or minimum when

$$\frac{d^2 f(a,b)}{dx^2}\frac{\partial^2 f(a,b)}{\partial y^2} - \left(\frac{\partial^2 f(a,b)}{\partial y \partial x}\right)^2 = 0$$

Attention: $\frac{\partial^2 f}{\partial y \partial x}$ is different from $\frac{d\{f\}}{dx} \cdot \frac{d\{f\}}{dy}$.

Basic derivative:

$$\frac{d\{x^\alpha\}}{dx} = \alpha x^{\alpha-1} \quad (46)$$

Attention: When you see a fraction, get rid of a fraction such as $\frac{1}{x^a}$ immediately by changing it to x^{-a}.

$$\frac{d\{x^a\}}{dx} = a \cdot x^{a-1} \quad (47)$$

$$\frac{d\{\mathfrak{e}^{kx}\}}{dx} = k\mathfrak{e}^{kx} \quad (48)$$

$$\frac{d\{\ln(kx)\}}{dx} = \frac{1}{x} \quad (49)$$

$$\frac{d\{\log_a(kx)\}}{dx} = \frac{1}{x \ln a} \quad (50)$$

$$\frac{d\{a^x\}}{dx} = a^x \ln a \quad (51)$$

$$\frac{d\{\sin kx\}}{dx} = k\cos kx \tag{52}$$

$$\frac{d\{\cos kx\}}{dx} = -k\sin kx \tag{53}$$

$$\frac{d\{\tan kx\}}{dx} = \frac{k}{\cos^2 kx} \tag{54}$$

VI. KEY POINTS ON INTEGRATION

Key points

1) Integral by Parts

$$\int_a^b f(x) \cdot g(x)dx \tag{55}$$
$$= \left[f(x) \cdot \int g(x)dx\right]_a^b - \int_a^b \left(\frac{d\{f(x)\}}{dx} \cdot \int g(x)dx\right)dx$$

Hint: Let $f(x)$ equate the polynomial part or logarithmic part of the intergral.

$\int \sin^n xdx$ and $\int \cos^n xdx$ can be obtained using "Integral by Parts" in order to reduce the power as follows

$$\begin{aligned}
\int \cos^n xdx &= \int \cos^{n-1} x \cdot \cos xdx \\
&= f(x) \cdot \int g(x)dx - \int \left(\frac{d\{f(x)\}}{dx} \cdot \int g(x)dx\right)dx \\
&= \cos^{n-1} x \cdot \int \cos xdx - \int \left(\frac{d\{\cos^{n-1} x\}}{dx} \cdot \int \cos xdx\right)dx \\
&= \cos^{n-1} x \cdot \sin x - \int \left((n-1)\cos^{n-2} x(-\sin x) \cdot \sin x\right)dx \\
&= \cos^{n-1} x \cdot \sin x + (n-1)\int \left(\cos^{n-2} x \cdot \sin^2 x\right)dx \\
&= \cos^{n-1} x \cdot \sin x + (n-1)\int \left(\cos^{n-2} x \cdot (1-\cos^2 x)\right)dx \\
&= \cos^{n-1} x \cdot \sin x + (n-1)\int \left(\cos^{n-2} x - \cos^n x\right)dx \\
&= \cos^{n-1} x \cdot \sin x + (n-1)\int \cos^{n-2} xdx - (n-1)\int \cos^n xdx \\
\therefore \int \cos^n xdx + (n-1)\int \cos^n xdx &= \cos^{n-1} x \cdot \sin x + (n-1)\int \cos^{n-2} xdx \\
\therefore n\int \cos^n xdx &= \cos^{n-1} x \cdot \sin x + (n-1)\int \cos^{n-2} xdx
\end{aligned}$$

$$\begin{aligned}
\int \sin^n xdx = \int \sin^{n-1} x \cdot \sin xdx &= f(x) \cdot \int g(x)dx - \int \left(\frac{d\{f(x)\}}{dx} \cdot \int g(x)dx\right)dx \\
&= \sin^{n-1} x \cdot \int \sin xdx - \int \left(\frac{d\{\sin^{n-1} x\}}{dx} \cdot \int \sin xdx\right)dx \\
&= \sin^{n-1} x \cdot (-\cos x) - \int \left((n-1)\sin^{n-2} x(\cos x) \cdot (-\cos x)\right)dx \\
&= -\sin^{n-1} x \cdot \cos x + \int \left((n-1)\sin^{n-2} x(\cos^2 x)\right)dx \\
&= -\sin^{n-1} x \cdot \cos x + (n-1)\int \left(\sin^{n-2} x(1-\sin^2 x)\right)dx \\
&= -\sin^{n-1} x \cdot \cos x + (n-1)\int \left(\sin^{n-2} x - \sin^n x\right)dx \\
&= -\sin^{n-1} x \cdot \cos x + (n-1)\int \left(\sin^{n-2} x\right)dx - (n-1)\int (\sin^n x)\,dx \\
\therefore \int \sin^n xdx + (n-1)\int (\sin^n x)\,dx &= -\sin^{n-1} x \cdot \cos x + (n-1)\int \left(\sin^{n-2} x\right)dx \\
\therefore n\int (\sin^n x)\,dx &= -\sin^{n-1} x \cdot \cos x + (n-1)\int \left(\sin^{n-2} x\right)dx
\end{aligned}$$

2) Integral by substitution

When a function $f(x)$ can be written as $h(g(x))\frac{d\{g(x)\}}{dx}$, you can let $t = g(x)$ therefore, $\frac{d\{t\}}{dx} = \frac{d\{g(x)\}}{dx}$.

$$\int f(x)dx = \int h(g(x))\frac{d\{g(x)\}}{dx}dx \tag{56}$$
$$= \int h(t)\frac{d\{t\}}{dx}dx = \int h(t)dt$$

- For $\int \sin^{2m+1} x dx$, set $t = \cos x$.
- For $\int \sin^{2m} x dx$, set $t = \sin x$.
- For $\int \cos^{2m+1} x dx$, set $t = \sin x$.
- For $\int \cos^{2m} x dx$, set $t = \cos x$.

where m is an integer. But in case of even power such as $2m$, it is better to decrease the power such as

$$\begin{aligned}\sin^4 x &= \left(\sin^2 x\right)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1-2\cos 2x+\cos^2 2x}{4} = \frac{1-2\cos 2x}{4} + \frac{1}{4}\cos^2 2x \\ &= \frac{1-2\cos 2x}{4} + \frac{1}{4}\frac{1+\cos 4x}{2} = \frac{2-4\cos 2x}{8} + \frac{1+\cos 4x}{8} = \frac{3-4\cos 2x+\cos 4x}{8}\end{aligned}$$

If the power is higher than 4, then use "Integral by Parts" as shown above.

When we carry out $\int_{x_\mathrm{L}}^{x_\mathrm{H}} f(x)dx$, the procedure of 'integral by substitution' is as follows

a) set the new variable θ for substitution such as $x = \dfrac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2}$

b) find the relationship between dx and $d\theta$ such as $dx = \dfrac{\mathfrak{e}^{\theta} + \mathfrak{e}^{-\theta}}{2} d\theta$

c) find the range for the new variable θ

$$\begin{aligned} x_\mathrm{L} &= \frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2} \rightarrow \theta_\mathrm{L} = \ln(x_\mathrm{L} + \sqrt{x_\mathrm{L}^2 + 1}) \\ x_\mathrm{H} &= \frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2} \rightarrow \theta_\mathrm{H} = \ln(x_\mathrm{H} + \sqrt{x_\mathrm{H}^2 + 1}) \end{aligned}$$

d) manipulate the original function $f(x)$ to remove x . $f(x) \Longrightarrow g(\theta)$

e) calculate the final modified integral such as $\int_{\theta_\mathrm{L}}^{\theta_\mathrm{H}} g(\theta) \dfrac{\mathfrak{e}^{\theta} + \mathfrak{e}^{-\theta}}{2} d\theta$

3) Integral of $f(x)^k \dfrac{d\{f(x)\}}{dx}$ for $k = -1$, *i.e.*, $\int \dfrac{f'(x)}{f(x)} dx$

$$\int \frac{1}{f(x)} \frac{d\{f(x)\}}{dx} dx = \ln|f(x)| + c \tag{57}$$

Proof:

$$\begin{aligned} \frac{d\{\ln|f(x)|\}}{dx} = \frac{d\{\ln|A|\}}{dx} (\because A \triangleq f(x)) = \frac{d\{A\}}{dx}\frac{d\{\ln|A|\}}{dA} = \frac{d\{f(x)\}}{dx}\frac{1}{A} = \frac{f'(x)}{f(x)} \\ \therefore \frac{f'(x)}{f(x)} = \frac{d\{\ln|f(x)|\}}{dx} \\ \therefore \int \frac{f'(x)}{f(x)} dx = \int \frac{d\{\ln|f(x)|\}}{dx} dx = \int \partial(\ln|f(x)|) = \ln|f(x)| \end{aligned}$$

4) Integral of $f(x)^k \dfrac{d\{f(x)\}}{dx}$ for $k \neq -1$

$$\int f(x)^k \cdot \frac{d\{f(x)\}}{dx} dx = \frac{1}{k+1} f(x)^{k+1} + c \tag{58}$$

5) $P(x)$ and $Q(x)$ are the mth and nth order polynomials, respectively.
   - When $m > n$, $\int \dfrac{P(x)}{Q(x)} dx$ can be obtained as follows:

     a) Find the answer of $A(x)$ and the remainer $R(x)$ of $\dfrac{P(x)}{Q(x)}$ which satisfy $P(x) = Q(x)A(x) + R(x)$

     b) Find the answer of C and the remainder of E of $\dfrac{R(x)}{Q'(x)}$ which satisfy $R(x) = C \cdot Q'(x) + E$

     c) $\int \dfrac{P(x)}{Q(x)} dx = \int \left(A(x) + C\dfrac{Q'(x)}{Q(x)} + \dfrac{E}{Q(x)}\right) dx = \int A(x)dx + C\ln|Q(x)| + \int \dfrac{E}{Q(x)} dx$
   - When $m < n$, $\int \dfrac{P(x)}{Q(x)} dx$ can be obtained as follows:

     a) Find the answer of C and the remainder of E of $\dfrac{P(x)}{Q'(x)}$ which satisfy $P(x) = C \cdot Q'(x) + E$

     b) $\int \dfrac{P(x)}{Q(x)} dx = \int \left(C\dfrac{Q'(x)}{Q(x)} + \dfrac{E}{Q(x)}\right) dx = C\ln|Q(x)| + \int \dfrac{E}{Q(x)} dx$

6) Calculation of Area(A), Arc-length (L), Surface area(S), Volume(V)
Example:

$$y = (x-1)^3 + 1 \Longleftrightarrow \qquad x = (y-1)^{\frac{1}{3}} + 1$$
$$\frac{d\{y\}}{dx} = 3(x-1)^2; \quad \frac{d\{x\}}{dy} = \frac{1}{3}(y-1)^{-\frac{2}{3}}$$

Using a parameter t, $y = (x-1)^3 + 1$ can be expressed as

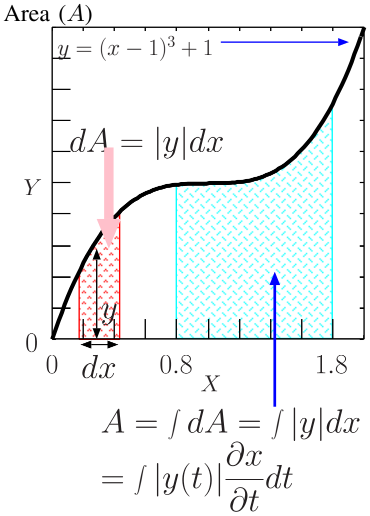
$$x = t + 1$$
$$y = t^3 + 1$$

x	0.8	1.8	y	1	2
t	-0.2	0.8	t	0	1

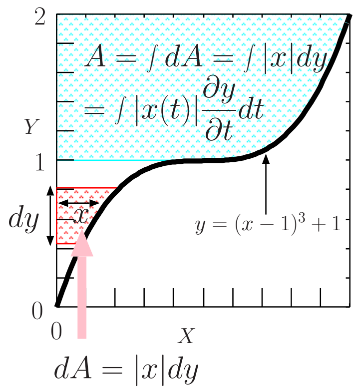
In this case

$$\frac{d\{x\}}{dt} = 1$$
$$\frac{d\{y\}}{dt} = 3t^2$$

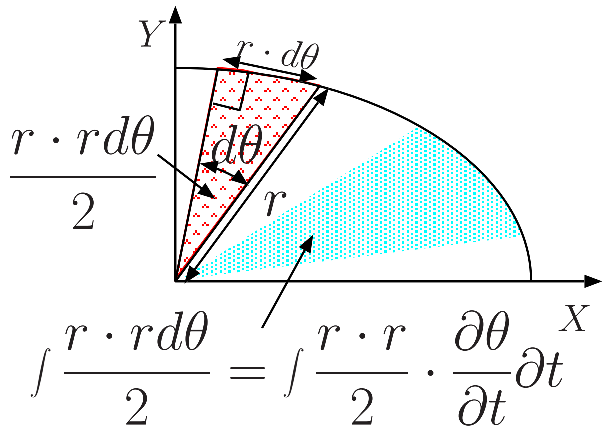
- Area (A)



Area bounded by the X-axis



Area bounded by the Y-axis



Area in polar coordinates

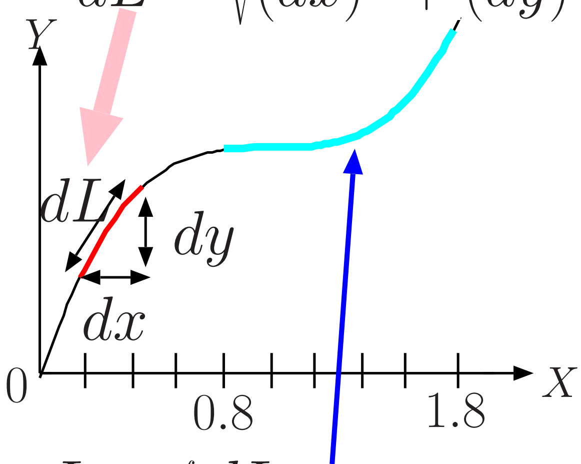
$$A = \int dA = \int_{0.8}^{1.8} ydx$$
$$= \int_{0.8}^{1.8} \left\{(x-1)^3 + 1\right\} dx$$

$$A = \int dA = \int_{1}^{2} xdy$$
$$= \int_{1}^{2} \left\{(y-1)^{\frac{1}{3}} + 1\right\} dy$$

$$A = \int dA$$
$$= \int_{-0.2}^{0.8} y(t)\frac{d\{x\}}{dt}dt$$
$$= \int_{-0.2}^{0.8} \left\{t^3 + 1\right\} \cdot 1 \cdot dt$$
$$= \int_{-0.2}^{0.8} \left\{t^3 + 1\right\} dt$$

$$A = \int dA$$
$$= \int_{0}^{1} x(t)\frac{d\{y\}}{dt}dt$$
$$= \int_{0}^{1} \{t + 1\} \cdot 3t^2 \cdot dt$$
$$= \int_{0}^{1} \left\{3t^3 + 3t^2\right\} dt$$

$$dL = \sqrt{(dx)^2 + (dy)^2}$$



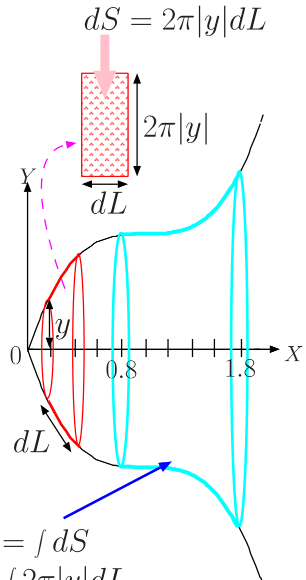
- Arc-length (L)

$$L = \int dL$$
$$= \int \sqrt{(dx)^2 + (dy)^2}$$
$$= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$
$$= \int \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy$$
$$= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$L = \int dL$$
$$= \int_{0.8}^{1.8} \sqrt{1 + \left(\frac{d\{y\}}{dx}\right)^2} dx$$
$$= \int_{0.8}^{1.8} \sqrt{1 + (3(x-1)^2)^2} dx$$
$$= \int_{0.8}^{1.8} \sqrt{1 + 9(x-1)^4} dx$$

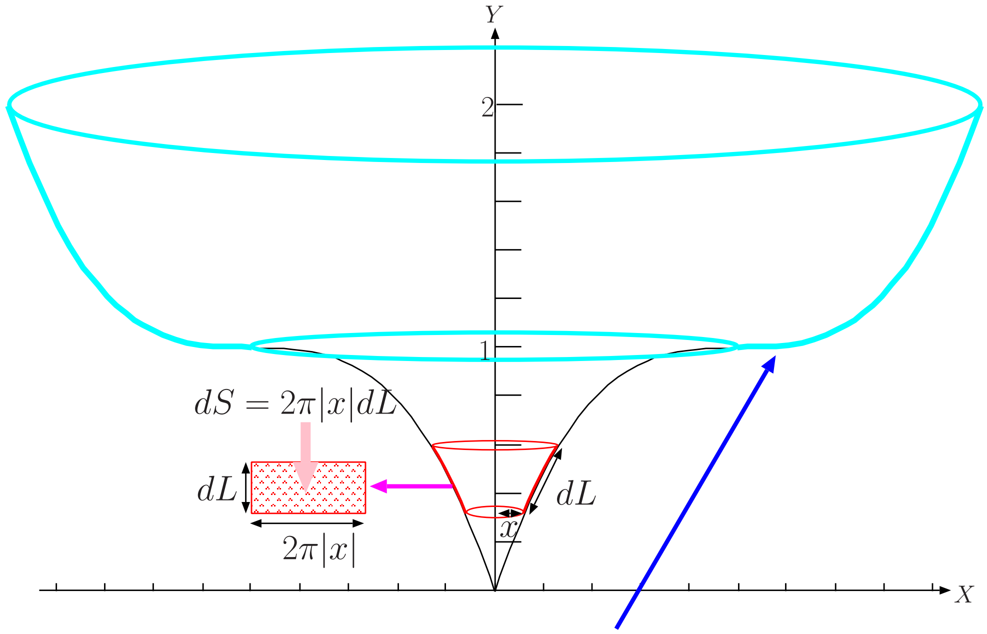
$$L = \int dL$$
$$= \int_{-0.2}^{0.8} \sqrt{\left(\frac{d\{x\}}{dt}\right)^2 + \left(\frac{d\{y\}}{dt}\right)^2} dt$$
$$= \int_{-0.2}^{0.8} \sqrt{(1)^2 + (3t^2)^2} dt$$
$$= \int_{-0.2}^{0.8} \sqrt{1 + 9t^4} dt$$

- Surface area (S) of solid of revolution



$S = \int dS$
$= \int 2\pi|y|dL$
$= \int 2\pi|y|\sqrt{(dx)^2 + (dy)^2}$
$= \int 2\pi|y|\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx$
$= \int 2\pi|y(t)|\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$

Rotation about the X-axis



$S = \int dS$
$= \int 2\pi|x|dL$
$= \int 2\pi|x|\sqrt{(dx)^2 + (dy)^2}$
$= \int 2\pi|x|\sqrt{\left(\frac{dx}{dy}\right)^2 + 1dy}$
$= \int 2\pi|x(t)|\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$

Rotation about the Y-axis

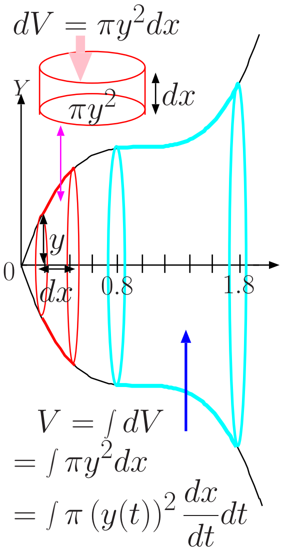
$$S = \int dS = \int_{0.8}^{1.8} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)}dx$$
$$= \int_{0.8}^{1.8} 2\pi\left\{(x-1)^3 + 1\right\}\sqrt{1 + 3(x-1)^2}dx$$

$$S = \int dS = \int_{1}^{2} 2\pi x\sqrt{\left(\frac{dx}{dy}\right) + 1dy}$$
$$= \int_{1}^{2} 2\pi\left\{(y-1)^{\frac{1}{3}} + 1\right\}\sqrt{\frac{1}{3}(y-1)^{-\frac{2}{3}} + 1dy}$$

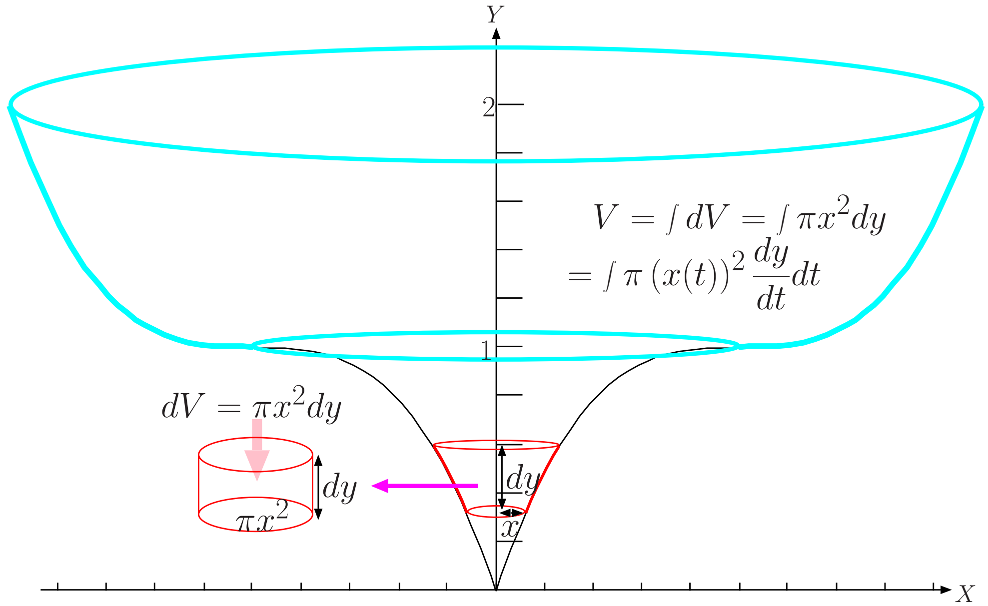
$$S = \int dS =$$
$$\int_{-0.2}^{0.8} 2\pi y(t)\sqrt{\left(\frac{d\{x\}}{dt}\right)^2 + \left(\frac{d\{y\}}{dt}\right)^2}dt$$
$$= \int_{-0.2}^{0.8} 2\pi(t^3 + 1)\sqrt{(1)^2 + (3t^2)^2}dt$$
$$= \int_{-0.2}^{0.8} 2\pi(t^3 + 1)\sqrt{1 + 9t^4}dt$$

$$S = \int dS =$$
$$\int_{0}^{1} 2\pi x(t)\sqrt{\left(\frac{d\{x\}}{dt}\right)^2 + \left(\frac{d\{y\}}{dt}\right)^2}dt$$
$$= \int_{0}^{1} 2\pi(t + 1)\sqrt{(1)^2 + (3t^2)^2}dt$$
$$= \int_{0}^{1} 2\pi(t + 1)\sqrt{1 + 9t^4}dt$$

- Volume (V) of solid of revolution



Rotation about the X-axis



Rotation about the Y-axis

$$V = \int dV = \int_{0.8}^{1.8} \pi y^2 dx$$
$$= \int_{0.8}^{1.8} \pi \left\{(x-1)^3 + 1\right\}^2 dx$$

$$V = \int dV = \int_{1}^{2} \pi x^2 dy$$
$$= \int_{1}^{2} \pi \left\{(y-1)^{\frac{1}{3}} + 1\right\}^2 dy$$

$$V = \int dV =$$
$$\int_{-0.2}^{0.8} \pi (y(t))^2 \frac{d\{x\}}{dt} dt$$
$$= \int_{-0.2}^{0.8} \pi \left(t^3 + 1\right)^2 \cdot 1 \cdot dt$$
$$= \int_{-0.2}^{0.8} \pi \left(t^3 + 1\right)^2 dt$$

$$V = \int dV =$$
$$\int_{0}^{1} \pi (x(t))^2 \frac{d\{y\}}{dt} dt$$
$$= \int_{0}^{1} \pi (t+1)^2 \cdot 3t^2 \cdot dt$$
$$= \int_{0}^{1} 3\pi \left(t^2 + t\right)^2 \cdot dt$$

7) Line integrals of a function which has dx,dy, and dz such as $I = \int_C (F_x dx + F_y dy + F_z dz)$.
Consider a curve C. The position vector of a point on the curve C is written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$
$$a \le t \le b$$

Denote

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and its derivative with respect to t as

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \\ \frac{d\{z\}}{dt} \end{pmatrix}.$$

When a vector function is expressed as

$$\mathbf{F}(\boldsymbol{r}) = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

a line integral of $\mathbf{F}(\boldsymbol{r})$ over a curve C is defined by

$$\int_C \mathbf{F}\cdot d\boldsymbol{r} = \int_{t=a}^{t=b} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \cdot \frac{d\{\boldsymbol{r}\}}{dt} dt$$

$$= \int_{t=a}^{t=b} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \cdot \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \\ \frac{d\{z\}}{dt} \end{pmatrix} dt$$

$$= \int_{t=a}^{t=b} (F_x \frac{d\{x\}}{dt} + F_y \frac{d\{y\}}{dt} + F_z \frac{d\{z\}}{dt}) dt \tag{59}$$

$$= \int (F_x dx + F_y dy + F_z dz) \tag{60}$$

$$= \int_{x=\hat{a}}^{x=\hat{b}} (F_x + F_y \frac{dy}{dx} + F_z \frac{dz}{dx}) dx \tag{61}$$

Thus the procedure to solve the line integral is

a) Express x, y, z on the curve C using t and set the range of t
b) Express $\mathbf{F}$ as the function of t
c) Express $\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \\ \frac{d\{z\}}{dt} \end{pmatrix}$ using t
d) Put all of them into $\int \mathbf{F} \cdot \frac{d\{\boldsymbol{r}\}}{dt} dt$

8) Line integrals of a function (which does not have dx, dy or dz explicitly) with respect to arc length such as $\int_C f(x, y, z) ds$.
Consider a curve C. The position vector of a point on the curve C is written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$
$$a \le t \le b$$

Denoting

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and its derivative with respect to t as

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \\ \frac{d\{z\}}{dt} \end{pmatrix}$$

the line integral of a function with respect to arc length is defined by

$$\int_C f(x, y, z) ds$$
$$= \int_{t=a}^{t=b} f(x, y, z) \sqrt{\left(\frac{d\{x\}}{dt}\right)^2 + \left(\frac{d\{y\}}{dt}\right)^2 + \left(\frac{d\{z\}}{dt}\right)^2} dt \tag{62}$$

where

$$ds = \sqrt{\left(\frac{d\{x\}}{dt}\right)^2 + \left(\frac{d\{y\}}{dt}\right)^2 + \left(\frac{d\{z\}}{dt}\right)^2} dt$$

The procedure to solve this type of the line integral is

a) Express x, y, z on the curve C using t and set the range of t
b) Express $f(x, y, z)$ as the function of t
c) Express $\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \\ \frac{d\{z\}}{dt} \end{pmatrix}$ using t

d) Put all of them into

$$\int_{t=a}^{t=b} f(x,y,z)\sqrt{\left(\frac{d\{x\}}{dt}\right)^2+\left(\frac{d\{y\}}{dt}\right)^2+\left(\frac{d\{z\}}{dt}\right)^2}dt$$

9) Multiple integration

$$I=\int_a^b\int_c^d\int_e^f f(x,y,z)dxdydz$$

has the following range:

$$\begin{array}{c} e\le x\le f \\ c\le y\le d \\ a\le z\le b \end{array}$$

The procedure for the calculation is

a)

$$A=\int_e^f f(x,y,z)dx$$

b)

$$B=\int_a^b\int_c^d Ady$$

c)

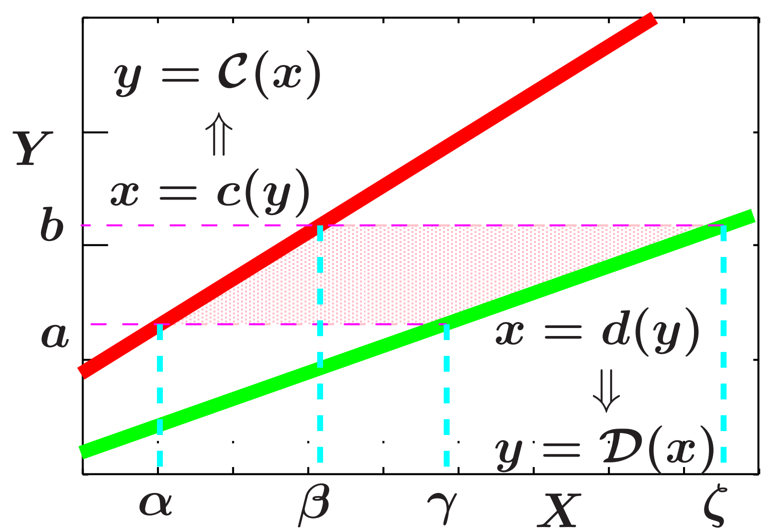
$$I=\int_a^b Bdz$$

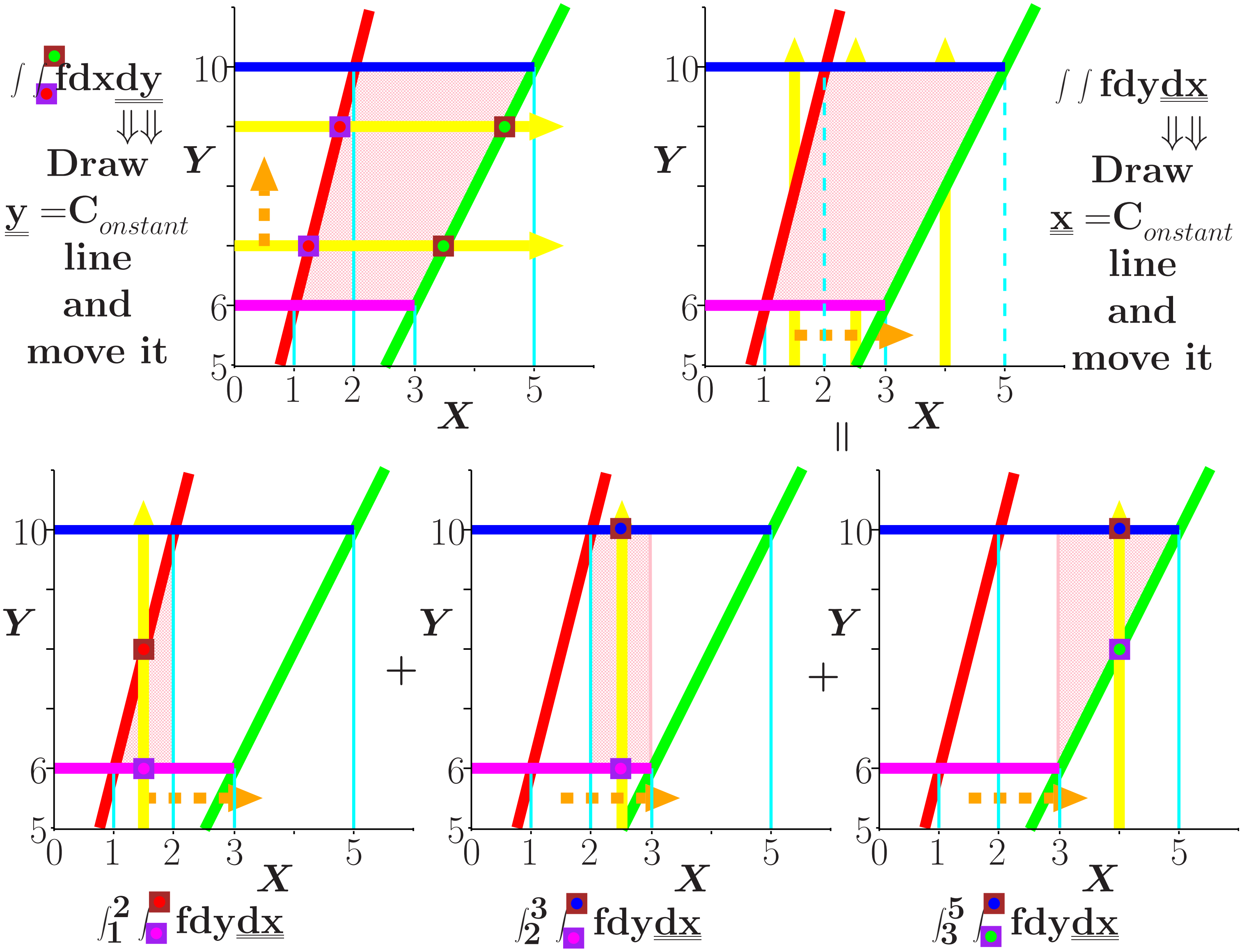
Please be aware that

$$\int_a^b\int_c^d\int_e^f fdxdydz \ne \int_a^b fdx\times\int_c^d fdy\times\int_e^f fdz$$

10) Reversing the order of multiple integration

- $$I=\int_a^b\int_{c(y)}^{d(y)} f(x,y)dxdy \qquad I=\int_6^{10}\int_{\frac{y}{4}-\frac{1}{2}}^{\frac{y}{2}} f(x,y)dxdy \text{ where } c(y)=\frac{y}{4}-\frac{1}{2}, \qquad d(y)=\frac{y}{2}$$





a) Find the original range of integration

$$c(y) \le x \le d(y) \qquad \frac{y}{4} - \frac{1}{2} \le x \le \frac{y}{2}$$
$$a \le y \le b \qquad 6 \le y \le 10$$

b) Sketch the range of integration

c) If necessary, manipulate the equation of $x = c(y)$ to make y the subject of the equation such as $y = \mathcal{C}(x)$. For example $x = \frac{y}{4} - \frac{1}{2}$ is changed to $y = 4x + 2$ and $x = \frac{y}{2}$ is changed to $y = 2x$

d) Find out the range of y when x is fixed such as

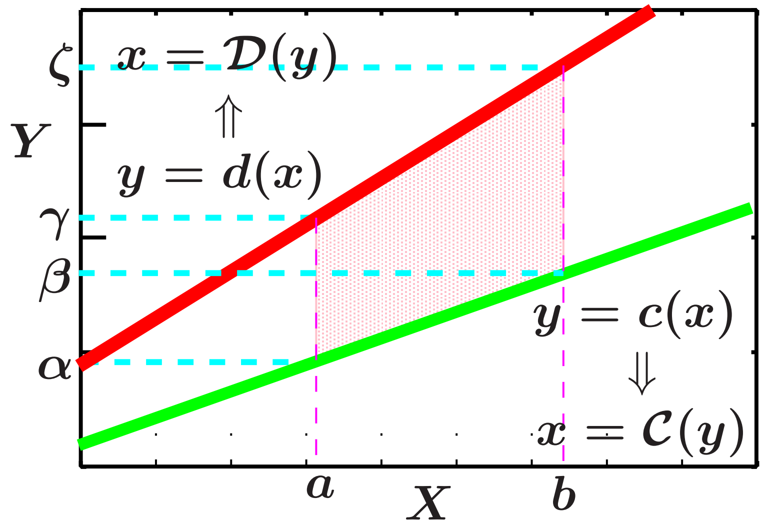
$$\begin{array}{lll} a \le y \le \mathcal{C}(x) & \text{for} & \alpha \le x \le \beta \\ a \le y \le b & \text{for} & \beta \le x \le \gamma \\ \mathcal{D}(x) \le y \le b & \text{for} & \gamma \le x \le \zeta \end{array} \qquad \begin{array}{lll} 6 \le y \le 4x + 2 & \text{for} & 1 \le x \le 2 \\ 6 \le y \le 10 & \text{for} & 2 \le x \le 3 \\ 2x \le y \le 10 & \text{for} & 3 \le x \le 5 \end{array}$$

e) Rewrite the integral such as

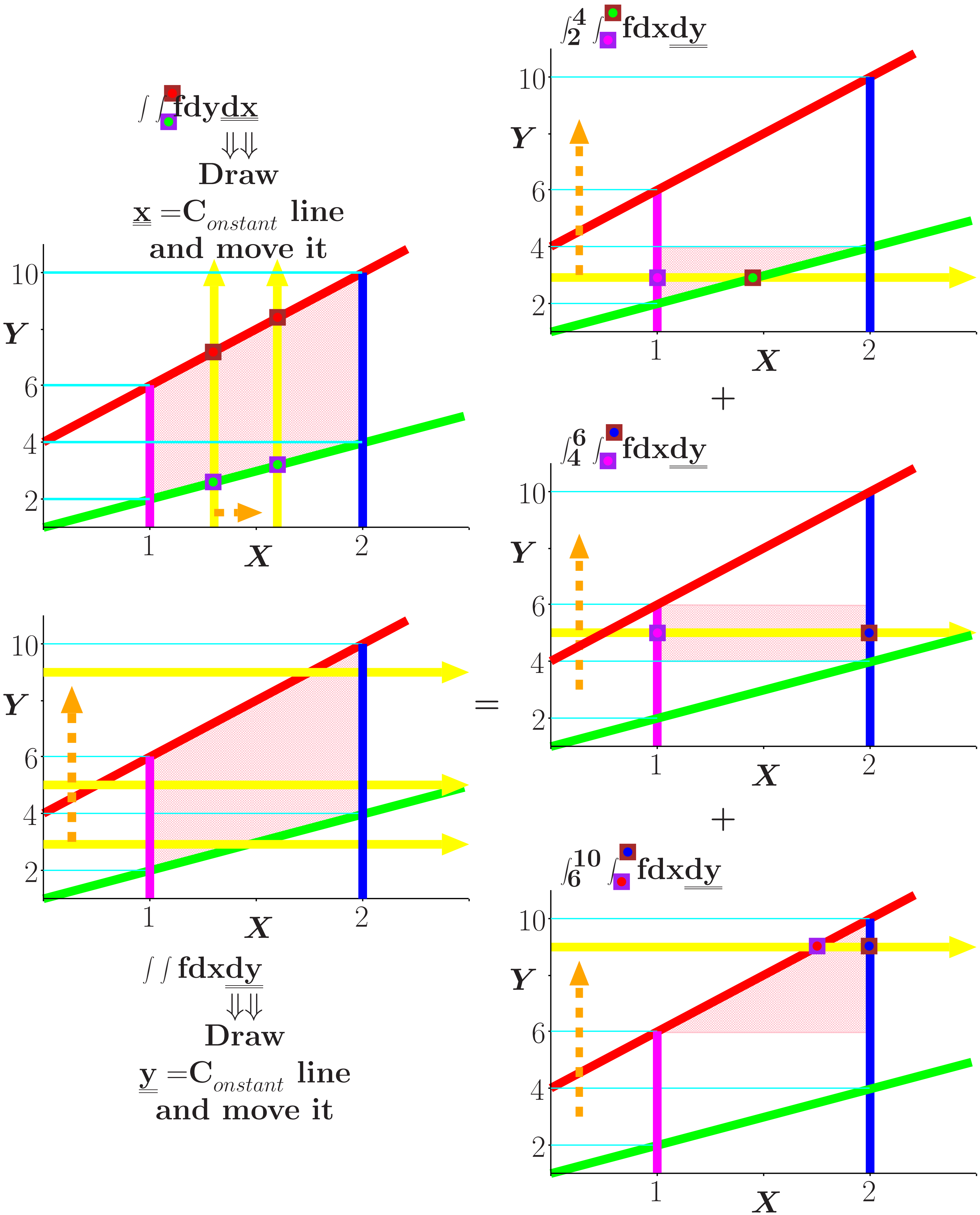
$$I = \int_\alpha^\beta \int_a^{\mathcal{C}(x)} f(x,y)dydx + \int_\beta^\gamma \int_a^b f(x,y)dydx + \int_\gamma^\zeta \int_{\mathcal{D}(x)}^b f(x,y)dydx$$
$$I = \int_1^2 \int_6^{4x+2} f(x,y)dydx + \int_2^3 \int_6^{10} f(x,y)dydx + \int_3^5 \int_{2x}^{10} f(x,y)dydx$$

•

$$I = \int_a^b \int_{c(x)}^{d(x)} f(x,y)dydx \qquad I = \int_1^2 \int_{2x}^{4x+2} f(x,y)dydx \text{ where } c(x) = 2x, \qquad d(x) = 4x + 2$$



ζ
$x = \mathcal{D}(y)$
$\Uparrow$
Y
$y = d(x)$
γ
β
$y = c(x)$
$\Downarrow$
α
$x = \mathcal{C}(y)$
a
X
b


∫ ∫ fdydx
⇓⇓
Draw
x =Constant line
and move it
∫ ∫ fdxdy
⇓⇓
Draw
y =Constant line
and move it
∫ 2 4 ∫ fdxdy
+
∫ 4 6 ∫ fdxdy
+
∫ 6 10 ∫ fdxdy
=
Y
X
10
6
4
2
1
2

a) Find the original range of integration

$$\begin{aligned} c(x) \le y \le d(x) \qquad & 2x \le y \le 4x+2 \\ a \le x \le b \qquad & 1 \le x \le 2 \end{aligned}$$

b) Sketch the range of integration

c) If necessary, manipulate the question of $y = c(x)$ to make x the subject of the equation such as $x = \mathcal{C}(y)$. For example $y = 2x$ is changed to $x = \dfrac{y}{2}$ and $y = 4x + 2$ is changed to $x = \dfrac{y}{4} - \dfrac{1}{2}$.

d) Find out the range of x when y is constant such as

$$\begin{aligned} a \le x \le \mathcal{C}(y) \text{ for } \alpha \le y \le \beta \qquad & 1 \le x \le \tfrac{y}{2} \text{ for } 2 \le y \le 4 \\ a \le x \le b \text{ for } \beta \le y \le \gamma \qquad & 1 \le x \le 2 \text{ for } 4 \le y \le 6 \\ \mathcal{D}(y) \le x \le b \text{ for } \gamma \le y \le \zeta \qquad & \tfrac{y}{4} - \tfrac{1}{2} \le x \le 2 \text{ for } 6 \le y \le 10 \end{aligned}$$

e) Rewrite the integral such as

$$I = \int_{\alpha}^{\beta}\int_{a}^{\mathcal{C}(y)} f(x,y)dxdy + \int_{\beta}^{\gamma}\int_{a}^{b} f(x,y)dxdy + \int_{\gamma}^{\zeta}\int_{\mathcal{D}(y)}^{b} f(x,y)dxdy$$

$$I = \int_{2}^{4}\int_{1}^{\frac{y}{2}} f(x,y)dxdy + \int_{4}^{6}\int_{1}^{2} f(x,y)dxdy + \int_{6}^{10}\int_{\frac{y}{4}-\frac{1}{2}}^{2} f(x,y)dxdy$$

11) Implicit multiple integration $I = \iint_D f(x,y)dA$

a) Sketch the integral region

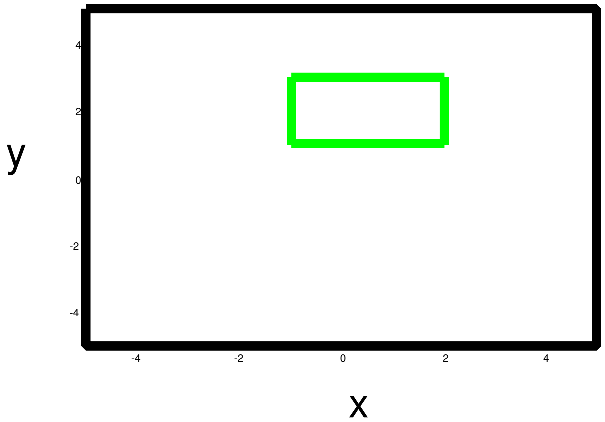
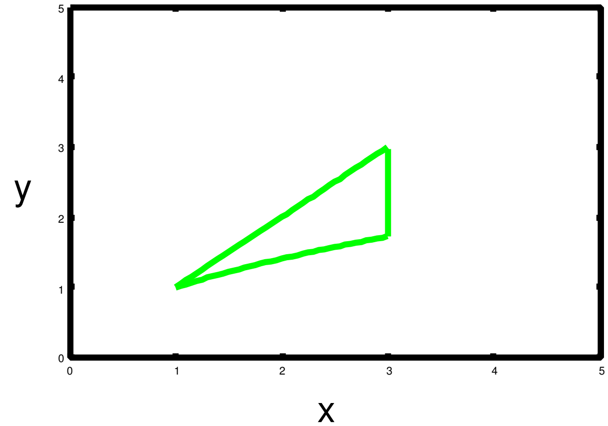
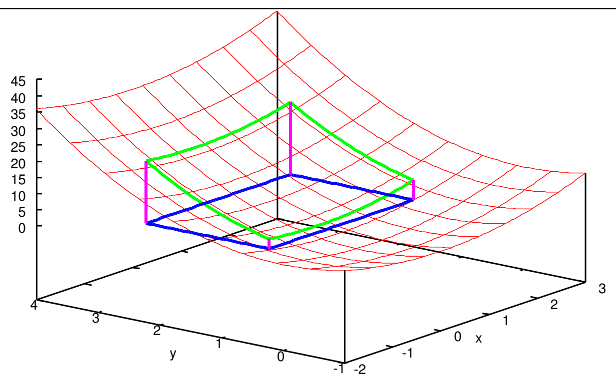
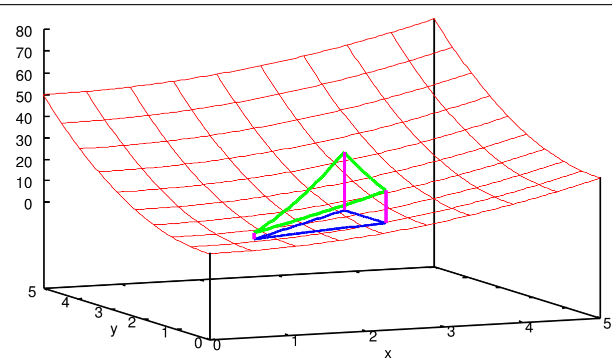
b) Set the range for x and y

The range of either x or y should be fixed without any variables.

c) Set the variable with fixed limits as the second integral variable and the other one as the first integral variable.

d) Do the first integral

e) Do the second integral

Function given	$I=\iint_D x^2+2y^2dA$	$I=\iint_D x^2+2y^2dA$
Ingegral region given	D is the region bounded by $-1\le x\le 2$ and $1\le y\le 3$.	D is the region bounded by $1\le x\le 3$ and $\sqrt{x}\le y\le x$.
Sketch the integral region		
Set the range of x and y	$-1\le x\le 2, 1\le y\le 3$	$1\le x\le 3$, $\sqrt{x}\le y\le x$
Set Set the variable with fixed limits as the <u>second</u> integral variable and the other one as the first integral variable.	$I=\int_1^3\int_{-1}^2 x^2+2y^2dxdy=\int_{-1}^2\int_1^3 x^2+2y^2dydx$	$I=\int_1^3\int_{\sqrt{x}}^x x^2+2y^2dydx$
Do the first integral	$I=\int_1^3\int_{-1}^2 x^2+2y^2dxdy$ $=\int_1^3[\frac{x^3}{3}+2y^2x]_{-1}^2dy$ $=\int_1^3[\frac{8}{3}+4y^2-\frac{1}{3}+2y^2]dy$ $=\int_1^3[\frac{7}{3}+6y^2]dy$	$I=\int_1^3\int_{\sqrt{x}}^x x^2+2y^2dydx$ $=\int_1^3[x^2y+\frac{2}{3}y^3]_{\sqrt{x}}^x dx$ $=\int_1^3[x^3+\frac{2}{3}x^3-x^{2.5}-\frac{2}{3}x^{1.5}]dx$ $=\int_1^3[\frac{5}{3}x^3-x^{2.5}-\frac{2}{3}x^{1.5}]dx$
Do the second integral	$I=\int_1^3[\frac{7}{3}+6y^2]dy$ $=[\frac{7}{3}y+2y^3]_1^3$ $=7+54-\frac{7}{3}-2=\frac{170}{3}$	$I=\int_1^3[\frac{5}{3}x^3-x^{2.5}-\frac{2}{3}x^{1.5}]dx$ $=[\frac{5}{12}x^4-\frac{1}{3.5}x^{3.5}-\frac{2}{7.5}x^{2.5}]_1^3$ $=\frac{80\cdot 5}{12}+\frac{1-3^{3.5}}{3.5}+2\frac{1-3^{2.5}}{7.5}$
What you got is the volume ; the red curve is $z=x^2+2y^2$		

Integrals of common functions.

Some are very similar to the fundamental functions for differentiation. So please do not mix up!, especially signs such as $+$ or $-$.

$$n \neq -1 \quad \text{and} \quad \int kx^n dx = \frac{1}{n+1} \cdot kx^{n+1} + c \tag{63}$$

$$n = -1 \quad \text{and} \quad \int kx^n dx = \int \frac{k}{x} dx = k \ln|x| + c \tag{64}$$

$$\int \cos kx dx = \frac{1}{k} \sin kx + c \tag{65}$$

$$\int \sin kx dx = -\frac{1}{k} \cos kx + c \tag{66}$$

$$\int \tan kx dx = -\frac{1}{k} \ln|\cos kx| + c \tag{67}$$

$$\int \mathfrak{e}^{kx} dx = \frac{1}{k} \mathfrak{e}^{kx} + c \tag{68}$$

$$\int a^{kx} dx = \frac{a^{kx}}{k \ln a} + c (a > 0) \tag{69}$$

$$\int \cos^2(kx) dx = \frac{1}{2k}(kx + \sin(kx)\cos(kx)) \tag{70}$$

$$\int \frac{1}{\cos^2(kx)} dx = \frac{\tan kx}{k} \tag{71}$$

$$\int \frac{1}{\sin^2(kx)} dx = -\frac{1}{k \tan kx} \tag{72}$$

$$\int \sin^2(kx) dx = \frac{1}{2k}(kx - \sin(kx)\cos(kx)) \tag{73}$$

$$\int \ln kx dx = x \ln kx - x \tag{74}$$

$$\int \frac{dx}{\sqrt{x^2 - k^2}} = \cosh^{-1}\left(\frac{x}{k}\right) \tag{75}$$

$$\int \frac{dx}{\sqrt{x^2 + k^2}} = \sinh^{-1}\left(\frac{x}{k}\right) \tag{76}$$

$$\int \frac{dx}{\sqrt{k^2 - x^2}} = \sin^{-1}\left(\frac{x}{k}\right) \tag{77}$$

$$\int \frac{dx}{x^2 + k^2} = \frac{1}{k} \tan^{-1}\left(\frac{x}{k}\right) \tag{78}$$

Proof of Equation (71)

$$t \triangleq \frac{1}{\tan(kx)} \quad ; \quad \therefore \tan(kx) = \frac{1}{t} \quad ; \quad \therefore \frac{d\left\{\tan(kx)\right\}}{dx} = \frac{d\left\{\frac{1}{t}\right\}}{dx} \quad ; \quad \therefore \frac{k}{\cos^2(kx)} = \frac{d\left\{t\right\}}{dx}\frac{d\left\{\frac{1}{t}\right\}}{dt}$$

$$\therefore \frac{k}{\cos^2(kx)} = -\frac{1}{t^2}\frac{d\left\{t\right\}}{dx} \quad ; \quad \therefore \frac{k}{\cos^2(kx)}dx = -\frac{1}{t^2}dt \quad ; \quad \therefore dx = -\frac{\cos^2(kx)}{kt^2}dt$$

$$\int \frac{1}{\cos^2(kx)}dx = \int \frac{1}{\cos^2(kx)}\left(-\frac{\cos^2(kx)}{kt^2}dt\right) = -\frac{1}{k}\int\left(\frac{1}{t^2}dt\right) = -\frac{1}{k}(-t^{-1}) = \frac{1}{k}\cdot\frac{1}{t} = \frac{1}{k}\cdot\tan(kx) = \frac{\tan(kx)}{k}$$

Proof of Equation (72)

$$t \triangleq \frac{1}{\tan(kx)} \quad ; \quad \therefore \tan(kx) = \frac{1}{t} \quad ; \quad \therefore \frac{d\left\{\tan(kx)\right\}}{dx} = \frac{d\left\{\frac{1}{t}\right\}}{dx} \quad ; \quad \therefore \frac{k}{\cos^2(kx)} = \frac{d\left\{t\right\}}{dx}\frac{d\left\{\frac{1}{t}\right\}}{dt}$$

$$\therefore \frac{k}{\cos^2(kx)} = -\frac{1}{t^2}\frac{d\left\{t\right\}}{dx} \quad ; \quad \therefore \frac{k}{\cos^2(kx)}dx = -\frac{1}{t^2}dt \quad ; \quad \therefore dx = -\frac{\cos^2(kx)}{kt^2}dt \quad ; \quad \int \frac{1}{\cos^2(kx)}dx$$

$$= \int \frac{1}{\sin^2(kx)}\left(-\frac{\cos^2(kx)}{kt^2}dt\right) = -\int \frac{\cos^2(kx)}{\sin^2(kx)}\left(\frac{1}{kt^2}dt\right) = -\int \frac{1}{\tan^2(kx)}\left(\frac{1}{kt^2}dt\right)$$

$$= -\int t^2\left(\frac{1}{kt^2}dt\right) = -\int\left(\frac{1}{k}dt\right) = -\frac{t}{k} = -\frac{1}{k\tan(kx)}$$

VII. Key points on sequences and series

Key points

1) Sequences and Series

   a) Arithmetic progressions. Consider a sequence that starts at r and we add d each time. This forms the Arithemtic series as follows.

$$\begin{aligned} a_1 &= r \\ a_2 &= r + d \\ a_3 &= r + 2d \\ a_4 &= r + 3d \\ &\dots \\ a_n &= r + (n-1)d \end{aligned}$$

   Here d is the difference or common difference between successive terms. The sum of an arthimetic progression is as follows.

$$S_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_n$$
$$S_n = r + (r + d) + (r + 2d) + \cdots + r + (n-1)d$$

$$S_n = rn + \frac{n(n-1)d}{2} \tag{79}$$

   b) Geometric progressions. Suppose we let the first term equal a and times each successive term by r then we get.

$$\begin{aligned} a_1 &= a \\ a_2 &= ar \\ a_3 &= ar^2 \\ a_4 &= ar^3 \\ a_5 &= ar^4 \\ &\dots \\ a_n &= ar^{n-1} \end{aligned}$$

   To find the sum of this progression to n terms, we sum all the terms up until n.

$$S_n = a + ar + ar^2 + ar^3 + ar^4, + \cdots + ar^{n-1}$$

   Since $r \cdot S_n$ is written as

$$rS_n = ar + ar^2 + ar^3 + ar^4, + \cdots + ar^{n-1} + ar^n$$

   Using these two equations, we calculate $S_n - rS_n$ as follows:

$$S_n - rS_n = a - ar^n$$

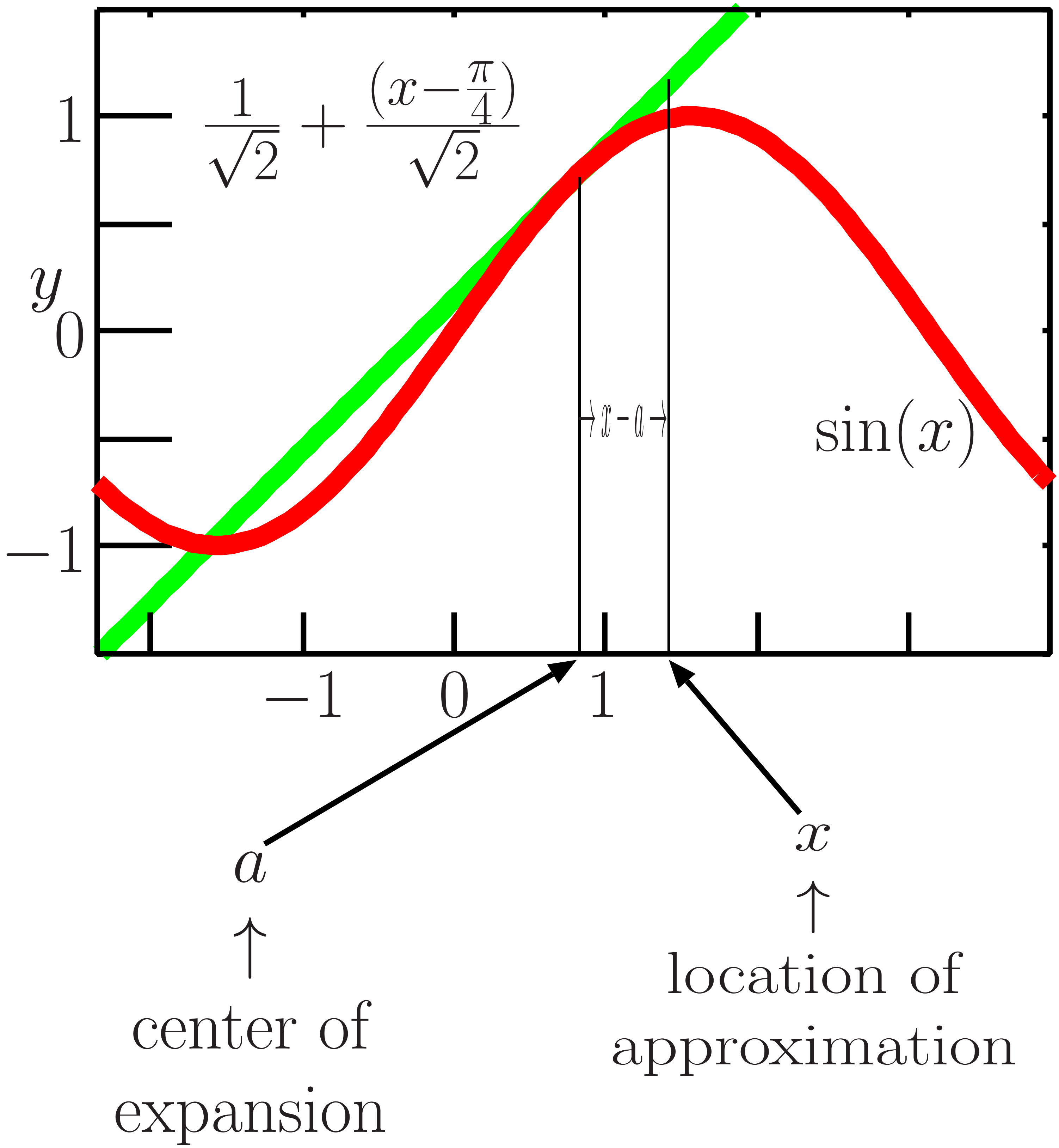
   This leads to :

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r} \tag{80}$$

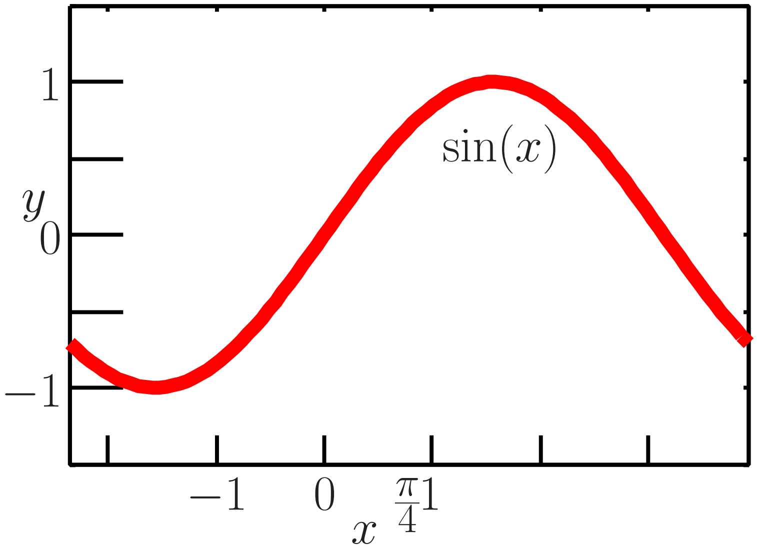
   If $-1 < r < 1$ therefore the sum to infinity of an geomteric series is given by the following

$$S_\infty = \frac{a}{1 - r} \tag{81}$$

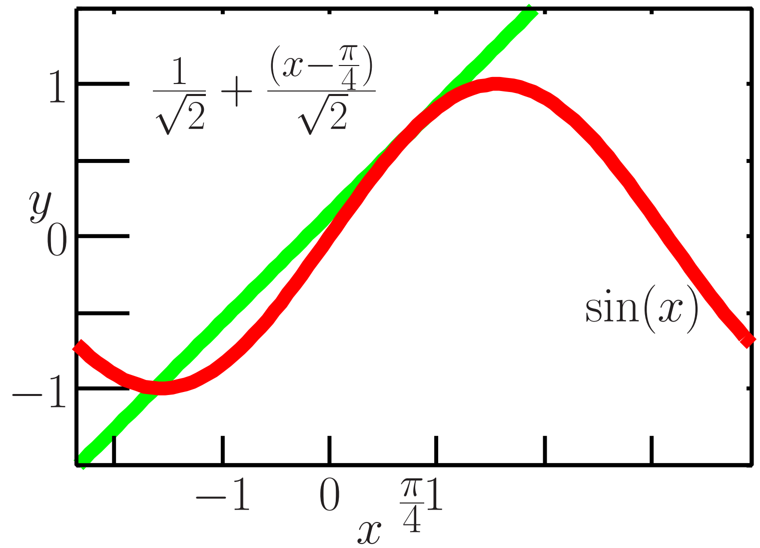
2) Taylor polynomial with one variable,*e.g.*, x. This is the example of one-dimensional Taylor series expansion as there is only one variable in the equation.



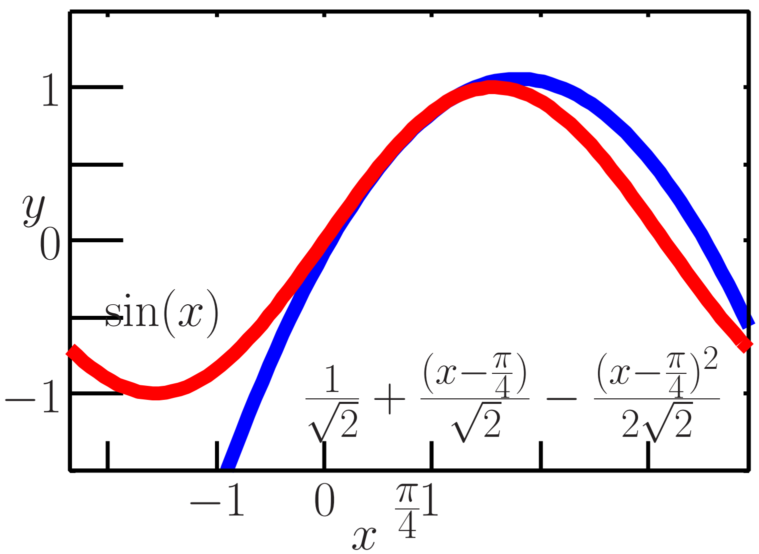
More information at https://www.scss.tcd.ie/Rozenn.Dahyot/CS1BA1/T2007_04_10_CS1BA1.pdf



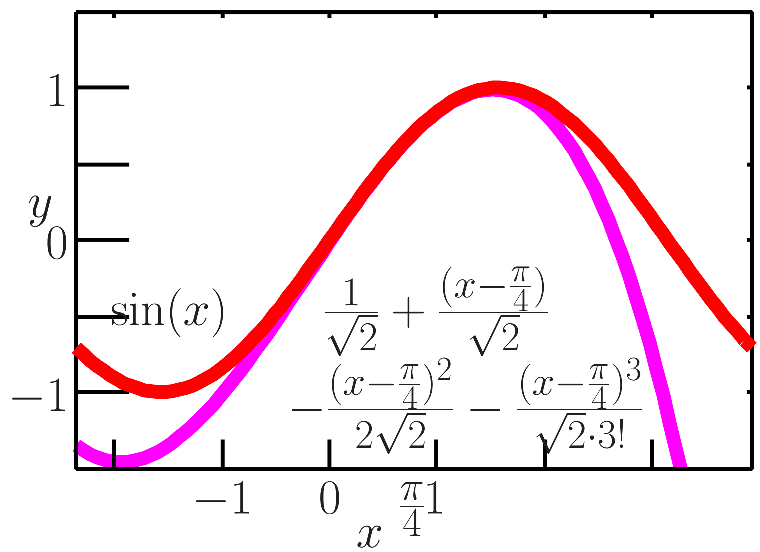
sin(x)



Linear approximation at $x = \frac{\pi}{4}$
degree one



Quadratic approximation at $x = \frac{\pi}{4}$
degree two



Cubic approximation at $x = \frac{\pi}{4}$
degree three

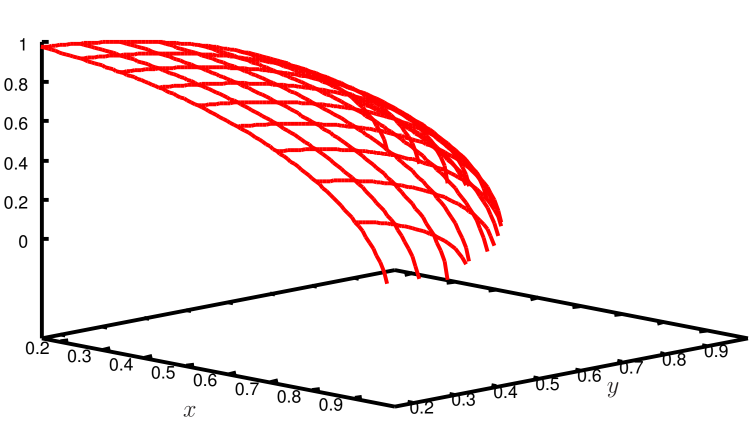
A Taylor series is a series expansion of a function about a point. The Taylor polynomial approximates/expresses the part of the function around $x = a$ using several polynomials.
A one-dimensional Taylor series is an expansion of a real function $f(x)$ about the point at $x = a$ up to degree n ($|x - a| \ll 1$) which is given by

$$f(x) = f(a) + (x-a)\left.\frac{\partial f}{\partial x}\right|_{x=a} + \frac{(x-a)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=a} + \frac{(x-a)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=a} + \cdots + \frac{(x-a)^n}{n!}\left.\frac{\partial^n f}{\partial x^n}\right|_{x=a} \tag{82}$$

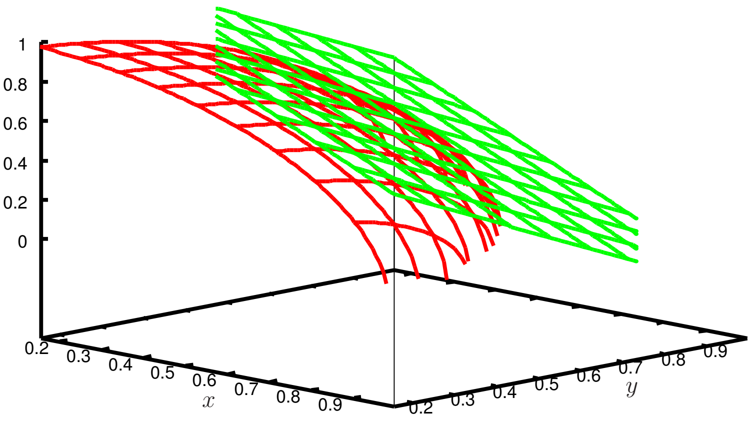
If $a = 0$, the expansion is known as a Maclaurin series.
In the end, in order to obtain the taylor series

a) Obtain $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, . . . $\frac{\partial^n f}{\partial x^n}$
b) Substitute $x = a$ into $f(x)$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, . . . $\frac{\partial^n f}{\partial x^n}$
c) Put all of them into Equation (82).

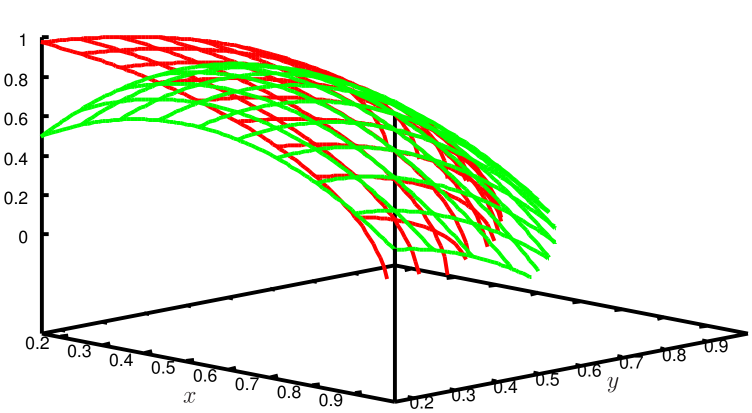
3) Taylor polynomial with two variable,*e.g.*, x and y. This is the example of two-dimensional Taylor series expansion as there are two variables in the equation.



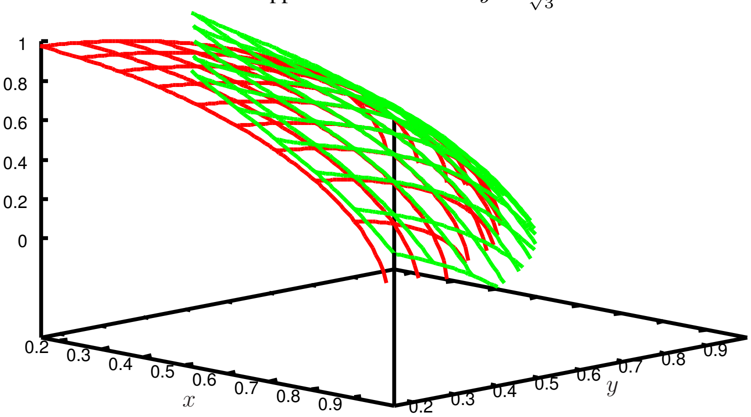
$z = (1 - x^2 - y^2)^{\frac{1}{2}}$



Linear approximation at $x = y = \frac{1}{\sqrt{3}}$



Quadratic approximation



Cubic approximation

The taylor series for two variables is very similar to that of one variable. The same method is used to find the series.
The Taylor polynomial approximates/expresses the part of the function around $(x, y) = (a, b)$ using several polynomials.
The Taylor series expansion about the point at $(x, y) = (a, b)$, where a and b are known constants, up to and including terms of degree three in $x - a$ and $y - b$ ($|x - a| \ll 1$ and $|y - b| \ll 1$) is expressed as

$$
\begin{aligned}
f(x,y) = {} & \\
& f(a,b) + (x-a)\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=a\\y=b}} + (y-b)\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=a\\y=b}} \\
& + \frac{1}{2!}\left[(x-a)^2\left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=a\\y=b}} + 2(x-a)(y-b)\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=a\\y=b}} \right. \\
& \left. + (y-b)^2\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=a\\y=b}}\right] \\
& + \frac{1}{3!}\left[(x-a)^3\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=a\\y=b}} + 3(x-a)^2(y-b)\left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=a\\y=b}} \right. \\
& \left. + 3(x-a)(y-b)^2\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=a\\y=b}} + (y-b)^3\left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=a\\y=b}}\right]
\end{aligned}
\tag{83}
$$

In the end, in order to obtain the taylor series

a) Obtain $\frac{d\{f(x,y)\}}{dx}, \frac{d\{f(x,y)\}}{dy}$ and if you need the second degree, then obtain $\frac{d^2 f(x,y)}{dx^2}, \frac{\partial^2 f(x,y)}{\partial y \partial x}, \frac{\partial^2 f(x,y)}{\partial y^2}$ as well, and if you need the third degree, then obtain $\frac{\partial^3 f(x,y)}{\partial x^3}, \frac{\partial^3 f(x,y)}{\partial y \partial x^2}, \frac{\partial^3 f(x,y)}{\partial y^2 \partial x}, \frac{\partial^3 f(x,y)}{\partial y^3}$ as well.
b) Substitute $x = a$ and $y = b$ into $\frac{d\{f(x,y)\}}{dx}, \frac{d\{f(x,y)\}}{dy}, \frac{d^2 f(x,y)}{dx^2}, \frac{\partial^2 f(x,y)}{\partial y \partial x}, \frac{\partial^2 f(x,y)}{\partial y^2}, \frac{\partial^3 f(x,y)}{\partial x^3}, \frac{\partial^3 f(x,y)}{\partial y \partial x^2}, \frac{\partial^3 f(x,y)}{\partial y^2 \partial x}, \frac{\partial^3 f(x,y)}{\partial y^3}$
c) Put all of them into Equation (83).

VIII. Key points on ordinary differential equations

Key points

1) The solution of the equation $\frac{d\{y\}}{dx} = f(x)g(y)$ may be found from separating the variables and integrating

$$\int \frac{1}{g(y)} dy = \int f(x)dx \tag{84}$$

Procedure:

a) Allocate $f(x)$ and $g(x)$
b) Calculate

$$\int \frac{1}{g(y)} dy = \int f(x)dx$$

2) When the differential equation can be written as $\frac{d\{y\}}{dx} + P(x)y = Q(x)$ then the answer is

$$y = \frac{1}{\Phi(x)} \left[\int \Phi(x)Q(x)dx + c \right] \tag{85}$$

where

$$\Phi(x) = \mathfrak{e}^{\int P(x)dx} \tag{86}$$

Procedure:

a) Allocate $P(x)$ and $Q(x)$
b) Calculate $A = \int P(x)dx$
c) Obtain $\Phi(x) = \mathfrak{e}^{A}$
d) Calculate $B = \int \Phi(x)Q(x)dx$
e) Obtain the general solution $y = \frac{1}{\Phi(x)} [B + c]$
f) Apply the condition to $y = \frac{1}{\Phi(x)} [B + c]$ in order to find out c and thus the particular solution

Proof:

When we multiply $\frac{d\{y\}}{dx} + P(x)y = Q(x)$ with $\Phi(x)$, we get:
$\Phi(x)\frac{d\{y\}}{dx} + \Phi(x)P(x)y = \Phi(x)Q(x)$. Since,

$$\begin{aligned}\frac{d\{\Phi(x)\}}{dx} &= \frac{d\left\{\mathfrak{e}^{\int P(x)dx}\right\}}{dx} \\ &= \mathfrak{e}^{\int P(x)dx}\frac{d\left\{\int P(x)dx\right\}}{dx} \\ &= \mathfrak{e}^{\int P(x)dx}P(x) \\ &= \Phi(x)P(x),\end{aligned}$$

$$\begin{aligned}\Phi(x)Q(x) &= \Phi(x)\frac{d\{y\}}{dx} + \Phi(x)P(x)y \\ &= \Phi(x)\frac{d\{y\}}{dx} + \frac{d\{\Phi(x)\}}{dx}y \\ &= \frac{d\{y\Phi(x)\}}{dx}\end{aligned}$$

because $\frac{d\{y\}}{dx} + P(x)y = Q(x)$ and $\frac{d\{\Phi(x)\}}{dx} = \Phi(x)P(x)$.
When we integrate $\frac{d\{y\Phi(x)\}}{dx} = \Phi(x)Q(x)$ with respect to x,

$$\begin{aligned}\int \frac{d\{y\Phi(x)\}}{dx}dx &= \int \Phi(x)Q(x)dx \\ \therefore y\Phi(x) &= \int \Phi(x)Q(x)dx + c \\ \therefore y &= \frac{1}{\Phi(x)}\left[\int \Phi(x)Q(x)dx + c\right]\end{aligned}$$

3) When f can be written as a function of $y/x \triangleq z$, the solution of the equation $\frac{d\{y\}}{dx} = f(y/x)$ may be found as

$$\int \frac{dz}{f(z) - z} = \int \frac{1}{x}dx = \ln x + c \tag{87}$$

Procedure:

a) Find $f(\frac{y}{x})$

b) Calculate

$$\int \frac{dz}{f(z)-z} \triangleq g(z)$$

c) Set $\ln(x)+c=g(z)$

d) Replace z with $\frac{y}{x}$ so that $\ln(x)+c=g(\frac{y}{x})$ is the answer

Proof: $y/x \triangleq z$ can be written as $y=zx$. Thus $\frac{d\{y\}}{dx}=\frac{d\{z\}}{dx}x+z\frac{d\{x\}}{dx}=x\frac{d\{z\}}{dx}+z$. Thus $\frac{d\{y\}}{dx}=f(y/x)=f(z)$ can be written as

$$x\frac{d\{z\}}{dx}+z=f(z)$$
$$\therefore x\frac{d\{z\}}{dx}=f(z)-z$$
$$\therefore \frac{1}{x}dx=\frac{1}{f(z)-z}dz$$
$$\therefore \int \frac{1}{f(z)-z}dz=\int \frac{1}{x}dx=\ln x+c$$

4) When the differential equation can be written as $f(x,y)dx+g(x,y)dy=0$ and if

$$\frac{d\{f(x,y)\}}{dy}=\frac{d\{g(x,y)\}}{dx}, \tag{88}$$

then there is a function $U(x,y)$ which satisfies

$$\begin{aligned} dU(x,y) &= \frac{d\{U(x,y)\}}{dx}dx+\frac{d\{U(x,y)\}}{dy}dy \\ &\equiv f(x,y)dx+g(x,y)dy=0 \end{aligned} \tag{89}$$

$dU(x,y)=0$ gives

$$U(x,y)=c \tag{90}$$

which is the answer. In order to find $U(x,y)$, we first perform

$$U(x,y)=\int f(x,y)dx+h(y) \tag{91}$$

then we find $h(y)$ from

$$\begin{aligned} \frac{d\{U(x,y)\}}{dy} &= \frac{d\left\{\int f(x,y)dx+h(y)\right\}}{dy} \\ &= g(x,y) \end{aligned} \tag{92}$$

The alternative approach to obtain $U(x,y)$ is

$$U(x,y)=\int_{x_0}^{x} f(x,y)dx+\int_{y_0}^{y} g(x_0,y)dy \tag{93}$$

where x_0 and y_0 are arbitrary constants. Please be aware of $g(\underline{x_0},y)$ which is not $g(\underline{x},y)$
x_0 and y_0 can be added into c in Equation (90) as they are arbitrary constants.

Procedure:

a) Allocate $f(x,y)$ and $g(x,y)$

b) Confirm

$$\frac{d\{f(x,y)\}}{dy}=\frac{d\{g(x,y)\}}{dx}$$

c) Apply $\int_{x_0}^{x} f(x,y)dx+\int_{y_0}^{y} g(x_0,y)dy=c$

d) Merge all the terms which have x_0 and y_0

Proof: Let's assume there is a function

$$U(x,y)=\int_{x_0}^{x} f(x,y)dx+\int_{y_0}^{y} g(x_0,y)dy=c \qquad ①$$

When you calculate $\int_{x_0}^{x} f(x,y)dx$, you assume y is a constant and let it be y_0. Thus we can write

$$\int f(x,y)dx \equiv \int f(x,y_0)dx \triangleq F(x,y_0) \qquad ②$$

In the similar way we can write

$$\int g(x_0,y)dy \triangleq G(x_0,y) \qquad ③$$

By putting ② and ③ into ①, we get

$$U(x,y) = F(x,y_0) - F(x_0,y_0) + G(x_0,y) - G(x_0,y_0) = c \qquad ④$$

Since $U(x,y) = c$ from ①, we can write

$$\partial U(x,y) = \frac{d\{U(x,y)\}}{dx}dx + \frac{d\{U(x,y)\}}{dy}dy = 0 \qquad ⑤$$

Using ④, we obtain $\frac{d\{U(x,y)\}}{dx}$ and $\frac{d\{U(x,y)\}}{dy}$ as follows:

$$\frac{d\{U(x,y)\}}{dx} = f(x,y_0) \qquad ⑥$$
$$\frac{d\{U(x,y)\}}{dy} = g(x_0,y) \qquad ⑦$$

By putting ⑥ and ⑦ into ⑤, we get

$$\frac{d\{U(x,y)\}}{dx}dx + \frac{d\{U(x,y)\}}{dy}dy = f(x,y_0)dx + g(x_0,y)dy = 0 \qquad ⑥$$

Now since

$$\frac{d\{f(x,y_0)\}}{dy} = \frac{d\{g(x_0,y)\}}{dx}(=0) \qquad ⑦$$

we can conclude that ① satisfies ⑥ and ⑦. In other words, when ⑥ and ⑦ are given, we can say ① is valid.

5) The solution of Jean Bernoulli equation

$$\frac{d\{y\}}{dx} + p(x)y = q(x)y^{\alpha} \qquad (\alpha \neq 0,1) \tag{94}$$

is obtained by solving

$$\frac{d\{Y\}}{dx} + (1-\alpha)p(x)Y = (1-\alpha)q(x) \tag{95}$$

where

$$Y = y^{1-\alpha}. \tag{96}$$

In other words, Y ($= y^{1-\alpha}$, be aware that this is not y but Y!!) is obtained from $Y = \frac{1}{\Phi(x)}\left[\int \Phi(x)Q(x)dx + c\right]$ where $\Phi(x) = \mathfrak{e}^{\int P(x)dx}$ and $P(x) = (1-\alpha)p(x)$ and $Q(x) = (1-\alpha)q(x)$. The steps to the solution are:

a) allocate $p(x)$ and $q(x)$
b) identify the value of α
c) allocate $P(x) = (1-\alpha)p(x)$ and $Q(x) = (1-\alpha)q(x)$
d) calculate $\int P(x)dx$
e) calculate $\Phi(x) = \mathfrak{e}^{\int P(x)dx}$
f) calculate $y^{1-\alpha} = \frac{1}{\Phi(x)}[\int \Phi(x)Q(x)dx + c]$

Proof:

$$\frac{d\{y\}}{dx} + p(x)y = q(x)y^{\alpha}$$
$$\therefore y^{-\alpha}\frac{d\{y\}}{dx} + p(x)y \cdot y^{-\alpha} = q(x)$$
$$\therefore y^{-\alpha}\frac{d\{y\}}{dx} + p(x)y^{1-\alpha} = q(x)$$

Since

$$\frac{d\left\{y^{1-\alpha}\right\}}{dx} = \frac{d\left\{y^{1-\alpha}\right\}}{dy}\frac{d\{y\}}{dx}$$
$$= (1-\alpha)y^{1-\alpha-1}\frac{d\{y\}}{dx}$$
$$= (1-\alpha)y^{-\alpha}\frac{d\{y\}}{dx}$$
$$\therefore \frac{1}{1-\alpha}\frac{d\left\{y^{1-\alpha}\right\}}{dx} = y^{-\alpha}\frac{d\{y\}}{dx}$$

we can manipulate the equation as follows:

$$
\begin{aligned}
y^{-\alpha}\frac{d\left\{y\right\}}{dx}+p(x)y^{1-\alpha}&=q(x)\\
\therefore\frac{1}{1-\alpha}\frac{d\left\{y^{1-\alpha}\right\}}{dx}+p(x)y^{1-\alpha}&=q(x)\\
\therefore\frac{d\left\{y^{1-\alpha}\right\}}{dx}+(1-\alpha)p(x)y^{1-\alpha}&\\
=(1-\alpha)q(x)&\\
\therefore\frac{d\left\{Y\right\}}{dx}+(1-\alpha)p(x)Y=(1-\alpha)q(x)&
\end{aligned}
$$

The answer can be obtained from Equation (85) where

$$P(x)=(1-\alpha)p(x) \tag{97}$$

$$Q(x)=(1-\alpha)q(x) \tag{98}$$

6) Clairaut type

$$y=x\frac{d\left\{y\right\}}{dx}+f\left(\frac{d\left\{y\right\}}{dx}\right) \tag{99}$$

can be solved as follows:

a) Allocate $f(\frac{d\left\{y\right\}}{dx})$
b) Write down the general solution of

$$y=ax+f(a)$$

which is the answer!. State a is a constant value.
c) Differentiate

$$y=ax+f(a)$$

with respect to a
d) Express a as a function of x, let's say $a=g(x)$
e) Insert $a=g(x)$ into the general solution to get a particular solution of

$$y=x\cdot g(x)+f\left(g(x)\right)$$

Proof:

$$
\begin{aligned}
\frac{d\left\{y\right\}}{dx}&=\frac{d\left\{x\frac{d\left\{y\right\}}{dx}+f\left(\frac{d\left\{y\right\}}{dx}\right)\right\}}{dx}\\
&=\frac{d\left\{x\right\}}{dx}\frac{d\left\{y\right\}}{dx}+x\frac{d^2y}{dx^2}+\frac{d\left\{f\left(\frac{d\left\{y\right\}}{dx}\right)\right\}}{dx}\\
&=\frac{d\left\{y\right\}}{dx}+x\frac{d^2y}{dx^2}+\frac{\partial\left\{f\left(\frac{d\left\{y\right\}}{dx}\right)\right\}}{\partial\left\{\frac{d\left\{y\right\}}{dx}\right\}}\frac{d\left\{\frac{d\left\{y\right\}}{dx}\right\}}{dx}\\
&=\frac{d\left\{y\right\}}{dx}+x\frac{d^2y}{dx^2}+\frac{\partial\left\{f\left(\frac{d\left\{y\right\}}{dx}\right)\right\}}{\partial\left\{\frac{d\left\{y\right\}}{dx}\right\}}\frac{d^2y}{dx^2}\\
\therefore 0&=x\frac{d^2y}{dx^2}+\frac{\partial\left\{f\left(\frac{d\left\{y\right\}}{dx}\right)\right\}}{\partial\left\{\frac{d\left\{y\right\}}{dx}\right\}}\frac{d^2y}{dx^2}\\
\therefore 0&=\left(x+\frac{\partial\left\{f\left(\frac{d\left\{y\right\}}{dx}\right)\right\}}{\partial\left\{\frac{d\left\{y\right\}}{dx}\right\}}\right)\frac{d^2y}{dx^2}
\end{aligned}
$$

Thus we obtain

$$\frac{d^2y}{dx^2}=0$$

or

$$x+\frac{\partial\left\{f\left(\frac{d\left\{y\right\}}{dx}\right)\right\}}{\partial\left\{\frac{d\left\{y\right\}}{dx}\right\}}=0$$

From $\frac{d^2y}{dx^2} = 0$ we obtain

$$\begin{aligned}
\frac{d^2y}{dx^2} &= 0 \\
\therefore \frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dx} &= 0 \\
\therefore \partial(\frac{d\{y\}}{dx}) &= 0 \cdot \partial x \\
\therefore \int d(\frac{d\{y\}}{dx}) &= \int 0 \cdot dx \\
\therefore \frac{d\{y\}}{dx} &= a \\
\therefore dy &= a \cdot dx \\
\therefore \int dy &= \int a \cdot dx \\
\therefore y &= ax + b \\
\therefore \frac{d\{y\}}{dx} = \frac{d\{ax+b\}}{dx} &= a
\end{aligned}$$

where a and b are the arbitrary constants. Substituting $y = ax + b$ and $\frac{d\{y\}}{dx} = a$ into the original equation, we get

$$\begin{aligned}
y &= x\frac{d\{y\}}{dx} + f\left(\frac{d\{y\}}{dx}\right) \\
\therefore ax + b &= x \cdot a + f(a) \\
\therefore b &= f(a)
\end{aligned}$$

Therefore

$$y = ax + f(a) \tag{100}$$

is a general solution with an arbitrary constant of a. Furthermore, when we take the differentiation of the equation with respect to a, we get

$$\begin{aligned}
\frac{\partial\{y\}}{\partial a} &= \frac{\partial\{ax + f(a)\}}{\partial a} \\
\therefore 0 &= \frac{\partial\{ax\}}{\partial a} + \frac{\partial\{f(a)\}}{\partial a} \\
\therefore 0 &= x + \frac{\partial\{f(a)\}}{\partial a}
\end{aligned}$$

We solve the equation for a. Let's assume $a = A(x)$ satisfies $x + \frac{\partial\{f(a)\}}{\partial a} = 0$. The resultant expression of a using x, which is $A(x)$ is put into $y = ax + f(a)$ to obtain a particular solution of Equation (101).

$$y = A(x) \cdot x + f(A(x)) \tag{101}$$

7) In order to solve second order differential equations

$$\frac{d^2y}{dx^2} + v\frac{d\{y\}}{dx} + wy = r(x), \tag{102}$$

where v, w are the constant values,

a) Production of an auxiliary equation by forcing $r(x)$ to 0
By substituting

$$\frac{d^2y}{dx^2} = \lambda^2, \frac{d\{y\}}{dx} = \lambda, y = \lambda^0 = 1 \tag{103}$$

into the original original equation, forcing $r(x)$ to zero, we solve the auxiliary equation of

$$\lambda^2 + v\lambda + w = 0 \tag{104}$$

and we obtain the answers $\lambda = \alpha$ and β.

b) Set complementary function as follows:

i) α and β are real and $\alpha \neq \beta$
Set the complementary function $Y_1(x)$ as

$$Y_1(x) = a\mathfrak{e}^{\alpha x} + b\mathfrak{e}^{\beta x} \tag{105}$$

where a, b are constant value which is found from the initial condition.

ii) α and β are real and $\alpha = \beta$
Set the complementary function $Y_1(x)$ as

$$Y_1(x) = a\mathfrak{e}^{\alpha x} + bx\mathfrak{e}^{\alpha x} \tag{106}$$

iii) α and β are complex numbers and $p \pm \jmath q$ (where p, q are real)
Set the complementary function $Y_1(x)$ as

$$Y_1(x) = \mathfrak{e}^{px}(a \cos qx + b \sin qx) \tag{107}$$

c) Check the characteristics of $r(x)$ and set the particular integral
  i) $r(x)$ is proportional to $\mathfrak{e}^{cx}$, where c is a constant value
    A) $\alpha \neq c$ and $\beta \neq c$
    Set the particular integral $Y_2(x)$ as

$$Y_2(x) = g\mathfrak{e}^{cx} \tag{108}$$

    where g is a constant value which is found from Equation (102).
    B) $\alpha = c$
    Set the particular integral $Y_2(x)$ as

$$Y_2(x) = gx^k\mathfrak{e}^{cx} \tag{109}$$

    where k is 1 or 2 or 3 ...
  ii) $r(x)$ is nth order polynomial
    A) $\alpha \neq 0$ and $\beta \neq 0$
    Set the particular integral $Y_2(x)$ as

$$Y_2(x) = \sum_{m=0}^{n} g_m x^m \tag{110}$$

    where g_m is a constant value which is found from Equation (102).
    B) $\alpha = 0$
    Set the particular integral $Y_2(x)$ as

$$Y_2(x) = x^k \left(\sum_{m=0}^{n} g_m x^m \right) \tag{111}$$

    where k is 1 or 2 or 3 ...
  iii) $r(x)$ is in the form of $P(x)\mathfrak{e}^{cx}$ where $P(x)$ is the nth order polynomial.
    A) $\alpha \neq c$ and $\beta \neq c$
    Set the particular integral $Y_2(x)$ as

$$Y_2(x) = \mathfrak{e}^{cx} \left(\sum_{m=0}^{n} g_m x^m \right) \tag{112}$$

    where g_m is a constant value which is found from Equation (102).
    B) $\alpha = c$
    Set the particular integral $Y_2(x)$ as

$$Y_2(x) = \mathfrak{e}^{cx} x^k \left(\sum_{m=0}^{n} g_m x^m \right) \tag{113}$$

    where k is 1 or 2 or 3 ...
  iv) $r(x)$ is the combination of $\cos \omega x$ and $\sin \omega x$
    A) $\alpha \neq \pm \jmath\omega$ and $\beta \neq \pm \jmath\omega$
    Set the particular integral $Y_2(x)$ as

$$Y_2(x) = g \cos \omega x + h \sin \omega x \tag{114}$$

    where g and h are constant values which is found from Equation (102).
    B) $\alpha = \pm \jmath\omega$
    Set the particular integral $Y_2(x)$ as

$$Y_2(x) = x^k(g \cos \omega x + h \sin \omega x) \tag{115}$$

    where k is 1 or 2 or 3 ...
  v) $r(x)$ is the combination of $\mathfrak{e}^{cx} \cos \omega x$ and $\mathfrak{e}^{cx} \sin \omega x$
    A) $\alpha \neq c \pm \jmath\omega$ and $\beta \neq c \pm \jmath\omega$
    Set the particular integral $Y_2(x)$ as

$$Y_2(x) = \mathfrak{e}^{cx}(g \cos \omega x + h \sin \omega x) \tag{116}$$

    where g and h are constant values which is found from Equation (102).
    B) $\alpha = c \pm \jmath\omega$
    Set the particular integral $Y_2(x)$ as

$$Y_2(x) = x^k\mathfrak{e}^{cx}(g \cos \omega x + h \sin \omega x) \tag{117}$$

    where k is 1 or 2 or 3 ...
d) Find the constant values g and h by

$$\frac{d^2Y_2(x)}{dx^2} + v\frac{d\{Y_2(x)\}}{dx} + wY_2(x) = r(x) \tag{118}$$

e) Get the general solution of The general solution is $y = Y_1(x) + Y_2(x)$ leaving a and b unknown.

f) Find the constant values a and b
Usually there are initial conditions for $y(0)$ and $\frac{d\{y\}}{dx}|_{x=0}$. Using these conditions, a and b are found.

g) The particular solution is $y = Y_1(x) + Y_2(x)$.

Summary Procedure of 2nd order ODE $\frac{d^2y}{dx^2} + v\frac{d\{y\}}{dx} + wy = r(x)$

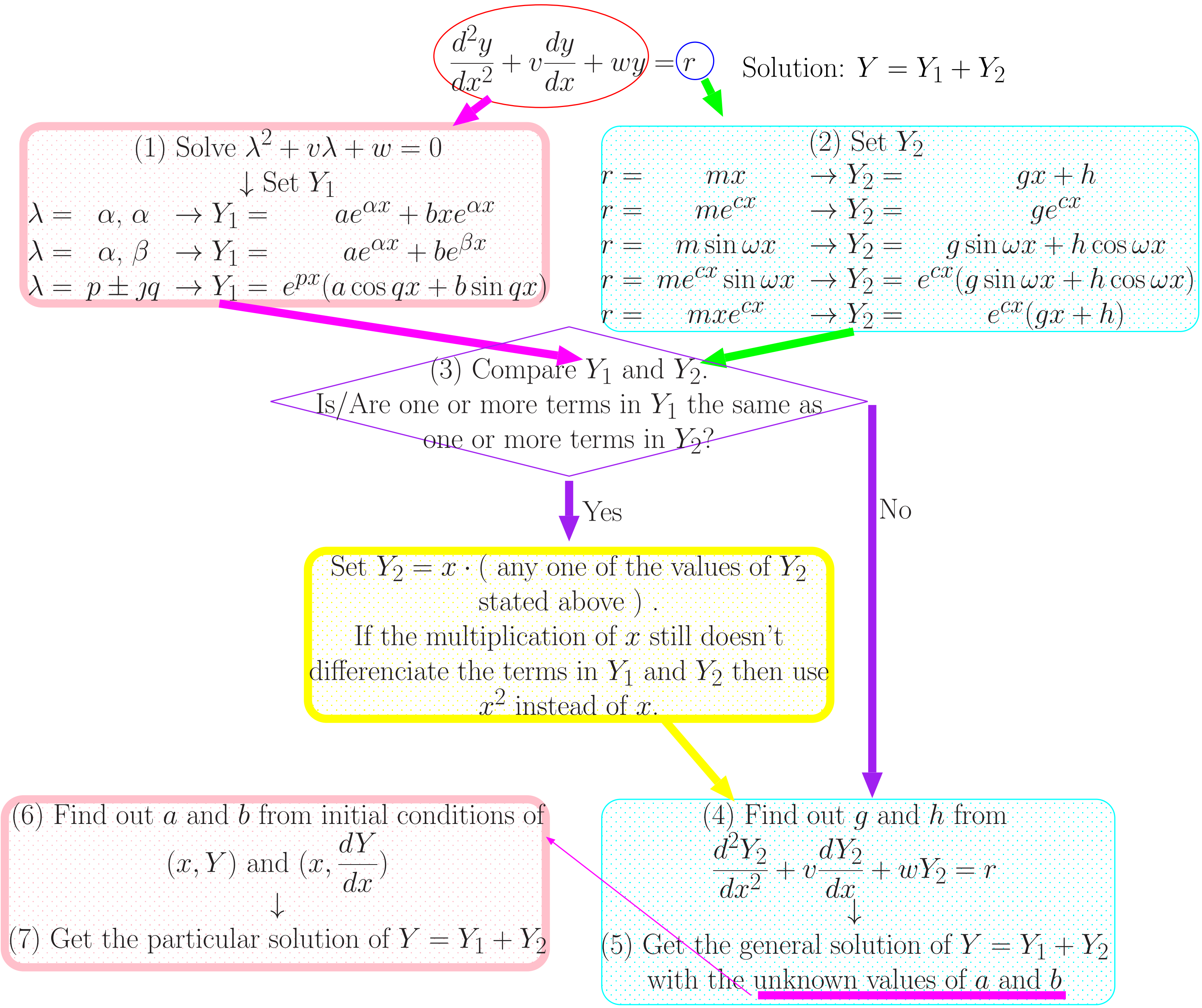
a) Produce and solve an auxiliary equation by setting $r(x) = 0$
b) Set the complementary function $Y_1(x)$ with the unknown variables a and b
c) Set particular integral $Y_2(x)$ with the unknown variables g and h
d) Find g and h from $\frac{d^2Y_2(x)}{dx^2} + v\frac{d\{Y_2(x)\}}{dx} + wY_2(x) = r(x)$
e) Get the general solution $y = Y_1(x) + Y_2(x)$ with unknown a and b
f) Find a and b using the initial conditions
g) Get the particular solution $y = Y_1(x) + Y_2(x)$ with known a and b

8) Lookup table for 2nd order ODE

$r(x)$	particular integral $Y_2(x)$
$\mathfrak{e}^{cx}, \alpha \neq c, \beta \neq c$	$g\mathfrak{e}^{cx}$
$\mathfrak{e}^{cx}, \alpha = c$	$gx^k\mathfrak{e}^{cx}$
$\sum_{m=0}^{n} \rho_m x^m, \alpha \neq 0, \beta \neq 0$	$\sum_{m=0}^{n} g_m x^m$
$\sum_{m=0}^{n} \rho_m x^m, \alpha = 0$	$x^k\left(\sum_{m=0}^{n} g_m x^m\right)$
$\mathfrak{e}^{cx}\sum_{m=0}^{n} \rho_m x^m, \alpha \neq c, \beta \neq c$	$\mathfrak{e}^{cx}\sum_{m=0}^{n} g_m x^m$
$\mathfrak{e}^{cx}\sum_{m=0}^{n} \rho_m x^m, \alpha = c$	$x^k\mathfrak{e}^{cx}\sum_{m=0}^{n} g_m x^m$
$\rho_1 \cos\omega x + \rho_2 \sin\omega x, \alpha \neq \pm\jmath\omega, \beta \neq \pm\jmath\omega$	$g\cos\omega x + h\sin\omega x$
$\rho_1 \cos\omega x + \rho_2 \sin\omega x, \alpha = \pm\jmath\omega$	$x^k(g\cos\omega x + h\sin\omega x)$
$\mathfrak{e}^{cx}(\rho_1 \cos\omega x + \rho_2 \sin\omega x), \alpha \neq c \pm \jmath\omega, \beta \neq c \pm \jmath\omega$	$\mathfrak{e}^{cx}(g\cos\omega x + h\sin\omega x)$
$\mathfrak{e}^{cx}(\rho_1 \cos\omega x + \rho_2 \sin\omega x), \alpha = c \pm \jmath\omega$	$x^k\mathfrak{e}^{cx}(g\cos\omega x + h\sin\omega x)$

TABLE I
PARTICULAR INTEGRAL FOR THE SECOND ORDER ODE

9) Summary for 2nd order ODE



10) Summary for 1st order ODE

Equation type	Procedure to follow
$\frac{d\{y\}}{dx} = f(x)g(y)$	a) Allocate $f(x)$ and $g(x)$ b) Calculate $$\int \frac{1}{g(y)}dy = \int f(x)dx$$
$\frac{d\{y\}}{dx} = f(\frac{y}{x})$	a) Find $f(\frac{y}{x})$ b) Calculate $$\int \frac{dz}{f(z)-z} \triangleq g(z)$$ c) Set $\ln(x)+c = g(z)$ d) Replace z with $\frac{y}{x}$ so that $\ln(x)+c = g(\frac{y}{x})$ is the answer
$\frac{d\{y\}}{dx} = -\frac{f(x,y)}{g(x,y)}$	a) Allocate $f(x,y)$ and $g(x,y)$ b) Confirm $$\frac{d\{f(x,y)\}}{dy} = \frac{d\{g(x,y)\}}{dx}$$ c) Apply $\int_{x_0}^{x} f(x,y)dx + \int_{y_0}^{y} g(x_0,y)dy = c$ d) Merge all the terms which have x_0 and y_0
$\frac{d\{y\}}{dx} = -P(x)y + Q(x)$	a) Allocate $P(x)$ and $Q(x)$ b) Calculate $\int P(x)dx$ c) Calculate $\Phi(x) = \mathfrak{e}^{\int P(x)dx}$ d) Calculate $y = \frac{1}{\Phi(x)}\left[\int \Phi(x)Q(x)dx + c\right]$
$\frac{d\{y\}}{dx} = -p(x)y + q(x)y^{\alpha}$	a) allocate $p(x)$ and $q(x)$ b) identify the value of α c) allocate $P(x) = (1-\alpha)p(x)$ and $Q(x) = (1-\alpha)q(x)$ d) calculate $\int P(x)dx$ e) calculate $\Phi(x) = \mathfrak{e}^{\int P(x)dx}$ f) calculate $y^{1-\alpha} = \frac{1}{\Phi(x)}[\int \Phi(x)Q(x)dx + c]$
$\frac{d\{y\}}{dx} = \frac{y}{x} + \frac{1}{x}f\left(\frac{d\{y\}}{dx}\right)$	a) Allocate $f(\frac{d\{y\}}{dx})$ b) Write down the general solution of $$y = ax + f(a)$$ which is the answer!. State a is a constant value. c) Differentiate $$y = ax + f(a)$$ with respect to a d) Express a as a function of x, let's say $a = g(x)$ e) Insert $a = g(x)$ into the general solution to get a particular solution of $$y = x \cdot g(x) + f(g(x))$$

IX. Exercises on Vectors

vectorall.tex

A. DAY1

1) **DAY1**
2) Evaluate

$$\sqrt{(-6)^2+5^2}$$

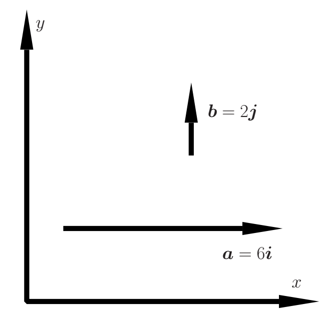
$$\begin{aligned}\sqrt{(-6)^2+5^2} &= \sqrt{36+25}\\ &= \sqrt{61}\end{aligned}$$

3) Evaluate

$$8\cdot(-4)+12\cdot(2)-(-1)\cdot(3)$$

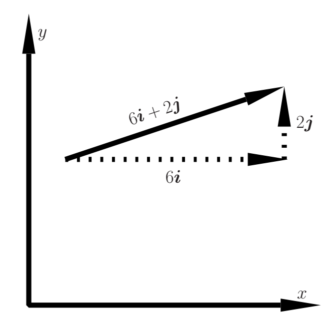
$$\begin{aligned}8\cdot(-4)+12\cdot(2)-(-1)\cdot(3)&\\ = -32+24+3&\\ = -5&\end{aligned}$$

4) Draw the vectors $\boldsymbol{a} = 6\boldsymbol{i}$ and $\boldsymbol{b} = 2\boldsymbol{j}$.

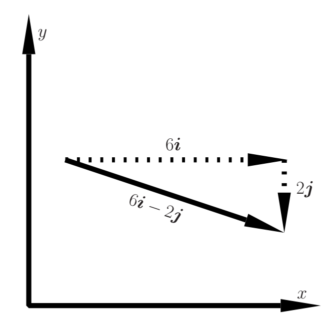


Then find and draw

a) $\boldsymbol{a}+\boldsymbol{b}$



b) $\boldsymbol{a}-\boldsymbol{b}$



5) If $\boldsymbol{p} = 9\boldsymbol{i} - 7\boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{q} = -8\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ express

a) $\boldsymbol{p}+\boldsymbol{q}$ in terms of $\boldsymbol{i}$,$\boldsymbol{j}$, and $\boldsymbol{k}$

$$\boldsymbol{p}+\boldsymbol{q} = \begin{pmatrix} 9 \\ -7 \\ 5 \end{pmatrix} + \begin{pmatrix} -8 \\ 3 \\ -2 \end{pmatrix}$$
$$= \begin{pmatrix} 9-8 \\ -7+3 \\ 5-2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$$

Therefore $\boldsymbol{p}+\boldsymbol{q} = \boldsymbol{i} - 4\boldsymbol{j} + 3\boldsymbol{k}$

b) $\boldsymbol{p}-\boldsymbol{q}$ in terms of $\boldsymbol{i}$,$\boldsymbol{j}$, and $\boldsymbol{k}$

$$\boldsymbol{p}-\boldsymbol{q} = \begin{pmatrix} 9 \\ -7 \\ 5 \end{pmatrix} - \begin{pmatrix} -8 \\ 3 \\ -2 \end{pmatrix}$$
$$= \begin{pmatrix} 9-(-8) \\ -7-(+3) \\ 5-(-2) \end{pmatrix} = \begin{pmatrix} 9+8 \\ -7-3 \\ 5+2 \end{pmatrix} = \begin{pmatrix} 17 \\ -10 \\ 7 \end{pmatrix}$$

Therefore $\boldsymbol{p}-\boldsymbol{q} = 17\boldsymbol{i} - 10\boldsymbol{j} + 7\boldsymbol{k}$

6) Sketch the position vectors

$$\boldsymbol{p} = 3\boldsymbol{i} + 4\boldsymbol{j}$$

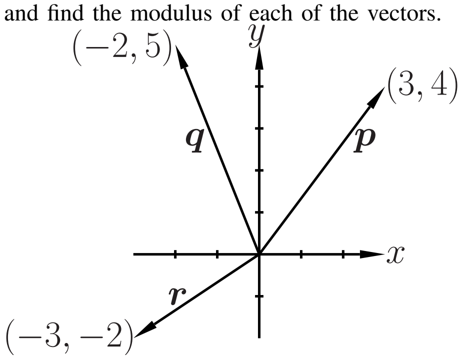
and

$$\boldsymbol{q} = -2\boldsymbol{i} + 5\boldsymbol{j}$$

and

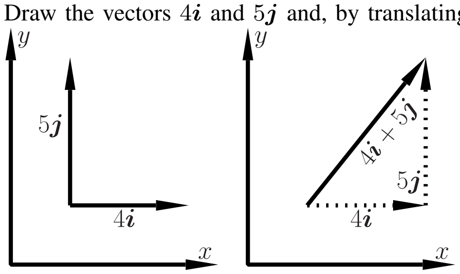
$$\boldsymbol{r} = -3\boldsymbol{i} - 2\boldsymbol{j}$$

and find the modulus of each of the vectors.



$$|\boldsymbol{p}| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$$
$$|\boldsymbol{q}| = \sqrt{(-2)^2+5^2} = \sqrt{4+25} = \sqrt{29}$$
$$|\boldsymbol{r}| = \sqrt{(-3)^2+(-2)^2} = \sqrt{9+4} = \sqrt{13}$$

7) Draw the vectors $4\boldsymbol{i}$ and $5\boldsymbol{j}$ and, by translating the vectors so that they lie head to tail, the vector sum $4\boldsymbol{i}+5\boldsymbol{j}$.



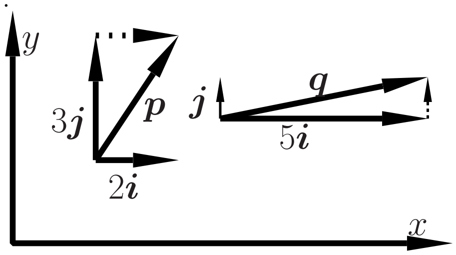
8) Answer the following set of problems

a) Draw an xy plane and show the vectors

$$\boldsymbol{p} = 2\boldsymbol{i} + 3\boldsymbol{j}$$

and

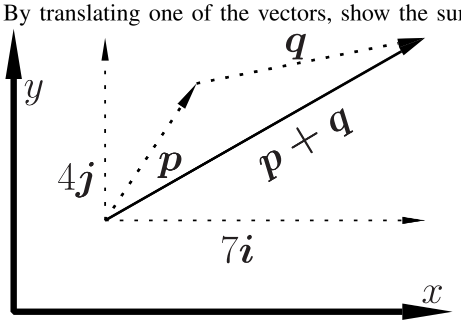
$$\boldsymbol{q} = 5\boldsymbol{i} + \boldsymbol{j}$$



b) Express $\boldsymbol{p}$ and $\boldsymbol{q}$ using column vector notation.

$$\boldsymbol{p} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \boldsymbol{q} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

c) By translating one of the vectors, show the sum $\boldsymbol{p}+\boldsymbol{q}$ on an xy plane.



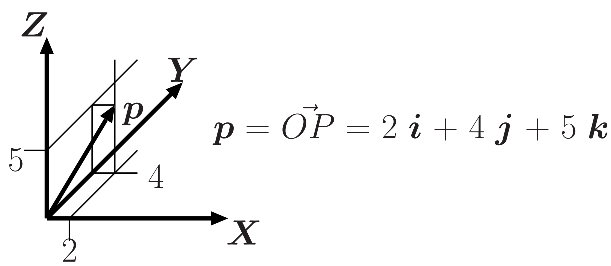
d) Express the resultant $\boldsymbol{p}+\boldsymbol{q}$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$

$$\boldsymbol{p}+\boldsymbol{q} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+5 \\ 3+1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

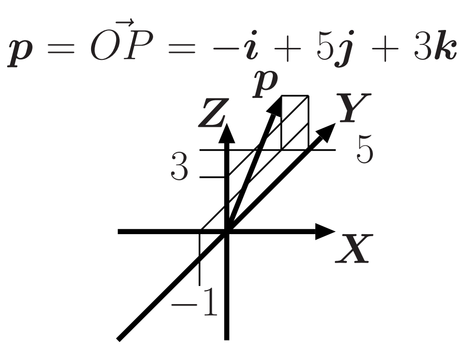
Therefore $\boldsymbol{p}+\boldsymbol{q} = 7\boldsymbol{i}+4\boldsymbol{j}$

9) State the position vectors of the points with the coordinates

a) P(2,4,5)
$\boldsymbol{p} = \overrightarrow{OP} = 2\boldsymbol{i}+4\boldsymbol{j}+5\boldsymbol{k}$

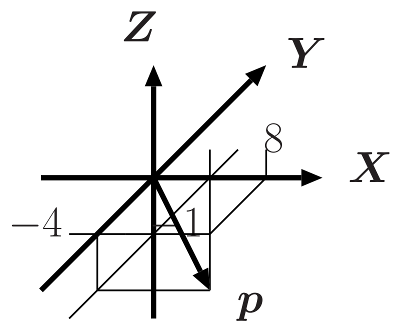


b) P(-1,5,3)
$\boldsymbol{p} = \overrightarrow{OP} = -\boldsymbol{i}+5\boldsymbol{j}+3\boldsymbol{k}$



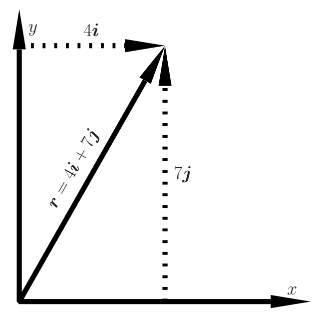
c) P(-2,-1,4)
$\boldsymbol{p} = \overrightarrow{OP} = -2\boldsymbol{i}-\boldsymbol{j}+4\boldsymbol{k}$

d) P(8,-4,-1)
$\boldsymbol{p} = \overrightarrow{OP} = 8\boldsymbol{i}-4\boldsymbol{j}-\boldsymbol{k}$

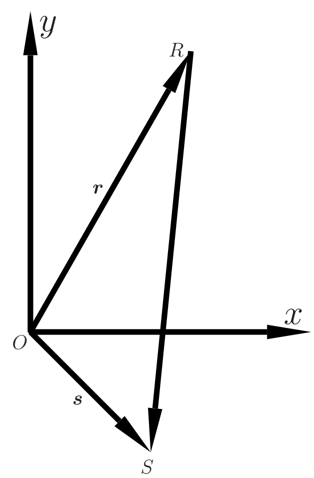


$$\boldsymbol{p} = \vec{OP} = 8\boldsymbol{i} - 4\,\boldsymbol{j} - \boldsymbol{k}$$

10) Consider the points $R = (4, 7)$ and $S = (3, -3)$.
Find
a) and draw the position vector of point R.



b) the vector $\overrightarrow{RS}$ expressed in column notation



$$\begin{aligned}\overrightarrow{RS} &= \overrightarrow{RO} + \overrightarrow{OS} \\ &= -\overrightarrow{OR} + \overrightarrow{OS} \\ &= -\boldsymbol{r} + \boldsymbol{s} \\ &= -\begin{pmatrix} 4 \\ 7 \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} -4+3 \\ -7-3 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ -10 \end{pmatrix}\end{aligned}$$

11) Consider the vectors $\boldsymbol{c} = 4\boldsymbol{i} - 5\boldsymbol{j} + 10\boldsymbol{k}$ and $\boldsymbol{d} = -6\boldsymbol{i} + \boldsymbol{j} - 7\boldsymbol{k}$ together with the scalar $\lambda = 3$.
Find
a) $\boldsymbol{c} - \lambda\boldsymbol{d}$ expressed in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$

$$\boldsymbol{c} - \lambda\boldsymbol{d} = \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix} - 3 \cdot \begin{pmatrix} -6 \\ 1 \\ -7 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix} - \begin{pmatrix} 3 \cdot (-6) \\ 3 \cdot 1 \\ 3 \cdot (-7) \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ -5 \\ 10 \end{pmatrix} - \begin{pmatrix} -18 \\ 3 \\ -21 \end{pmatrix}$$
$$= \begin{pmatrix} 4-(-18) \\ -5-3 \\ 10-(-21) \end{pmatrix}$$
$$= \begin{pmatrix} 22 \\ -8 \\ 31 \end{pmatrix}$$
$$= 22\boldsymbol{i} - 8\boldsymbol{j} + 31\boldsymbol{k}$$

b) the magnitude of $\boldsymbol{c}$

$$\begin{aligned} |\boldsymbol{c}| &= \sqrt{4^2 + (-5)^2 + 10^2} \\ &= \sqrt{16+25+100} \\ &= \sqrt{141} \end{aligned}$$

c) a unit vector parallel to $\boldsymbol{c}$

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{c}}{|\boldsymbol{c}|} = \frac{1}{\sqrt{141}}(4\boldsymbol{i} - 5\boldsymbol{j} + 10\boldsymbol{k})$$

12) Point P has coordinates $(7, -4, -2)$. Point Q has coordinates $(-2, -5, -1)$

a) State the position vectors of P and Q

$$\boldsymbol{p} = 7\boldsymbol{i} - 4\boldsymbol{j} - 2\boldsymbol{k}$$
$$\boldsymbol{q} = -2\boldsymbol{i} - 5\boldsymbol{j} - \boldsymbol{k}$$

b) Find an expression for $\overrightarrow{PQ}$

$$\overrightarrow{PQ}$$
$$= \overrightarrow{PO} + \overrightarrow{OQ}$$
$$= -\overrightarrow{OP} + \overrightarrow{OQ}$$
$$= -\boldsymbol{p} + \boldsymbol{q}$$
$$= -\begin{pmatrix} 7 \\ -4 \\ -2 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} -7-2 \\ 4-5 \\ 2-1 \end{pmatrix}$$
$$= \begin{pmatrix} -9 \\ -1 \\ 1 \end{pmatrix}$$
$$= -9\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$$

Or alternatively,

$$\overrightarrow{PQ}$$
$$= \overrightarrow{PO} + \overrightarrow{OQ}$$
$$= -\overrightarrow{OP} + \overrightarrow{OQ}$$
$$= -\boldsymbol{p} + \boldsymbol{q}$$
$$= -(7\boldsymbol{i} - 4\boldsymbol{j} - 2\boldsymbol{k}) + (-2\boldsymbol{i} - 5\boldsymbol{j} - \boldsymbol{k})$$
$$= -7\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k} - 2\boldsymbol{i} - 5\boldsymbol{j} - \boldsymbol{k}$$
$$= (-7-2)\boldsymbol{i} + (4-5)\boldsymbol{j} + (2-1)\boldsymbol{k}$$
$$= (-7-2)\boldsymbol{i} + (-1)\boldsymbol{j} + (1)\boldsymbol{k}$$
$$= -9\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$$

c) Find $\left|\overrightarrow{PQ}\right|$

Since

$$\overrightarrow{PQ} = -9\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k},$$

$$
\begin{aligned}
&\left|\overrightarrow{PQ}\right| \\
&= \sqrt{9^2 + (-1)^2 + 1^2} \\
&= \sqrt{81 + 1 + 1} \\
&= \sqrt{83}
\end{aligned}
$$

B. DAY2

13) Simplify

$$2^8 \cdot 2^8$$

$$\begin{aligned} & 2^8 \cdot 2^8 \\ &= 2^{8+8} \\ &= 2^{16} \end{aligned}$$

14) Simplify

$$3x + 2y - z + t(7x + 5y + z - (3x + 2y - z))$$

$$\begin{aligned} & 3x + 2y - z + t(7x + 5y + z - (3x + 2y - z)) \\ &= 3x + 2y - z + t(7x + 5y + z - 3x - 2y + z) \\ &= 3x + 2y - z + t(4x + 3y + 2z) \\ &= 3x + 2y - z + 4tx + 3ty + 2tz \\ &= x(4t + 3) + y(3t + 2) + z(2t - 1) \end{aligned}$$

15) Evaluate

$$9 - 7(12 - 5^2)$$

$$\begin{aligned} & 9 - 7(12 - 5^2) \\ &= 9 - 7(12 - 25) \\ &= 9 - 7(-13) \\ &= 9 + 7 \cdot 13 \\ &= 9 + 91 \\ &= 100 \end{aligned}$$

16) Find the angle between the vectors

$$\boldsymbol{p} = -3\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$$

and

$$\boldsymbol{q} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$$

$$\boldsymbol{p} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}, \boldsymbol{q} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$\begin{aligned} & \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \\ &= (-3) \cdot 2 + (-1) \cdot (3) + (2) \cdot (1) \\ &= -6 - 3 + 2 = -7 \end{aligned}$$

On the other hand

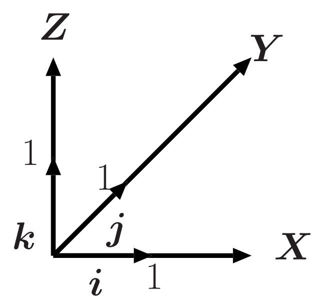
$$\begin{aligned} & |\boldsymbol{p}| \\ &= \sqrt{(-3)^2 + (-1)^2 + 2^2} \\ &= \sqrt{9 + 1 + 4} \\ &= \sqrt{14} \end{aligned}$$

$$\begin{aligned} & |\boldsymbol{q}| \\ &= \sqrt{(2)^2 + (3)^2 + 1^2} \\ &= \sqrt{4 + 9 + 1} \\ &= \sqrt{14} \end{aligned}$$

Therefore

$$\begin{aligned}
\boldsymbol{p}\cdot\boldsymbol{q} &= |\boldsymbol{p}|\cdot|\boldsymbol{q}|\cdot\cos\theta\\
\therefore -7 &= \sqrt{14}\cdot\sqrt{14}\cos\theta\\
\therefore -7 &= 14\cos\theta\\
\therefore \frac{-7}{14} &= \frac{14\cos\theta}{14}\\
\therefore \frac{-1}{2} &= \cos\theta\\
\therefore \theta = 120^o &= \frac{2\pi}{3}
\end{aligned}$$

17) Find $\boldsymbol{i}\cdot\boldsymbol{i}$ and $\boldsymbol{i}\cdot\boldsymbol{j}$ and $\boldsymbol{i}\cdot\boldsymbol{k}$



$$\begin{aligned}
&\boldsymbol{i}\cdot\boldsymbol{i}\\
&= |\boldsymbol{i}|\cdot|\boldsymbol{i}|\cdot\cos 0^o\\
&= 1\cdot 1\cdot 1\\
&= 1
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{i}\cdot\boldsymbol{j}\\
&= |\boldsymbol{i}|\cdot|\boldsymbol{j}|\cdot\cos 90^o\\
&= 1\cdot 1\cdot 0\\
&= 0
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{i}\cdot\boldsymbol{k}\\
&= |\boldsymbol{i}|\cdot|\boldsymbol{k}|\cdot\cos 90^o\\
&= 1\cdot 1\cdot 0\\
&= 0
\end{aligned}$$

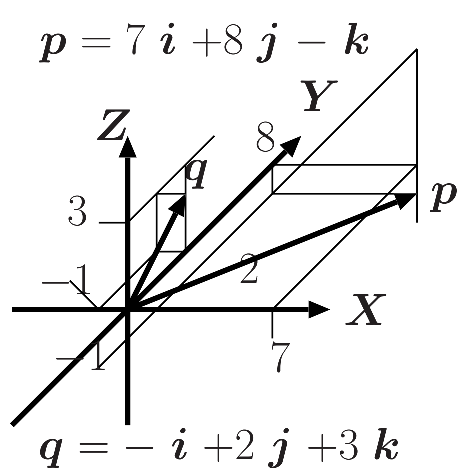
18) If

$$\boldsymbol{p} = 7\boldsymbol{i} + 8\boldsymbol{j} - \boldsymbol{k}$$

and

$$\boldsymbol{q} = -\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$$

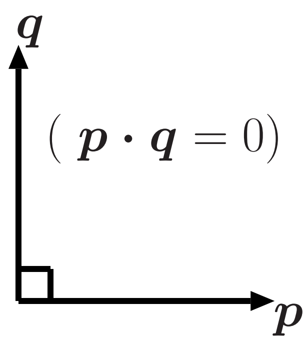
find the scalar product $\boldsymbol{p}\cdot\boldsymbol{q}$



$$p = \begin{pmatrix} 7 \\ 8 \\ -1 \end{pmatrix}, q = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{aligned} &\begin{pmatrix} 7 \\ 8 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \\ &= 7 \cdot (-1) + 8 \cdot (2) + (-1) \cdot (3) \\ &= -7 + 16 - 3 = 6 \end{aligned}$$

19) If $\boldsymbol{p}$ and $\boldsymbol{q}$ are perpendicular, simplify $(\boldsymbol{p} - 2\boldsymbol{q}) \cdot (3\boldsymbol{p} + 5\boldsymbol{q})$



$$\begin{aligned} &(\boldsymbol{p} - 2\boldsymbol{q}) \cdot (3\boldsymbol{p} + 5\boldsymbol{q}) \\ &= 3\boldsymbol{p} \cdot \boldsymbol{p} + 5\boldsymbol{q} \cdot \boldsymbol{p} - 2\boldsymbol{q} \cdot 3\boldsymbol{p} - 2\boldsymbol{q} \cdot 5\boldsymbol{q} \\ &= 3|\boldsymbol{p}|^2 + 5\boldsymbol{q} \cdot \boldsymbol{p} - 6\boldsymbol{q} \cdot \boldsymbol{p} - 10|\boldsymbol{q}|^2 \\ &= 3|\boldsymbol{p}|^2 - \boldsymbol{q} \cdot \boldsymbol{p} - 10|\boldsymbol{q}|^2 \\ &= 3|\boldsymbol{p}|^2 - 10|\boldsymbol{q}|^2 \\ &(\because \boldsymbol{p} \cdot \boldsymbol{q} = 0) \end{aligned}$$

20) Points R, S, and T have coordinates $(-4, 0, -1)$, $(5, 3, -5)$ and $(2, -7, -3)$ respectively. Find
   a) the scalar product $\overrightarrow{RS} \cdot \overrightarrow{RT}$.

$$\begin{aligned} \overrightarrow{RS} &= \overrightarrow{RO} + \overrightarrow{OS} \\ &= -\overrightarrow{OR} + \overrightarrow{OS} \\ &= -\boldsymbol{r} + \boldsymbol{s} \\ &= -\begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} -(-4) + 5 \\ 3 \\ -(-1) - 5 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 3 \\ -4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \overrightarrow{RT} &= \overrightarrow{RO} + \overrightarrow{OT} \\ &= -\overrightarrow{OR} + \overrightarrow{OT} \\ &= -\boldsymbol{r} + \boldsymbol{t} \\ &= -\begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} -(-4) + 2 \\ -7 \\ -(-1) - 3 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -7 \\ -2 \end{pmatrix} \end{aligned}$$

$$\therefore \overrightarrow{RS} \cdot \overrightarrow{RT} = \begin{pmatrix} 9 \\ 3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -7 \\ -2 \end{pmatrix}$$
$$= 9 \cdot 6 + 3 \cdot (-7) + (-4) \cdot (-2)$$
$$= 54 - 21 + 8 = 41$$

b) the vector product $\overrightarrow{RS} \times \overrightarrow{RT}$.

$$\begin{pmatrix} \overrightarrow{RS} & \overrightarrow{RT} \end{pmatrix} = \begin{pmatrix} 9 & 6 \\ 3 & -7 \\ -4 & -2 \end{pmatrix}$$
$$\overrightarrow{RS} \times \overrightarrow{RT} = \begin{vmatrix} 3 & -7 \\ -4 & -2 \end{vmatrix} \boldsymbol{i}$$
$$+ \begin{vmatrix} -4 & -2 \\ 9 & 6 \end{vmatrix} \boldsymbol{j}$$
$$+ \begin{vmatrix} 9 & 6 \\ 3 & -7 \end{vmatrix} \boldsymbol{k}$$
$$= \{3 \cdot (-2) - (-4) \cdot (-7)\}\boldsymbol{i}$$
$$+\{(-4) \cdot 6 - 9 \cdot (-2)\}\boldsymbol{j}$$
$$+\{9 \cdot (-7) - 3 \cdot 6\}\boldsymbol{k}$$
$$= \{-6 - 28\}\boldsymbol{i}$$
$$+\{-24 - (-18)\}\boldsymbol{j}$$
$$+\{-63 - 18)\}\boldsymbol{k}$$
$$= -34\boldsymbol{i} - 6\boldsymbol{j} - 81\boldsymbol{k}$$

c) the angle between the vectors $\overrightarrow{RS}$ and $\overrightarrow{RT}$.

$$\left|\overrightarrow{RS}\right| = \sqrt{9^2 + 3^2 + (-4)^2}$$
$$= \sqrt{81 + 9 + 16}$$
$$= \sqrt{106}$$
$$\left|\overrightarrow{RT}\right| = \sqrt{6^2 + (-7)^2 + (-2)^2}$$
$$= \sqrt{36 + 49 + 4}$$
$$= \sqrt{89}$$

$$\therefore \overrightarrow{RS} \cdot \overrightarrow{RT} = \left|\overrightarrow{RS}\right| \cdot \left|\overrightarrow{RT}\right| \cdot \cos\theta$$
$$\therefore 41 = \sqrt{106} \cdot \sqrt{89} \cdot \cos\theta$$
$$\therefore \frac{41}{\sqrt{106} \cdot \sqrt{89}} = \cos\theta$$
$$\therefore \theta = \cos^{-1} \frac{41}{\sqrt{106} \cdot \sqrt{89}} = 1.12363\text{radians}$$

21) Find a vector which is perpendicular to both of the vectors

$$\boldsymbol{c} = 5\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$$

and

$$\boldsymbol{d} = \boldsymbol{i} - 4\boldsymbol{j} - 6\boldsymbol{k}.$$

A vector which is orthogonal to $\boldsymbol{c}$ and $\boldsymbol{d}$ is $\boldsymbol{c} \times \boldsymbol{d}$.

$$\boldsymbol{c} = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}, \boldsymbol{d} = \begin{pmatrix} 1 \\ -4 \\ -6 \end{pmatrix}$$

$$\therefore \begin{pmatrix} \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & -4 \\ -3 & -6 \end{pmatrix}$$

Thus

$$\begin{aligned} &\boldsymbol{c}\times\boldsymbol{d} \\ =&\begin{vmatrix} 1 & -4 \\ -3 & -6 \end{vmatrix}\boldsymbol{i} \\ +&\begin{vmatrix} -3 & -6 \\ 5 & 1 \end{vmatrix}\boldsymbol{j} \\ +&\begin{vmatrix} 5 & 1 \\ 1 & -4 \end{vmatrix}\boldsymbol{k} \\ =&\{1\cdot(-6)-(-3)\cdot(-4)\}\boldsymbol{i} \\ &+\{-3\cdot(1)-5\cdot(-6)\}\boldsymbol{j} \\ &+\{5\cdot(-4)-1\cdot 1\}\boldsymbol{k} \\ =&\{-6-12\}\boldsymbol{i} \\ &+\{-3+30\}\boldsymbol{j} \\ &+\{-20-1\}\boldsymbol{k} \\ =&-18\boldsymbol{i}+27\boldsymbol{j}-21\boldsymbol{k} \end{aligned}$$

22) Points P, Q, and R have coordinates $(9,1,-2)$, $(3,1,3)$, and $(1,0,-1)$ respectively. Find $\overrightarrow{PQ}\times\overrightarrow{PR}$.

$$\begin{aligned} &\overrightarrow{PQ} \\ =&\overrightarrow{PO}+\overrightarrow{OQ} \\ =&-\overrightarrow{OP}+\overrightarrow{OQ} \\ =&-\begin{pmatrix} 9 \\ 1 \\ -2 \end{pmatrix}+\begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} \\ =&\begin{pmatrix} -9+3 \\ -1+1 \\ +2+3 \end{pmatrix} \\ =&\begin{pmatrix} -6 \\ 0 \\ 5 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &\overrightarrow{PR} \\ =&\overrightarrow{PO}+\overrightarrow{OR} \\ =&-\overrightarrow{OP}+\overrightarrow{OR} \\ =&-\begin{pmatrix} 9 \\ 1 \\ -2 \end{pmatrix}+\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ =&\begin{pmatrix} -9+1 \\ -1+0 \\ +2-1 \end{pmatrix} \\ =&\begin{pmatrix} -8 \\ -1 \\ 1 \end{pmatrix} \end{aligned}$$

$$\therefore\begin{pmatrix} \overrightarrow{PQ} & \overrightarrow{PR} \end{pmatrix}=\begin{pmatrix} -6 & -8 \\ 0 & -1 \\ 5 & 1 \end{pmatrix}$$

Thus

$$\begin{aligned} &\overrightarrow{PQ}\times\overrightarrow{PR} \\ =&\begin{vmatrix} 0 & -1 \\ 5 & 1 \end{vmatrix}\boldsymbol{i} \\ +&\begin{vmatrix} 5 & 1 \\ -6 & -8 \end{vmatrix}\boldsymbol{j} \\ +&\begin{vmatrix} -6 & -8 \\ 0 & -1 \end{vmatrix}\boldsymbol{k} \\ =&\{0\cdot(1)-(-1)\cdot 5\}\boldsymbol{i} \\ &+\{5\cdot(-8)-(1)\cdot(-6)\}\boldsymbol{j} \\ &+\{-6\cdot(-1)-(-8)\cdot(0)\}\boldsymbol{k} \end{aligned}$$

$$
\begin{aligned}
&= \{0+5\}\boldsymbol{i}\\
&+\{-40+6\}\boldsymbol{j}\\
&+\{6+0\}\boldsymbol{k}\\
&= 5\boldsymbol{i} - 34\boldsymbol{j} + 6\boldsymbol{k}
\end{aligned}
$$

23) Evaluate the vector product $\boldsymbol{p} \times \boldsymbol{q}$ if

$$\boldsymbol{p} = 3\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$$

and

$$\boldsymbol{q} = 7\boldsymbol{i} + 4\boldsymbol{j} - 8\boldsymbol{k}$$

$$\boldsymbol{p} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}, \boldsymbol{q} = \begin{pmatrix} 7 \\ 4 \\ -8 \end{pmatrix}$$

$$\therefore \begin{pmatrix} \boldsymbol{p} & \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ -2 & 4 \\ 5 & -8 \end{pmatrix}$$

Thus

$$
\begin{aligned}
&\boldsymbol{p} \times \boldsymbol{q}\\
&= \begin{vmatrix} -2 & 4 \\ 5 & -8 \end{vmatrix} \boldsymbol{i}\\
&+ \begin{vmatrix} 5 & -8 \\ 3 & 7 \end{vmatrix} \boldsymbol{j}\\
&+ \begin{vmatrix} 3 & 7 \\ -2 & 4 \end{vmatrix} \boldsymbol{k}\\
&= \{-2 \cdot (-8) - 4 \cdot 5\}\boldsymbol{i}\\
&+\{5 \cdot (7) - (-8) \cdot 3\}\boldsymbol{j}\\
&+\{3 \cdot (4) - (7) \cdot (-2)\}\boldsymbol{k}\\
&= \{16 - 20\}\boldsymbol{i}\\
&+\{35 + 24\}\boldsymbol{j}\\
&+\{12 + 14\}\boldsymbol{k}\\
&= -4\boldsymbol{i} + 59\boldsymbol{j} + 26\boldsymbol{k}
\end{aligned}
$$

24) Find the vector product of

$$\boldsymbol{p} = -2\boldsymbol{i} - 3\boldsymbol{j}$$

and

$$\boldsymbol{q} = 4\boldsymbol{i} + 7\boldsymbol{j}$$

$$\boldsymbol{p} = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix}, \boldsymbol{q} = \begin{pmatrix} 4 \\ 7 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} \boldsymbol{p} & \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ -3 & 7 \\ 0 & 0 \end{pmatrix}$$

Thus

$$
\begin{aligned}
&\boldsymbol{p}\times\boldsymbol{q}\\
&=\begin{vmatrix} -3 & 7 \\ 0 & 0 \end{vmatrix}\boldsymbol{i}\\
&+\begin{vmatrix} 0 & 0 \\ -2 & 4 \end{vmatrix}\boldsymbol{j}\\
&+\begin{vmatrix} -2 & 4 \\ -3 & 7 \end{vmatrix}\boldsymbol{k}\\
&=\{-3\cdot(0)-7\cdot 0\}\boldsymbol{i}\\
&+\{0\cdot(4)-(0)\cdot(-2)\}\boldsymbol{j}\\
&+\{-2\cdot(7)-(4)\cdot(-3)\}\boldsymbol{k}\\
&=\{0\}\boldsymbol{i}\\
&+\{0\}\boldsymbol{j}\\
&+\{-14+12\}\boldsymbol{k}\\
&=-2\boldsymbol{k}
\end{aligned}
$$

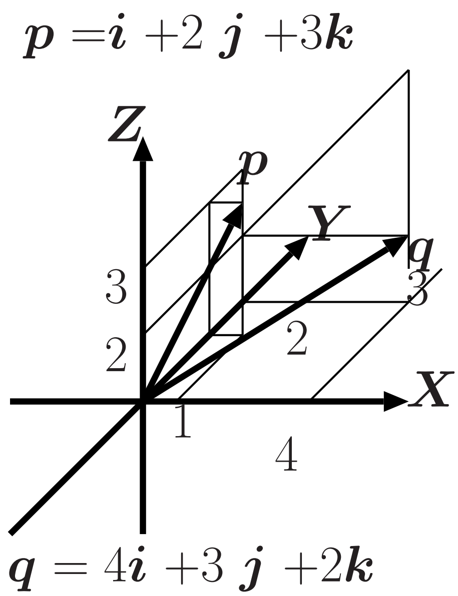
25) If

$$\boldsymbol{p}=\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}$$

and

$$\boldsymbol{q}=4\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k},$$

find $\boldsymbol{p}\times\boldsymbol{q}$ and $\boldsymbol{q}\times\boldsymbol{p}$



$$\boldsymbol{p}=\begin{pmatrix}1\\2\\3\end{pmatrix},\boldsymbol{q}=\begin{pmatrix}4\\3\\2\end{pmatrix}$$

$$\therefore\left(\begin{array}{cc}\boldsymbol{p} & \boldsymbol{q}\end{array}\right)=\begin{pmatrix}1 & 4\\2 & 3\\3 & 2\end{pmatrix}$$

Thus

$$
\begin{aligned}
&\boldsymbol{p}\times\boldsymbol{q}\\
&=\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}\boldsymbol{i}\\
&+\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}\boldsymbol{j}\\
&+\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}\boldsymbol{k}\\
&=\{2\cdot(2)-3\cdot 3\}\boldsymbol{i}
\end{aligned}
$$

$$
\begin{aligned}
&+\{3\cdot(4)-(1)\cdot(2)\}\boldsymbol{j}\\
&+\{1\cdot(3)-(2)\cdot(4)\}\boldsymbol{k}\\
&=\{4-9\}\boldsymbol{i}\\
&+\{12-2\}\boldsymbol{j}\\
&+\{3-8\}\boldsymbol{k}\\
&=-5\boldsymbol{i}+10\boldsymbol{j}-5\boldsymbol{k}
\end{aligned}
$$

$$
\therefore \begin{pmatrix} \boldsymbol{q} & \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \\ 2 & 3 \end{pmatrix}
$$

Thus

$$
\begin{aligned}
&\boldsymbol{p}\times\boldsymbol{q}\\
&=\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}\boldsymbol{i}\\
&+\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}\boldsymbol{j}\\
&+\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}\boldsymbol{k}\\
&=\{3\cdot(3)-2\cdot 2\}\boldsymbol{i}\\
&+\{2\cdot(1)-(4)\cdot(3)\}\boldsymbol{j}\\
&+\{4\cdot(2)-(1)\cdot(3)\}\boldsymbol{k}\\
&=\{9-4\}\boldsymbol{i}\\
&+\{2-12\}\boldsymbol{j}\\
&+\{8-3\}\boldsymbol{k}\\
&=5\boldsymbol{i}-10\boldsymbol{j}+5\boldsymbol{k}
\end{aligned}
$$

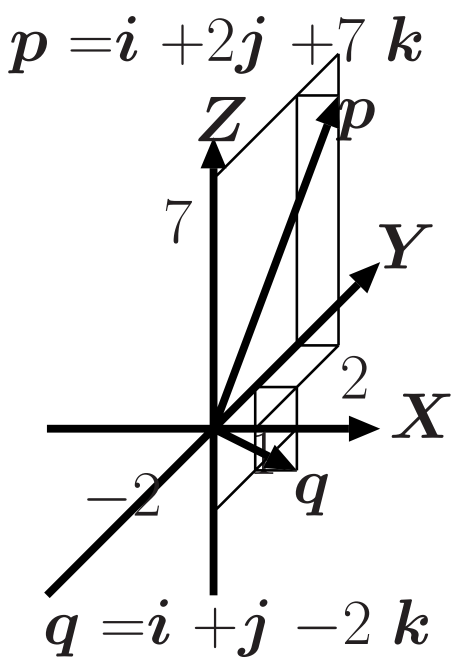
26) Find a vector which is perpendicular to both of the vectors

$$\boldsymbol{p}=\boldsymbol{i}+2\boldsymbol{j}+7\boldsymbol{k}$$

and

$$\boldsymbol{q}=\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}.$$

Hence find a unit vector which is perpendicular to both $\boldsymbol{p}$ and $\boldsymbol{q}$.



A vector which is orthogonal to $\boldsymbol{p}$ and $\boldsymbol{q}$ is $\boldsymbol{p}\times\boldsymbol{q}$.

$$
\boldsymbol{p}=\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}, \boldsymbol{q}=\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}
$$

$$\therefore \begin{pmatrix} \boldsymbol{p} & \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 7 & -2 \end{pmatrix}$$

Thus

$$\begin{aligned} &\boldsymbol{p} \times \boldsymbol{q} \\ &= \begin{vmatrix} 2 & 1 \\ 7 & -2 \end{vmatrix} \boldsymbol{i} \\ &+ \begin{vmatrix} 7 & -2 \\ 1 & 1 \end{vmatrix} \boldsymbol{j} \\ &+ \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \boldsymbol{k} \\ &= \{2 \cdot (-2) - 1 \cdot 7\}\boldsymbol{i} \\ &+\{7 \cdot (1) - (-2) \cdot (1)\}\boldsymbol{j} \\ &+\{1 \cdot (1) - (1) \cdot (2)\}\boldsymbol{k} \\ &= \{-4 - 7\}\boldsymbol{i} \\ &+\{7 + 2\}\boldsymbol{j} \\ &+\{1 - 2\}\boldsymbol{k} \\ &= -11\boldsymbol{i} + 9\boldsymbol{j} - \boldsymbol{k} \end{aligned}$$

In order to get the unit vector, we need the modulus of $-11\boldsymbol{i} + 9\boldsymbol{j} - \boldsymbol{k}$. The modulus is

$$\begin{aligned} &\sqrt{11^2 + 9^2 + (-1)^2} \\ &= \sqrt{121 + 81 + 1} \\ &= \sqrt{203} \end{aligned}$$

Thus the unit vector is

$$\frac{-11\boldsymbol{i} + 9\boldsymbol{j} - \boldsymbol{k}}{\sqrt{203}}.$$

27) For the vectors

$$\boldsymbol{p} = 4\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$$

$$\boldsymbol{q} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k},$$

and

$$\boldsymbol{r} = 3\boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k},$$

evaluate both $\boldsymbol{p} \times (\boldsymbol{q} \times \boldsymbol{r})$ and $(\boldsymbol{p} \times \boldsymbol{q}) \times \boldsymbol{r}$.

$$\boldsymbol{p} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \boldsymbol{q} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \boldsymbol{r} = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix}$$

$$\therefore \begin{pmatrix} \boldsymbol{q} & \boldsymbol{r} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -2 & -3 \\ 1 & 4 \end{pmatrix}$$

Thus

$$\begin{aligned} &\boldsymbol{q} \times \boldsymbol{r} \\ &= \begin{vmatrix} -2 & -3 \\ 1 & 4 \end{vmatrix} \boldsymbol{i} \\ &+ \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} \boldsymbol{j} \\ &+ \begin{vmatrix} 1 & 3 \\ -2 & -3 \end{vmatrix} \boldsymbol{k} \\ &= \{-2 \cdot (4) - (-3) \cdot 1\}\boldsymbol{i} \\ &+\{1 \cdot (3) - (4) \cdot (1)\}\boldsymbol{j} \\ &+\{1 \cdot (-3) - (3) \cdot (-2)\}\boldsymbol{k} \\ &= \{-8 + 3\}\boldsymbol{i} \\ &+\{3 - 4\}\boldsymbol{j} \\ &+\{-3 + 6\}\boldsymbol{k} \end{aligned}$$

$$= -5\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$$
$$= \begin{pmatrix} -5 \\ -1 \\ 3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} \boldsymbol{p} & \boldsymbol{q} \times \boldsymbol{r} \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 2 & -1 \\ 1 & 3 \end{pmatrix}$$

$$\therefore \boldsymbol{p} \times (\boldsymbol{q} \times \boldsymbol{r})$$
$$= \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} \boldsymbol{i}$$
$$+ \begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix} \boldsymbol{j}$$
$$+ \begin{vmatrix} 4 & -5 \\ 2 & -1 \end{vmatrix} \boldsymbol{k}$$
$$= \{2 \cdot (3) - (-1) \cdot 1\}\boldsymbol{i}$$
$$+\{1 \cdot (-5) - (3) \cdot (4)\}\boldsymbol{j}$$
$$+\{4 \cdot (-1) - (-5) \cdot (2)\}\boldsymbol{k}$$
$$= \{6 + 1\}\boldsymbol{i}$$
$$+\{-5 - 12\}\boldsymbol{j}$$
$$+\{-4 + 10\}\boldsymbol{k}$$
$$= 7\boldsymbol{i} - 17\boldsymbol{j} + 6\boldsymbol{k}$$

$$\therefore \begin{pmatrix} \boldsymbol{p} & \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & -2 \\ 1 & 1 \end{pmatrix}$$

Thus

$$\boldsymbol{p} \times \boldsymbol{q}$$
$$= \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} \boldsymbol{i}$$
$$+ \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} \boldsymbol{j}$$
$$+ \begin{vmatrix} 4 & 1 \\ 2 & -2 \end{vmatrix} \boldsymbol{k}$$
$$= \{2 \cdot (1) - (-2) \cdot 1\}\boldsymbol{i}$$
$$+\{1 \cdot (1) - (1) \cdot (4)\}\boldsymbol{j}$$
$$+\{4 \cdot (-2) - (1) \cdot (2)\}\boldsymbol{k}$$
$$= \{2 + 2\}\boldsymbol{i}$$
$$+\{1 - 4\}\boldsymbol{j}$$
$$+\{-8 - 2\}\boldsymbol{k}$$
$$= 4\boldsymbol{i} - 3\boldsymbol{j} - 10\boldsymbol{k}$$
$$= \begin{pmatrix} 4 \\ -3 \\ -10 \end{pmatrix}$$

$$\therefore \begin{pmatrix} \boldsymbol{p} \times \boldsymbol{q} & \boldsymbol{r} \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ -3 & -3 \\ -10 & 4 \end{pmatrix}$$

$$\therefore (\boldsymbol{p} \times \boldsymbol{q}) \times \boldsymbol{r}$$
$$= \begin{vmatrix} -3 & -3 \\ -10 & 4 \end{vmatrix} \boldsymbol{i}$$
$$+ \begin{vmatrix} -10 & 4 \\ 4 & 3 \end{vmatrix} \boldsymbol{j}$$
$$+ \begin{vmatrix} 4 & 3 \\ -3 & -3 \end{vmatrix} \boldsymbol{k}$$
$$= \{-3 \cdot (4) - (-3) \cdot (-10)\}\boldsymbol{i}$$

$$
\begin{aligned}
&+\{-10 \cdot (3) - (4) \cdot (4)\}\boldsymbol{j} \\
&+\{4 \cdot (-3) - (3) \cdot (-3)\}\boldsymbol{k} \\
&= \{-12 - 30\}\boldsymbol{i} \\
&+\{-30 - 16\}\boldsymbol{j} \\
&+\{-12 + 9\}\boldsymbol{k} \\
&= -42\boldsymbol{i} - 46\boldsymbol{j} - 3\boldsymbol{k}
\end{aligned}
$$

Thus, in general, the vector product is not associative.

C. DAY3

28) Solve the following equation for θ.

$$6 = 12\cos\theta$$

$$\begin{aligned} 6 &= 12\cos\theta \\ \therefore \frac{6}{12} &= \cos\theta \\ \therefore \frac{1}{2} &= \cos\theta \\ \therefore \theta = \cos^{-1}\left(\frac{1}{2}\right) &= \frac{\pi}{3} \end{aligned}$$

29) Expand and simplify

$$(p-3q)\cdot(2p-8q)$$

$$\begin{aligned} (p-3q)\cdot(2p-8q) &= 2p^2 - 8pq - 6pq + 24q^2 \\ &= 2p^2 - 14pq + 24q^2 \end{aligned}$$

30) Calculate the area of a triangle which has a base of 12λ and a height of 8λ.

$$\begin{aligned} A &= \frac{1}{2}(b\cdot h) \\ &= \frac{1}{2}(12\lambda\cdot 8\lambda) \\ &= \frac{1}{2}(96\lambda^2) \\ &= \frac{96\lambda^2}{2} \\ &= 48\lambda^2 \end{aligned}$$

31) Evaluate

$$(-2)^4 + 4|7-4(4-7)|$$

$$\begin{aligned} &(-2)^4 + 4|7-4(4-7)| \\ &= 16 + 4|7-4(-3)| \\ &= 16 + 4|7+12| \\ &= 16 + 4|19| \\ &= 16 + 76 \\ &= 92 \end{aligned}$$

32) Simplify

$$\frac{(c^5d^3)(c^3d^4)}{-5c^6d^3}$$

$$\begin{aligned} &\frac{(c^5d^3)(c^3d^4)}{-5c^6d^3} \\ &= \frac{c^{5+3}d^{3+4}}{-5c^6d^3} \\ &= \frac{c^8d^7}{-5c^6d^3} \\ &= \frac{c^{8-6}d^{7-3}}{-5} \\ &= \frac{c^2d^4}{-5} \end{aligned}$$

33) Simplify

$$15p - 2(6p^2 - 15p + 24) + 8p^2 + 20$$

$$
\begin{aligned}
&15p-2(6p^2-15p+24)+8p^2+20\\
&=15p-12p^2+30p-48+8p^2+20\\
&=45p-4p^2-28
\end{aligned}
$$

34) Solve the following equation for t.

$$(4-6t)\cdot(-2)+21s\cdot 0+4t\cdot 6=0$$

$$
\begin{aligned}
(4-6t)\cdot(-2)+21s\cdot 0+4t\cdot 6&=0\\
\therefore -8+12t+24t&=0\\
\therefore 36t-8&=0\\
\therefore 36t&=8\\
\therefore t&=\frac{8}{36}\\
&=\frac{2}{9}
\end{aligned}
$$

35) Simplify

$$\sqrt{\left(\frac{12}{15}\right)^2+\left(\frac{8}{15}\right)^2}$$

$$
\begin{aligned}
&\sqrt{\left(\frac{12}{15}\right)^2+\left(\frac{8}{15}\right)^2}\\
&=\sqrt{\frac{12^2}{15^2}+\frac{8^2}{15^2}}\\
&=\sqrt{\frac{12^2+8^2}{15^2}}\\
&=\sqrt{\frac{144+64}{15^2}}\\
&=\sqrt{\frac{208}{15^2}}\\
&=\frac{\sqrt{208}}{15}\\
&=\frac{\sqrt{16}\cdot\sqrt{13}}{15}\\
&=\frac{4\sqrt{13}}{15}
\end{aligned}
$$

36) Simplify

$$\boldsymbol{p}=\begin{pmatrix}1\\4\\-2\end{pmatrix}+s\left(-\begin{pmatrix}1\\4\\-2\end{pmatrix}+\begin{pmatrix}4\\2\\-3\end{pmatrix}\right)$$

$$
\begin{aligned}
\boldsymbol{p}&=\begin{pmatrix}1\\4\\-2\end{pmatrix}+s\left(-\begin{pmatrix}1\\4\\-2\end{pmatrix}+\begin{pmatrix}4\\2\\-3\end{pmatrix}\right)\\
&=\begin{pmatrix}1\\4\\-2\end{pmatrix}+s\begin{pmatrix}-1+4\\-4+2\\2-3\end{pmatrix}=\begin{pmatrix}1\\4\\-2\end{pmatrix}+s\begin{pmatrix}3\\-2\\-1\end{pmatrix}=\begin{pmatrix}1\\4\\-2\end{pmatrix}+\begin{pmatrix}3s\\-2s\\-s\end{pmatrix}=\begin{pmatrix}1+3s\\4-2s\\-2-s\end{pmatrix}
\end{aligned}
$$

37) Find the vector equation of the line passing through $P(9,1,2)$ and which is parallel to the vector $(1,2,3)$. Furthermore find the Cartesian equation of this line.
We assume t is a variable number. When $\boldsymbol{r}$ is on the line in question, $\boldsymbol{r}$ goes through $\boldsymbol{p}$ with the direction of $(1,2,3)$.

$$\boldsymbol{r}=\begin{pmatrix}9\\1\\2\end{pmatrix}+t\begin{pmatrix}1\\2\\3\end{pmatrix}$$

In Cartesian coordinates, $\boldsymbol{r}$ can be expressed as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. By substituting

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

into the equation above,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

This can be re-written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 9 \\ 1 \\ 2 \end{pmatrix} = t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Thus we obtain

$$\begin{aligned} x - 9 &= t \\ y - 1 &= 2t \\ z - 2 &= 3t \end{aligned}$$

These can be re-written as

$$\begin{aligned} x - 9 &= t \\ \frac{y-1}{2} &= t \\ \frac{z-2}{3} &= t \end{aligned}$$

By removing t from these equations,

$$x - 9 = \frac{y-1}{2} = \frac{z-2}{3}$$

38) Write down the vector equation of the line which passes through the points with position vectors

$$\boldsymbol{p} = 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$$

and

$$\boldsymbol{q} = 7\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}$$

. Also express the equation in column vector form.
We assume t is a variable number. When $\boldsymbol{r}$ is on the line in question, $\boldsymbol{r}$ goes through $\boldsymbol{p}$ or $\boldsymbol{q}$ with the direction of

$$\begin{aligned} &\overrightarrow{PQ} \\ &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= -\overrightarrow{OP} + \overrightarrow{OQ} \\ &= -\boldsymbol{p} + \boldsymbol{q} \end{aligned}$$

Thus using the variable t,

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{p} + t\overrightarrow{PQ} \\ &= \boldsymbol{p} + t(-\boldsymbol{p} + \boldsymbol{q}) \\ &= \boldsymbol{p} + t(-\boldsymbol{p} + \boldsymbol{q}) \\ &= 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} \\ &+t(7\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k} - (3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})) \\ &= 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} \\ &+t(7\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k} - 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) \\ &= 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} \\ &+t((7-3)\boldsymbol{i} + (5-2)\boldsymbol{j} + (1+1)\boldsymbol{k}) \\ &= 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} \\ &+t(4\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}) \end{aligned}$$

Using column vector notation, this can be re-written as

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + t \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

39) Write down the vector $\overrightarrow{PQ}$ joining the points P and Q with coordinates $(3, 2, 7)$ and $(-1, 2, 3)$ respectively. Then find the equation of the straight line through P and Q.
We assume t is a variable number. When $\boldsymbol{r}$ is on the line in question, $\boldsymbol{r}$ goes through $\boldsymbol{p}$ or $\boldsymbol{q}$ with the direction of

$$\begin{aligned} &\overrightarrow{PQ} \\ &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= -\overrightarrow{OP} + \overrightarrow{OQ} \\ &= -\boldsymbol{p} + \boldsymbol{q} \end{aligned}$$

Thus using the variable t,

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{p} + t\overrightarrow{PQ} \\ &= \boldsymbol{p} + t(\boldsymbol{q} - \boldsymbol{p}) \\ &= \boldsymbol{p} + t(\boldsymbol{q} - \boldsymbol{p}) \\ &= 3\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k} \\ +t(-\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} - (3\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k})) \\ &= 3\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k} \\ +t(-\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} - 3\boldsymbol{i} - 2\boldsymbol{j} - 7\boldsymbol{k}) \\ &= 3\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k} \\ +t((-1-3)\boldsymbol{i} + (2-2)\boldsymbol{j} + (3-7)\boldsymbol{k}) \\ &= 3\boldsymbol{i} + 2\boldsymbol{j} + 7\boldsymbol{k} \\ +t(-4\boldsymbol{i} - 4\boldsymbol{k}) \end{aligned}$$

Using column vector notation, this can be re-written as

$$\boldsymbol{r} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} + t\begin{pmatrix} -4 \\ 0 \\ -4 \end{pmatrix}$$

The vector $\overrightarrow{PQ}$ is $-4\boldsymbol{i} - 4\boldsymbol{k}$

40) There are four points $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, 3)$ and $D(2, 2, 3)$. The point P divides AB internally in the ratio $s : (1-s)$. The point Q divides PC internally in the ratio $t : (1-t)$. A line l is parallel to $\boldsymbol{v} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$, passing through D. Find s and t when Q is on the line l.
$\overrightarrow{OP} \triangleq \boldsymbol{p}$ is expressed using $\overrightarrow{OA} \triangleq \boldsymbol{a}$ and $\overrightarrow{OB} \triangleq \boldsymbol{b}$ as follows:

$$\begin{aligned} \boldsymbol{p} &= (1-s)\boldsymbol{a} + s\boldsymbol{b} \\ \therefore \boldsymbol{p} &= \boldsymbol{a} - s\boldsymbol{a} + s\boldsymbol{b} \\ \therefore \boldsymbol{p} &= \boldsymbol{a} + s(-\boldsymbol{a} + \boldsymbol{b}) \\ \therefore \boldsymbol{p} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\left(-\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right) \\ \therefore \boldsymbol{p} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\left(\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right) \\ \therefore \boldsymbol{p} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1+0 \\ 0+2 \\ 0+0 \end{pmatrix} \\ \therefore \boldsymbol{p} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \\ \therefore \boldsymbol{p} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \cdot s \\ 2 \cdot s \\ 0 \cdot s \end{pmatrix} \\ \therefore \boldsymbol{p} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -s \\ 2s \\ 0 \end{pmatrix} \\ \therefore \boldsymbol{p} &= \begin{pmatrix} 1-s \\ 0+2s \\ 0+0 \end{pmatrix} \\ \therefore \boldsymbol{p} &= \begin{pmatrix} 1-s \\ 2s \\ 0 \end{pmatrix} \end{aligned}$$

In the similar manner, $\overrightarrow{OQ} \triangleq \boldsymbol{q}$ is expressed using $\overrightarrow{OP} \triangleq \boldsymbol{p}$ and $\overrightarrow{OC} \triangleq \boldsymbol{c}$ as follows:

$$\boldsymbol{q} = (1-t)\boldsymbol{p} + t\boldsymbol{c}$$
$$\therefore \boldsymbol{q} = \boldsymbol{p} - t\boldsymbol{p} + t\boldsymbol{c}$$
$$\therefore \boldsymbol{q} = \boldsymbol{p} + t(-\boldsymbol{p} + \boldsymbol{c})$$
$$\therefore \boldsymbol{q} = \begin{pmatrix} 1-s \\ 2s \\ 0 \end{pmatrix} + t\left(-\begin{pmatrix} 1-s \\ 2s \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}\right)$$
$$\therefore \boldsymbol{q} = \begin{pmatrix} 1-s \\ 2s \\ 0 \end{pmatrix} + t\left(\begin{pmatrix} -(1-s) \\ -2s \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}\right)$$
$$\therefore \boldsymbol{q} = \begin{pmatrix} 1-s \\ 2s \\ 0 \end{pmatrix} + t\begin{pmatrix} -1+s+0 \\ -2s+0 \\ 0+3 \end{pmatrix}$$
$$\therefore \boldsymbol{q} = \begin{pmatrix} 1-s \\ 2s \\ 0 \end{pmatrix} + t\begin{pmatrix} -1+s \\ -2s \\ 3 \end{pmatrix}$$
$$\therefore \boldsymbol{q} = \begin{pmatrix} 1-s \\ 2s \\ 0 \end{pmatrix} + \begin{pmatrix} (-1+s)\cdot t \\ -2s\cdot t \\ 3\cdot t \end{pmatrix}$$
$$\therefore \boldsymbol{q} = \begin{pmatrix} 1-s \\ 2s \\ 0 \end{pmatrix} + \begin{pmatrix} -t+st \\ -2st \\ 3t \end{pmatrix}$$
$$\therefore \boldsymbol{q} = \begin{pmatrix} 1-s-t+st \\ 2s-2st \\ 0+3t \end{pmatrix}$$
$$\therefore \boldsymbol{q} = \begin{pmatrix} 1-s-t+st \\ 2s-2st \\ 3t \end{pmatrix}$$

Using an arbitrary variable u, the line l can be expressed as

$$\overrightarrow{OD} + u\boldsymbol{v}$$
$$= \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + u\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -u \\ -u \\ -u \end{pmatrix}$$
$$= \begin{pmatrix} 2-u \\ 2-u \\ 3-u \end{pmatrix}$$

Since Q is on the line l,

$$\boldsymbol{q} = \begin{pmatrix} 1-s-t+st \\ 2s-2st \\ 3t \end{pmatrix} \equiv \begin{pmatrix} 2-u \\ 2-u \\ 3-u \end{pmatrix}$$

Now we obtain three equations for three unknown variables.

$$\begin{aligned} 1 - s - t + st &= 2 - u \\ 2s - 2st &= 2 - u \\ 3t &= 3 - u \end{aligned}$$

Using these three equations, first we remove u as follows:

$$\begin{array}{rccc} & 2s - 2st & = & 2 - u \\ -) & 3t & = & 3 - u \\ \hline & 2s - 2st - 3t & = & 2 - u - (3 - u) \\ & 2s - 2st - 3t & = & 2 - u - 3 + u \\ & 2s - 2st - 3t & = & -1 \end{array}$$

and

$$\begin{array}{rccc} & 1 - s - t + st & = & 2 - u \\ -) & 2s - 2st & = & 2 - u \\ \hline & & & \\ & 1 - s - t + st - 2s + 2st & = & 0 \\ & 1 - 3s - t + 3st & = & 0 \end{array}$$

Second, using $1 - 3s - t + 3st = 0$ and $2s - 2st - 3t = -1$ we remove s as follows.

$$\begin{aligned} 1 - 3s - t + 3st &= 0 \\ \therefore s(-3 + 3t) + 1 - t &= 0 \\ \therefore s(-3 + 3t) &= -1 + t \end{aligned}$$

When $-3 + 3t = 0$, $3t = 3$, *i.e.,*$t = 1$. When $t = 1$ is put into $2s - 2st - 3t = -1$, $2s - 2s - 3 = -1$*i.e.,* $-3 = -1$ which is incorrect. Thus $t \neq 1$. Thus we can divide $s(-3 + 3t) = -1 + t$ by $-3 + 3t$.

$$\begin{aligned} \therefore s &= \frac{-1 + t}{-3 + 3t} \\ \therefore s &= \frac{-1 + t}{3(-1 + t)} \\ \therefore s &= \frac{-1 + t}{3(-1 + t)} \\ \therefore s &= \frac{1}{3} \end{aligned}$$

Substituting $s = \frac{1}{3}$ into $2s - 2st - 3t = -1$, we get

$$\begin{aligned} 2 \cdot \frac{1}{3} - 2t \cdot \frac{1}{3} - 3t &= -1 \\ \therefore 2 - 2t - 3t \cdot 3 &= -1 \cdot 3 \\ \therefore 2 - 2t - 9t &= -3 \\ \therefore -11t &= -3 - 2 \\ \therefore -11t &= -5 \\ \therefore t &= \frac{5}{11} \end{aligned}$$

D. DAY4

41) Solve the following simultaneous linear equations

$$2s - 3t = 9 \qquad ①$$

and

$$3s - 18t = 21 \qquad ②$$

①×6

$$\begin{aligned} 12s - 18t &= 54 \qquad ③ \\ 3s - 18t &= 21 \qquad ② \end{aligned}$$

③ − ②

$$\begin{aligned} 9s &= 33 \\ s &= \frac{33}{9} \\ &= \frac{11}{3} \end{aligned}$$

Sub $s = \frac{11}{3}$ into ①

$$\begin{aligned} 2\left(\frac{11}{3}\right) - 3t &= 9 \\ \therefore \frac{22}{3} - 3t &= 9 \\ \therefore \frac{22}{3} - 9 &= 3t \\ \therefore \frac{22}{3} - \frac{27}{3} &= 3t \\ \therefore \frac{-5}{3} &= 3t \\ \therefore \frac{-5}{3} \times \frac{1}{3} &= t \\ \therefore t &= \frac{-5}{9} \end{aligned}$$

42) Solve the following equation for t in terms of c.

$$-5 + 8\left(\frac{6-4t}{14}\right) + (3+c^2)t = 0$$

$$\begin{aligned} -5 + 8\left(\frac{6-4t}{14}\right) + (3+c^2)t &= 0 \\ \therefore -5 + \frac{48-32t}{14} + 3t + c^2t &= 0 \\ \therefore -5 + \frac{24-16t}{7} + 3t + c^2t &= 0 \\ \therefore -5 + \frac{24}{7} - \frac{16}{7}t + 3t + c^2t &= 0 \\ \therefore -\frac{35}{7} + \frac{24}{7} - \frac{16}{7}t + \frac{21}{7}t + c^2t &= 0 \\ \therefore -\frac{11}{7} + \frac{5}{7} + c^2t &= 0 \\ \therefore -\frac{11}{7} + t\left(c^2 + \frac{5}{7}\right) &= 0 \\ \therefore t\left(c^2 + \frac{5}{7}\right) &= \frac{11}{7} \\ \therefore t &= \frac{11}{7}\left(\frac{1}{c^2 + \frac{5}{7}}\right) \\ &= \frac{11}{7\left(c^2 + \frac{5}{7}\right)} \\ &= \frac{11}{7c^2 + 5} \end{aligned}$$

43) Simplify

$$r \cdot (r+3) + r \cdot (r-1) + (-r) \cdot (-r+5) = \sqrt{3r^2(3r^2+6)} \cdot \frac{\sqrt{3}}{2}$$

$$\begin{aligned}
r \cdot (r+3) + r \cdot (r-1) + (-r) \cdot (-r+5) &= \sqrt{3r^2(3r^2+6)} \cdot \frac{\sqrt{3}}{2} \\
r^2 + 3r + r^2 - r + r^2 - 5r &= \sqrt{3r^2(3r^2+6)} \cdot \frac{\sqrt{3}}{2} \\
3r^2 - 3r &= \sqrt{3r^2(3r^2+6)} \cdot \frac{\sqrt{3}}{2} \\
\frac{2}{\sqrt{3}} \cdot (3r^2 - 3r) &= \sqrt{3r^2(3r^2+6)} \\
\left(\frac{2}{\sqrt{3}} \cdot (3r^2 - 3r)\right)^2 &= 3r^2(3r^2+6) \\
\left(\frac{2}{\sqrt{3}}\right)^2 \cdot (3r^2 - 3r)^2 &= 3r^2(3r^2+6) \\
\frac{4}{3} \cdot ((3r^2 - 3r)(3r^2 - 3r)) &= 3r^2(3r^2+6) \\
\frac{4}{3} \cdot (9r^4 - 18r^3 + 9r^2) &= 3r^2(3r^2+6) \\
4r^2(3r^2 - 6r + 3) &= 3r^2(3r^2+6) \\
12r^2(r^2 - 2r + 1) &= 3r^2(3r^2+6) \\
12(r^2 - 2r + 1) &= 3(3r^2+6) \\
12r^2 - 24r + 12 &= 9r^2 + 18 \\
3r^2 - 24r - 6 &= 0 \\
r^2 - 8r - 2 &= 0
\end{aligned}$$

44) Solve the following equation

$$8 - 2(1+7x) - 6(-7-x) = 36$$

$$\begin{aligned}
8 - 2(1+7x) - 6(-7-x) &= 36 \\
\therefore 8 - 2 - 14x + 42 + 6x &= 36 \\
\therefore -14x + 6x &= 36 - 8 + 2 - 42 \\
\therefore -8x &= -12 \\
\therefore x &= \frac{-12}{-8} = \frac{3}{2}
\end{aligned}$$

45) Solve the following inequality

$$2 + 3(x-1) < -2x + 4$$

$$\begin{aligned}
2 + 3(x-1) &< -2x + 4 \\
\therefore 2 + 3x - 3 &< -2x + 4 \\
\therefore 3x + 2x &< 4 - 2 + 3 \\
\therefore 5x &< 5 \\
\therefore x &< 1
\end{aligned}$$

46) A line l is parallel to $\boldsymbol{p} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$, passing through $A(4,0,-2)$. A line m is parallel to $\boldsymbol{q} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, passing through $B(0,a,-4)$. Find a when l and m are crossing each other, the Cartesian coordinate of the crossing point and $\cos\theta$, where θ is the angle of l and m $(0 \leq \theta \leq \frac{\pi}{2})$.

Using an arbitrary variable s, the line l can be expressed as

$$\begin{aligned} &\overrightarrow{OA} + s\boldsymbol{p} \\ = &\begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} + s\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \\ = &\begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \cdot s \\ -3 \cdot s \\ -1 \cdot s \end{pmatrix} \\ = &\begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 2s \\ -3s \\ -s \end{pmatrix} \\ = &\begin{pmatrix} 4+2s \\ 0-3s \\ -2-s \end{pmatrix} \\ = &\begin{pmatrix} 4+2s \\ -3s \\ -2-s \end{pmatrix} \end{aligned}$$

Using an arbitrary variable t, the line m can be expressed as

$$\begin{aligned} &\overrightarrow{OB} + t\boldsymbol{q} \\ = &\begin{pmatrix} 0 \\ a \\ -4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \\ = &\begin{pmatrix} 0 \\ a \\ -4 \end{pmatrix} + \begin{pmatrix} 2 \cdot t \\ 1 \cdot t \\ 3 \cdot t \end{pmatrix} \\ = &\begin{pmatrix} 0 \\ a \\ -4 \end{pmatrix} + \begin{pmatrix} 2t \\ t \\ 3t \end{pmatrix} \\ = &\begin{pmatrix} 0+2t \\ a+t \\ -4+3t \end{pmatrix} \\ = &\begin{pmatrix} 2t \\ a+t \\ -4+3t \end{pmatrix} \end{aligned}$$

When l and m are crossing, there exists s and t which satisfy

$$\begin{pmatrix} 4+2s \\ -3s \\ -2-s \end{pmatrix} = \begin{pmatrix} 2t \\ a+t \\ -4+3t \end{pmatrix}$$

From this, we obtain the following 3 equations:

$$\begin{aligned} 4+2s &= 2t \\ -3s &= a+t \\ -2-s &= -4+3t \\ \therefore -s &= -2+3t \\ \therefore s &= 2-3t \end{aligned}$$

Using the first and third equations, we can obtain s and t. $s = 2 - 3t$ is substituted into the first equation:

$$\begin{aligned} 4+2s &= 2t \\ \therefore 4+2(2-3t) &= 2t \\ \therefore 4+4-6t &= 2t \\ \therefore 8 &= 2t+6t \\ \therefore 8 &= 8t \\ \therefore 1 &= t \end{aligned}$$

Substituting $t = 1$ into $s = 2 - 3t$, we obtain

$$s = 2 - 3 \cdot 1 = 2 - 3 = -1$$

By substituting $(s,t)=(-1,1)$ into $-3s=a+t$

$$\begin{aligned} -3s &= a+t \\ \therefore -3\cdot(-1) &= a+1 \\ \therefore 3 &= a+1 \\ \therefore 3-1 &= a \\ \therefore a &= 2 \end{aligned}$$

Thus the Cartesian coordinate of the crossing point is

$$\begin{aligned} &\begin{pmatrix} 4+2s \\ -3s \\ -2-s \end{pmatrix} \\ = &\begin{pmatrix} 4+2\cdot(-1) \\ -3\cdot(-1) \\ -2-(-1) \end{pmatrix} \\ = &\begin{pmatrix} 4-2 \\ 3 \\ -2+1 \end{pmatrix} \\ = &\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \end{aligned}$$

Since $\cos\theta = \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}|\cdot|\boldsymbol{q}|}$, we first find $\boldsymbol{p}\cdot\boldsymbol{q}$ and the modulus of $\boldsymbol{p}$ and $\boldsymbol{q}$.

$$\begin{aligned} \boldsymbol{p}\cdot\boldsymbol{q} &= \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}\cdot\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \\ &= 2\cdot 2+(-3)\cdot 1+(-1)\cdot 3 \\ &= 4-3-3=-2 \end{aligned}$$

$$\begin{aligned} |\boldsymbol{p}| &= \sqrt{2^2+(-3)^2+(-1)^2} = \sqrt{4+9+1} = \sqrt{14} \\ |\boldsymbol{q}| &= \sqrt{2^2+(1)^2+(3)^2} = \sqrt{4+1+9} = \sqrt{14} \end{aligned}$$

Therefore

$$\begin{aligned} \cos\theta &= \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}|\cdot|\boldsymbol{q}|} \\ &= \frac{-2}{\sqrt{14}\sqrt{14}} \\ &= \frac{-2}{14} \\ &= \frac{-1}{7} < 0 \end{aligned}$$

θ we can get from $\cos\theta=-1/7$ is $\pi/2 \le \theta = 1.71414 \le \pi$. However we need to find θ which is $0\le\theta\le\pi/2$. When θ is an angle of two lines, $\pi-\theta$ is also the angle between two lines. Therefore we need

$$\pi-\theta = 3.14159-1.71414 = 1.427448758$$

radians.

47) There are three points $A(-1,1,1)$, $B(1,-1,1)$, $C(1,1,-1)$. Prove that the triangle ABC is the equilateral triangle. Find a unit vector $\hat{\boldsymbol{n}}$ which is orthogonal to $\overrightarrow{AB}$ and $\overrightarrow{AC}$

First we find out $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AC}$ to find the length of AB, BC, AC.

$$\begin{aligned} &\overrightarrow{AB} \\ &= \overrightarrow{AO}+\overrightarrow{OB} \\ &= -\overrightarrow{OA}+\overrightarrow{OB} \\ &= -\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}+\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}+\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+1 \\ -1-1 \\ -1+1 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AC}$$

$$= \overrightarrow{AO} + \overrightarrow{OC}$$

$$= -\overrightarrow{OA} + \overrightarrow{OC}$$

$$= -\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+1 \\ -1+1 \\ -1-1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

$$\overrightarrow{BC}$$

$$= \overrightarrow{BO} + \overrightarrow{OC}$$

$$= -\overrightarrow{OB} + \overrightarrow{OC}$$

$$= -\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1+1 \\ 1+1 \\ -1-1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$$

The modulus of these three vectors are calculated as follows:

$$|\overrightarrow{AB}| = \left| \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \right|$$

$$= \sqrt{2^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8}$$

$$|\overrightarrow{AC}| = \left| \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \right|$$

$$= \sqrt{2^2 + (-2)^2} = \sqrt{8}$$

$$|\overrightarrow{BC}| = \left| \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \right|$$

$$= \sqrt{2^2 + (-2)^2} = \sqrt{8}$$

Thus $|\overrightarrow{AB}| = |\overrightarrow{AC}| = |\overrightarrow{BC}|$.

- Solution 1 Since $\hat{\boldsymbol{n}} \perp \overrightarrow{AB}$ and $\hat{\boldsymbol{n}} \perp \overrightarrow{AC}$, $\hat{\boldsymbol{n}}$ is parallel to $\overrightarrow{AB} \times \overrightarrow{AC}$.Thus first we find $\overrightarrow{AB} \times \overrightarrow{AC}$.

$$\begin{pmatrix} \overrightarrow{AB} & \overrightarrow{AC} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} \boldsymbol{i}$$

$$+ \begin{vmatrix} 0 & -2 \\ 2 & 2 \end{vmatrix} \boldsymbol{j}$$

$$+\begin{vmatrix} 2 & 2 \\ -2 & 0 \end{vmatrix}\boldsymbol{k}$$
$$= 4\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}$$

The modulus of $4\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}$ is $\sqrt{4^2 + 4^2 + 4^2} = 4\sqrt{3}$. Thus

$$\hat{\boldsymbol{n}} = \frac{4\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k}}{4\sqrt{3}} = \frac{\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}}{\sqrt{3}}$$

- Solution 2 When we define $\hat{\boldsymbol{n}} \triangleq \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ where $|\hat{\boldsymbol{n}}|^2 = p^2 + q^2 + r^2 = 1$ ($\because \hat{\boldsymbol{n}}$ is a unit vector), $\hat{\boldsymbol{n}}$ satisfies $\hat{\boldsymbol{n}} \perp \overrightarrow{AB}$ and $\hat{\boldsymbol{n}} \perp \overrightarrow{AC}$.Thus

$$\begin{aligned} \hat{\boldsymbol{n}} &\perp \overrightarrow{AB} \\ \therefore \begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} &= 0 \\ \therefore 2p - 2q + 0 \cdot r &= 0 \\ \therefore 2p - 2q &= 0 \\ \therefore 2p &= 2q \\ \therefore p &= q \end{aligned}$$

and

$$\begin{aligned} \hat{\boldsymbol{n}} &\perp \overrightarrow{AC} \\ \therefore \begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} &= 0 \\ \therefore 2p + q \cdot 0 - 2r &= 0 \\ \therefore 2p - 2r &= 0 \\ \therefore 2p &= 2r \\ \therefore p &= r \end{aligned}$$

We now find $p = q = r$. Since

$$p^2 + q^2 + r^2 = 1,$$

by substituting $q = p$,$r = p$ into this equation,

$$\begin{aligned} p^2 + p^2 + p^2 = 3p^2 &= 1 \\ \therefore p^2 &= \frac{1}{3} \\ \therefore p = \pm\sqrt{\frac{1}{3}} &= \pm\frac{1}{\sqrt{3}} \end{aligned}$$

Thus

$$\begin{aligned} \hat{\boldsymbol{n}} &= \begin{pmatrix} p \\ q \\ r \end{pmatrix} \\ &= \begin{pmatrix} p \\ p \\ p \end{pmatrix} \\ &= p\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \pm\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

48) There are three points $A(1, 0, 0)$, $B(1, 2, 0)$, $C(0, 2, 2)$. A point P is on a line which goes through O and B. A point Q is on a line which goes through A and C. $\overrightarrow{PQ}$ is parallel to $\hat{\boldsymbol{n}}$.

a) Find a unit vector $\hat{\boldsymbol{n}}$ which is orthogonal to $\overrightarrow{OB}$ and $\overrightarrow{AC}$ and the x component of $\hat{\boldsymbol{n}}$ is positive.

$$\overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

and

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC} \\ &= -\overrightarrow{OA} + \overrightarrow{OC} \\ &= -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -1+0 \\ 0+2 \\ 0+2 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}\end{aligned}$$

The vector which is orthogonal to $\overrightarrow{OB}$ and $\overrightarrow{AC}$ is $\overrightarrow{OB} \times \overrightarrow{AC}$ which we find as follows:
Using

$$\begin{pmatrix} \overrightarrow{OB} & \overrightarrow{AC} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 0 & 2 \end{pmatrix}$$

we get

$$\begin{aligned}&\overrightarrow{OB} \times \overrightarrow{AC} \\ &= \begin{vmatrix} 2 & 2 \\ 0 & 2 \end{vmatrix} \boldsymbol{i} \\ &+ \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} \boldsymbol{j} \\ &+ \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} \boldsymbol{k} \\ &= \{2 \cdot (2) - (2) \cdot 0\}\boldsymbol{i} \\ &+\{0 \cdot (-1) - (2) \cdot (1)\}\boldsymbol{j} \\ &+\{1 \cdot (2) - (-1) \cdot (2)\}\boldsymbol{k} \\ &= \{4\}\boldsymbol{i} + \{-2\}\boldsymbol{j} + \{4\}\boldsymbol{k}\end{aligned}$$

Thus we can get the modulus of $\overrightarrow{OB} \times \overrightarrow{AC}$ as
$\sqrt{4^2 + 2^2 + 4^2} = 6$. Therefore the unit vector of $\overrightarrow{OB} \times \overrightarrow{AC}$ is

$$\frac{4\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}}{6} = \frac{2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}}{3}$$

Alternative method:

When we define $\hat{\boldsymbol{n}} \triangleq \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ where $|\hat{\boldsymbol{n}}|^2 = p^2 + q^2 + r^2 = 1$ ($\because \hat{\boldsymbol{n}}$ is a unit vector), $\hat{\boldsymbol{n}}$ satisfies $\hat{\boldsymbol{n}} \perp \overrightarrow{OB}$ and $\hat{\boldsymbol{n}} \perp \overrightarrow{AC}$. Thus

$$\begin{aligned}&\hat{\boldsymbol{n}} \cdot \overrightarrow{OB} \\ &= \begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \\ &= p + 2q + r \cdot 0 \\ &= p + 2q = 0 \\ &\therefore p = -2q\end{aligned}$$

$$\begin{aligned}&\hat{\boldsymbol{n}} \cdot \overrightarrow{AC} \\ &= \begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \\ &= -p + 2q + 2r = 0\end{aligned}$$

When $p=-2q$ is substituted into $-p+2q+2r=0$,

$$\begin{aligned} -p+2q+2r&=0\\ \therefore -(-2q)+2q+2r&=0\\ \therefore 2q+2q+2r&=0\\ \therefore 4q+2r&=0\\ \therefore 2q+r&=0\\ \therefore r&=-2q \end{aligned}$$

By substituting $r=-2q$ and $p=-2q$ into

$$p^2+q^2+r^2=1$$

, we get

$$\begin{aligned} p^2+q^2+r^2&=1\\ \therefore (-2q)^2+q^2+(-2q)^2&=1\\ \therefore 4q^2+q^2+4q^2&=1\\ \therefore 9q^2&=1\\ \therefore q^2&=\frac{1}{9}\\ \therefore q=\pm\sqrt{\frac{1}{9}}&=\pm\frac{1}{3} \end{aligned}$$

Thus $\hat{\boldsymbol{n}}$ is obtained as follows:

$$\begin{aligned} \hat{\boldsymbol{n}}&=\begin{pmatrix} p\\ q\\ r \end{pmatrix}\\ &=\begin{pmatrix} -2q\\ q\\ -2q \end{pmatrix}\\ &=-q\begin{pmatrix} 2\\ -1\\ 2 \end{pmatrix}\\ &=\mp\frac{1}{3}\begin{pmatrix} 2\\ -1\\ 2 \end{pmatrix} \end{aligned}$$

Since x component of $\hat{\boldsymbol{n}}$ is supposed to be positive,

$$\hat{\boldsymbol{n}}=\frac{1}{3}\begin{pmatrix} 2\\ -1\\ 2 \end{pmatrix}$$

b) Express $\overrightarrow{OP}$ and $\overrightarrow{AQ}$ using $\overrightarrow{OB}$ and $\overrightarrow{AC}$.
Since P is on the line OB,

$$\begin{aligned} \overrightarrow{OP}&=s\overrightarrow{OB}\\ &=s\begin{pmatrix} 1\\ 2\\ 0 \end{pmatrix}\\ &=\begin{pmatrix} 1\cdot s\\ 2\cdot s\\ 0\cdot s \end{pmatrix}\\ &=\begin{pmatrix} s\\ 2s\\ 0 \end{pmatrix} \end{aligned}$$

where s is an arbitrary real number. Since Q is on the line AC,

$$\begin{aligned} \overrightarrow{OQ}&=\overrightarrow{OA}+t\overrightarrow{AC}\\ &=\begin{pmatrix} 1\\ 0\\ 0 \end{pmatrix}+t\begin{pmatrix} -1\\ 2\\ 2 \end{pmatrix}\\ &=\begin{pmatrix} 1\\ 0\\ 0 \end{pmatrix}+\begin{pmatrix} -1\cdot t\\ 2\cdot t\\ 2\cdot t \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -t \\ 2t \\ 2t \end{pmatrix}$$
$$= \begin{pmatrix} 1-t \\ 2t \\ 2t \end{pmatrix}$$

where t is an arbitrary real number. Thus $\overrightarrow{PQ}$ can be expressed as

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$$
$$= -\overrightarrow{OP} + \overrightarrow{OQ}$$
$$= -\begin{pmatrix} s \\ 2s \\ 0 \end{pmatrix} + \begin{pmatrix} 1-t \\ 2t \\ 2t \end{pmatrix}$$
$$= \begin{pmatrix} -s \\ -2s \\ 0 \end{pmatrix} + \begin{pmatrix} 1-t \\ 2t \\ 2t \end{pmatrix}$$
$$= \begin{pmatrix} -s+1-t \\ -2s+2t \\ 2t \end{pmatrix}$$

Since $\overrightarrow{PQ}$ is parallel to $\hat{\boldsymbol{n}}$, we can express $\overrightarrow{PQ}$ using an arbitrary real variable u as $\overrightarrow{PQ} = u\hat{\boldsymbol{n}}$

$$\begin{pmatrix} -s+1-t \\ -2s+2t \\ 2t \end{pmatrix} = u \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$
$$\therefore \begin{pmatrix} -s+1-t \\ -2s+2t \\ 2t \end{pmatrix} = \begin{pmatrix} 2 \cdot u \\ -1 \cdot u \\ 2 \cdot u \end{pmatrix}$$
$$\therefore \begin{pmatrix} -s+1-t \\ -2s+2t \\ 2t \end{pmatrix} = \begin{pmatrix} 2u \\ -u \\ 2u \end{pmatrix}$$

The above can be obtained using

$$\overrightarrow{OP} = s\overrightarrow{OB}$$
$$\overrightarrow{AQ} = t\overrightarrow{AC}$$
$$\overrightarrow{PQ} = u \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$
$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -s\overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{AQ} = -s\overrightarrow{OB} + \overrightarrow{OA} + t\overrightarrow{AC} = u \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

From this, we can get three equations for three variables:

$$-s+1-t = 2u$$
$$-2s+2t = -u$$
$$2t = 2u$$

From the third equation of $2t = 2u$, we get $u = t$. When we substituting $u = t$ into the first two equations,

$$-s+1-t = 2t$$
$$-2s+2t = -t$$

Therefore

$$-s+1 = 2t+t$$
$$-2s = -t-2t$$

Therefore

$$-s+1 = 3t$$
$$-2s = -3t$$

Now by adding these two equations:

$$\begin{array}{rrcr} & -s+1 & = & 3t \\ +) & -2s & = & -3t \\ \hline & -s+1-2s & = & 0 \\ \therefore & 1-3s & = & 0 \end{array}$$

$$\therefore 1 = 3s$$
$$\therefore \frac{1}{3} = s$$

Thus

$$\overrightarrow{OP} = \frac{1}{3}\overrightarrow{OB}.$$

By substituting $s = 1/3$ into $2s = 3t$

$$\begin{aligned} 2s &= 3t \\ \therefore 2 \cdot \frac{1}{3} &= 3t \\ \therefore 2 \cdot \frac{1}{3} \cdot \frac{1}{3} &= t \\ \therefore 2 \cdot \frac{1}{9} &= t \\ \therefore \frac{2}{9} &= t = u \end{aligned}$$

Since we found out $t = u = \frac{2}{9}$, we can now get $\overrightarrow{AQ}$

$$\begin{aligned} \overrightarrow{AQ} &= \overrightarrow{AO} + \overrightarrow{OQ} \\ &= -\overrightarrow{OA} + \overrightarrow{OQ} \\ &= -\overrightarrow{OA} + \overrightarrow{OA} + t\overrightarrow{AC} \\ &= t\overrightarrow{AC} \\ &= \frac{2}{9}\overrightarrow{AC} \end{aligned}$$

E. DAY5

49) Find a and c which satisfy

$$\begin{aligned} 4-a+3c &= 0 \\ -3-c+a &= 0 \end{aligned}$$

$$\begin{array}{rcl} & 4-a+3c & =0 \\ +) & -3-c+a & =0 \\ \hline & 4-a+3c-3-c+a & =0 \\ \therefore & 1+2c & =0 \\ \therefore & 2c & =-1 \\ \therefore & c & =-1/2 \end{array}$$

By substituting $c=-1/2$ into $-3-c+a=0$

$$\begin{aligned} -3-c+a &= 0 \\ \therefore -3-(-\frac{1}{2})+a &= 0 \\ \therefore -3+\frac{1}{2}+a &= 0 \\ \therefore -3\cdot\frac{2}{2}+\frac{1}{2}+a &= 0 \\ \therefore \frac{-3\cdot 2}{2}+\frac{1}{2}+a &= 0 \\ \therefore \frac{-6}{2}+\frac{1}{2}+a &= 0 \\ \therefore \frac{-6+1}{2}+a &= 0 \\ \therefore \frac{-5}{2}+a &= 0 \\ \therefore a &= \frac{5}{2} \end{aligned}$$

50) Find p and q which satisfy

$$\begin{aligned} 0 &= 1+q \\ 2p &= 3-2q \\ p &= -1+q \end{aligned}$$

From the first equation

$$0=1+q \therefore -1=q$$

By substituting $q=-1$ into the second and third equations:

$$\begin{aligned} 2p &= 3-2\cdot(-1) \\ p &= -1+(-1) \end{aligned}$$

Therefore

$$\begin{aligned} 2p &= 3+2 \\ p &= -1-1 \end{aligned}$$

Thus

$$\begin{aligned} 2p &= 5 \\ p &= -2 \end{aligned}$$

In other words

$$\begin{aligned} p &= 5/2 \\ p &= -2 \end{aligned}$$

Since these two equations give the different value for p there is no p and q which satisfy those three equations.

51) Find p,q and r which satisfy

$$\begin{aligned} 1+q &= 4r && ① \\ 3-2q-2p &= r && ② \\ -1+q-p &= -2r && ③ \end{aligned}$$

In order to solve simultaneous equations by hand, the golden approach is to reduce the number of variables in equations. ① can be modified as

$$q = 4r - 1 \qquad ④$$

When we substitute ④ into ② we obtain

$$\begin{aligned} 3 - 2(4r-1) - 2p &= r \\ \therefore 3 - 8r + 2 - 2p &= r \\ \therefore 5 - 9r - 2p &= 0 \qquad ⑤ \end{aligned}$$

When we substitute ④ into ③ we obtain

$$\begin{aligned} -1 + q - p &= -2r \\ \therefore -1 + 4r - 1 - p &= -2r \\ \therefore -2 + 6r - p &= 0 \qquad ⑥ \end{aligned}$$

⑥ is re-written as

$$p = -2 + 6r \qquad ⑦$$

By substituting ⑦ into ⑤ we obtain

$$\begin{aligned} 5 - 9r - 2p &= 0 \\ \therefore 5 - 9r - 2(-2+6r) &= 0 \\ \therefore 5 - 9r + 4 - 12r &= 0 \\ \therefore 9 - 21r &= 0 \\ \therefore r = \frac{9}{21} &= \frac{3}{7} \qquad ⑧ \end{aligned}$$

By substituting ⑧ into ⑦ and ④ we obtain

$$\begin{aligned} p = -2 + 6r = -2 + 6 \cdot \frac{3}{7} = \frac{-14+18}{7} &= \frac{4}{7} \\ q = 4r - 1 = 4 \cdot \frac{3}{7} - 1 = \frac{12-7}{7} &= \frac{5}{7} \end{aligned}$$

52) A line l is parallel to $\boldsymbol{l} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, passing through the origin. A line m is parallel to $\boldsymbol{m} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, passing through $(1, 3, 0)$. The point A is on the line l. The point G moves on the line m and at the point B, the length of $|AG|$ becomes minimum. The point C is on the line m. The point F moves on the line l and at the point D, the length of $|CF|$ becomes minimum. Find the Cartesian coordinate of A and C and the length of AC when $A = D$ and $B = C$.

A point A on the line l can be expressed as

$$\overrightarrow{OA} = a \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

where a is an arbitrary variable.

A point F on the line l can be expressed as

$$\overrightarrow{OF} = f \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

where f is an arbitrary variable. f becomes d for the point D.

A point G on the line m can be expressed as

$$\overrightarrow{OG} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + g \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

where g is an arbitrary variable. g becomes b for the point B.

A point C on the line m can be expressed as

$$\overrightarrow{OC} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

where c is an arbitrary variable.
Since $|AG|$ is minimum at the point B, $\overrightarrow{AB} \perp \boldsymbol{m}$. This can be expressed as

$$\begin{aligned}
\overrightarrow{AB} &\perp \boldsymbol{m} \\
\therefore \overrightarrow{AB} \cdot \boldsymbol{m} &= 0 \\
\therefore (\overrightarrow{AO} + \overrightarrow{OB}) \cdot \boldsymbol{m} &= 0 \\
\therefore (-\overrightarrow{OA} + \overrightarrow{OB}) \cdot \boldsymbol{m} &= 0 \\
\therefore \left(-a\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}1\\3\\0\end{pmatrix} + b\begin{pmatrix}1\\1\\-1\end{pmatrix}\right) \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} &= 0 \\
\therefore \left(\begin{pmatrix}0\cdot(-a)\\1\cdot(-a)\\0\cdot(-a)\end{pmatrix} + \begin{pmatrix}1\\3\\0\end{pmatrix} + \begin{pmatrix}1\cdot b\\1\cdot b\\-1\cdot b\end{pmatrix}\right) \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} &= 0 \\
\therefore \left(\begin{pmatrix}0\\-a\\0\end{pmatrix} + \begin{pmatrix}1\\3\\0\end{pmatrix} + \begin{pmatrix}b\\b\\-b\end{pmatrix}\right) \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} &= 0 \\
\therefore \left(\begin{pmatrix}0+1+b\\-a+3+b\\0+0-b\end{pmatrix}\right) \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} &= 0 \\
\therefore \left(\begin{pmatrix}1+b\\-a+3+b\\-b\end{pmatrix}\right) \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} &= 0 \\
\therefore 1\cdot(1+b) & \\
+1\cdot(-a+3+b) & \\
+(-1)\cdot(-b) &= 0 \\
\therefore 1+b-a+3+b+b &= 0 \\
\therefore 4-a+3b &= 0
\end{aligned}$$

Since $|CF|$ is minimum at the point D, $\overrightarrow{CD} \perp \boldsymbol{l}$.

$$\begin{aligned}
\overrightarrow{CD} &\perp \boldsymbol{l} \\
\therefore \overrightarrow{CD} \cdot \boldsymbol{l} &= 0 \\
\therefore (\overrightarrow{CO} + \overrightarrow{OD}) \cdot \boldsymbol{l} &= 0 \\
\therefore (-\overrightarrow{OC} + \overrightarrow{OD}) \cdot \boldsymbol{l} &= 0 \\
\therefore \left(-\left\{\begin{pmatrix}1\\3\\0\end{pmatrix} + c\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\} + d\begin{pmatrix}0\\1\\0\end{pmatrix}\right) \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} &= 0 \\
\therefore \left(-\left\{\begin{pmatrix}1\\3\\0\end{pmatrix} + \begin{pmatrix}1\cdot c\\1\cdot c\\-1\cdot c\end{pmatrix}\right\} + \begin{pmatrix}0\cdot d\\1\cdot d\\0\cdot d\end{pmatrix}\right) \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} &= 0 \\
\therefore \left(-\left\{\begin{pmatrix}1\\3\\0\end{pmatrix} + \begin{pmatrix}c\\c\\-c\end{pmatrix}\right\} + \begin{pmatrix}0\\d\\0\end{pmatrix}\right) \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} &= 0 \\
\therefore \left(-\begin{pmatrix}1+c\\3+c\\0-c\end{pmatrix} + \begin{pmatrix}0\\d\\0\end{pmatrix}\right) \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} &= 0 \\
\therefore \left(\begin{pmatrix}(1+c)\cdot(-1)\\(3+c)\cdot(-1)\\(0-c)\cdot(-1)\end{pmatrix} + \begin{pmatrix}0\\d\\0\end{pmatrix}\right) \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} &= 0 \\
\therefore \left(\begin{pmatrix}-1-c\\-3-c\\c\end{pmatrix} + \begin{pmatrix}0\\d\\0\end{pmatrix}\right) \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} &= 0 \\
\therefore \begin{pmatrix}-1-c+0\\-3-c+d\\c+0\end{pmatrix} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} &= 0 \\
\therefore \begin{pmatrix}-1-c\\-3-c+d\\c\end{pmatrix} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} &= 0 \\
\therefore (-1-c)\cdot 0 &
\end{aligned}$$

$$
\begin{aligned}
&+(-3-c+d)\cdot 1\\
&+c\cdot 0 = 0\\
&\therefore -3-c+d=0
\end{aligned}
$$

when $A = D$ and $B = C$, the two equations

$$
\begin{aligned}
4-a+3b&=0\\
-3-c+d&=0
\end{aligned}
$$

can be re-written as

$$
\begin{aligned}
4-a+3c&=0\\
-3-c+a&=0
\end{aligned}
$$

$$
\begin{array}{lcr}
 & 4-a+3c & =0\\
+) & -3-c+a & =0\\
\hline
 & 4-a+3c-3-c+a & =0\\
\therefore & 1+2c & =0\\
\therefore & 2c & =-1\\
\therefore & c & =-1/2
\end{array}
$$

By substituting $c = -1/2$ into $-3 - c + a = 0$

$$
\begin{aligned}
-3-c+a&=0\\
\therefore -3-(-\frac{1}{2})+a&=0\\
\therefore -3+\frac{1}{2}+a&=0\\
\therefore -3\cdot\frac{2}{2}+\frac{1}{2}+a&=0\\
\therefore \frac{-3\cdot 2}{2}+\frac{1}{2}+a&=0\\
\therefore \frac{-6}{2}+\frac{1}{2}+a&=0\\
\therefore \frac{-6+1}{2}+a&=0\\
\therefore \frac{-5}{2}+a&=0\\
\therefore a&=\frac{5}{2}
\end{aligned}
$$

The Cartesian coordinate of A is

$$\overrightarrow{OA} = a\begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{5}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\\frac{5}{2}\\0\end{pmatrix}$$

$$
\begin{aligned}
\overrightarrow{OC} &= \begin{pmatrix}1\\3\\0\end{pmatrix} + c\begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\3\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\3\\0\end{pmatrix} + \begin{pmatrix}1\cdot(-\frac{1}{2})\\1\cdot(-\frac{1}{2})\\-1\cdot(-\frac{1}{2})\end{pmatrix} = \begin{pmatrix}1\\3\\0\end{pmatrix} + \begin{pmatrix}-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\\
&= \begin{pmatrix}1-\frac{1}{2}\\3-\frac{1}{2}\\0+\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{2}{2}-\frac{1}{2}\\3\cdot\frac{2}{2}-\frac{1}{2}\\0+\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{2}{2}-\frac{1}{2}\\\frac{3\cdot 2}{2}-\frac{1}{2}\\0+\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{2-1}{2}\\\frac{6}{2}-\frac{1}{2}\\\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\\frac{6-1}{2}\\\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\\frac{5}{2}\\\frac{1}{2}\end{pmatrix}
\end{aligned}
$$

The distance $|AC|$ is obtained by getting $\overrightarrow{AC}$

$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\overrightarrow{OA} + \overrightarrow{OC} = -\begin{pmatrix}0\\\frac{5}{2}\\0\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\\frac{5}{2}\\\frac{1}{2}\end{pmatrix} = \begin{pmatrix}0\\-\frac{5}{2}\\0\end{pmatrix} + \begin{pmatrix}\frac{1}{2}\\\frac{5}{2}\\\frac{1}{2}\end{pmatrix} = \begin{pmatrix}0+\frac{1}{2}\\-\frac{5}{2}+\frac{5}{2}\\0+\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\0\\\frac{1}{2}\end{pmatrix}$$

Thus the modulus of $\overrightarrow{AC}$ is

$$\sqrt{\left(\frac{1}{2}\right)^2 + 0 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

53) There are 4 points $O(0,0,0)$, $A(0,2,1)$, $B(1,3,-1)$, and $C(2,1,0)$.
A line l goes through the points O and A.
A line m goes through the points B and C.

a) Prove that the line l and the line m do not share a point.
All we have to do is to try to find out the crossing points of l and m. The point P on the line l is expressed as

$$\overrightarrow{OP} = p\overrightarrow{OA} = p\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \cdot p \\ 2 \cdot p \\ 1 \cdot p \end{pmatrix} = \begin{pmatrix} 0 \\ 2p \\ p \end{pmatrix}$$

The point Q on the line m is expressed as

$$\overrightarrow{OQ} = \overrightarrow{OB} + q\overrightarrow{BC} = \overrightarrow{OB} + q(\overrightarrow{BO} + \overrightarrow{OC}) = \overrightarrow{OB} + q(-\overrightarrow{OB} + \overrightarrow{OC})$$

$$= \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + q\left(-\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + q\left(\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + q\begin{pmatrix} -1+2 \\ -3+1 \\ 1+0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + q\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \cdot q \\ -2 \cdot q \\ 1 \cdot q \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} q \\ -2q \\ q \end{pmatrix} = \begin{pmatrix} 1+q \\ 3-2q \\ -1+q \end{pmatrix}$$

When P and Q are at the same location,

$$\overrightarrow{OP} = \overrightarrow{OQ}$$

$$\therefore \begin{pmatrix} 0 \\ 2p \\ p \end{pmatrix} = \begin{pmatrix} 1+q \\ 3-2q \\ -1+q \end{pmatrix}$$

From this, we obtain the following three equations:

$$0 = 1 + q$$
$$2p = 3 - 2q$$
$$p = -1 + q$$

From the first equation

$$0 = 1 + q \therefore -1 = q$$

By substituting $q = -1$ into the second and third equations:

$$2p = 3 - 2 \cdot (-1)$$
$$p = -1 + (-1)$$

Therefore

$$2p = 3 + 2$$
$$p = -1 - 1$$

Thus

$$2p = 5$$
$$p = -2$$

In other words

$$p = 5/2$$
$$p = -2$$

Since these two equations give the different value for p, $\overrightarrow{OP} = \overrightarrow{OQ}$ is invalid.

b) There is a line n which is orthogonal to the line l and the line m.
Find the crossing points between n and l and between n and m.

- Solution 1 We assume the line n is parallel to $\boldsymbol{n}$. The line l is parallel to $\boldsymbol{l} = \overrightarrow{OA}$. The line m is parallel to $\boldsymbol{m} = \overrightarrow{BC}$. As $\boldsymbol{n} \perp \boldsymbol{l}$ and $\boldsymbol{n} \perp \boldsymbol{m}$ we can say

$$\begin{pmatrix} \overrightarrow{OA} & \overrightarrow{BC} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & -2 \\ 1 & 1 \end{pmatrix}$$

$$\overrightarrow{OA} \times \overrightarrow{BC} = \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} \boldsymbol{i}$$
$$+ \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \boldsymbol{j}$$
$$+ \begin{vmatrix} 0 & 1 \\ 2 & -2 \end{vmatrix} \boldsymbol{k}$$
$$= 4\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$$

As the point P is on l, we can say

$$\overrightarrow{OP} = \begin{pmatrix} 0 \\ 2p \\ p \end{pmatrix}$$

As the point Q is on m, we can say

$$\overrightarrow{OQ} = \overrightarrow{OB} + q\overrightarrow{BC} = \begin{pmatrix} 1+q \\ 3-2q \\ -1+q \end{pmatrix}$$

Therefore $\overrightarrow{PQ}$ can be written as

$$\begin{aligned} \overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= -\overrightarrow{OP} + \overrightarrow{OQ} \\ &= \begin{pmatrix} 1+q \\ 3-2q-2p \\ -1+q-p \end{pmatrix} \end{aligned}$$

On the other hand, we know that $\overrightarrow{PQ} \parallel \overrightarrow{OA} \times \overrightarrow{BC}$. Therefore $\overrightarrow{PQ}$ can be expressed as

$$\begin{aligned} \overrightarrow{PQ} &= r\overrightarrow{OA} \times \overrightarrow{BC} \\ &= \begin{pmatrix} 4r \\ r \\ -2r \end{pmatrix} \end{aligned}$$

Thus we can write

$$\begin{pmatrix} 1+q \\ 3-2q-2p \\ -1+q-p \end{pmatrix} = \begin{pmatrix} 4r \\ r \\ -2r \end{pmatrix}$$

This equation gives us

$$\begin{aligned} 1+q &= 4r \\ 3-2q-2p &= r \\ -1+q-p &= -2r \end{aligned}$$

The solution of the simultaneous equation is $(p, q, r) = (\frac{4}{7}, \frac{5}{7}, \frac{3}{7})$. Thus

$$\overrightarrow{OP} = \begin{pmatrix} 0 \\ 2p \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{8}{7} \\ \frac{4}{7} \end{pmatrix}$$

and

$$\overrightarrow{OQ} = \overrightarrow{OB} + q\overrightarrow{BC} = \begin{pmatrix} 1+q \\ 3-2q \\ -1+q \end{pmatrix} = \begin{pmatrix} \frac{12}{7} \\ \frac{11}{7} \\ \frac{-2}{7} \end{pmatrix}$$

- Solution 2 The line l is parallel to $\overrightarrow{OA} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \triangleq \boldsymbol{l}$.

  The line m is parallel to $\overrightarrow{BC} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \triangleq \boldsymbol{m}$.

  $\overrightarrow{PQ}$ is orthogonal to l. $\overrightarrow{PQ}$ is:

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\overrightarrow{OP} + \overrightarrow{OQ} = -\begin{pmatrix} 0 \\ 2p \\ p \end{pmatrix} + \begin{pmatrix} 1+q \\ 3-2q \\ -1+q \end{pmatrix} = \begin{pmatrix} 0 \\ -2p \\ -p \end{pmatrix} + \begin{pmatrix} 1+q \\ 3-2q \\ -1+q \end{pmatrix} = \begin{pmatrix} 1+q \\ -2p+3-2q \\ -p-1+q \end{pmatrix}$$

  Therefore the orthogonality is expressed as

$$\begin{aligned} \overrightarrow{PQ} &\perp \boldsymbol{l} \\ \therefore \overrightarrow{PQ} \cdot \boldsymbol{l} &= 0 \\ \therefore \begin{pmatrix} 1+q \\ -2p+3-2q \\ -p-1+q \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} &= 0 \\ \therefore (1+q) \cdot 0 + (-2p+3-2q) \cdot 2 + (-p-1+q) \cdot 1 &= 0 \\ \therefore -4p+6-4q-p-1+q &= 0 \\ \therefore -5p+5-3q &= 0 \end{aligned}$$

$\overrightarrow{PQ}$ is orthogonal to m.
The orthogonality is expressed as

$$\begin{aligned}\overrightarrow{PQ} &\perp \boldsymbol{m}\\ \therefore \overrightarrow{PQ}\cdot\boldsymbol{m} &= 0\\ \therefore \begin{pmatrix}1+q\\-2p+3-2q\\-p-1+q\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix} &= 0\\ \therefore (1+q)\cdot 1+(-2p+3-2q)\cdot(-2)+(-p-1+q)\cdot 1 &= 0\\ \therefore 1+q+4p-6+4q-p-1+q &= 0\\ \therefore 3p-6+6q &= 0\\ \therefore p-2+2q &= 0\\ \therefore p &= 2-2q\end{aligned}$$

$p=2-2q$ is put into $-5p+5-3q=0$:

$$\begin{aligned}-5p+5-3q &= 0\\ \therefore -5(2-2q)+5-3q &= 0\\ \therefore -10+10q+5-3q &= 0\\ \therefore -5+7q &= 0\\ \therefore 7q &= 5\\ \therefore q &= \frac{5}{7}\end{aligned}$$

By substituting $q=\frac{5}{7}$ into $p=2-2q$,

$$p=2-2q=2-2\cdot\frac{5}{7}=2\cdot\frac{7}{7}+\frac{-10}{7}=\frac{14}{7}+\frac{-10}{7}=\frac{14-10}{7}=\frac{4}{7}$$

The point$P$ is

$$\begin{pmatrix}0\\2p\\p\end{pmatrix}=\begin{pmatrix}0\\2\cdot\frac{4}{7}\\\frac{4}{7}\end{pmatrix}=\begin{pmatrix}0\\\frac{8}{7}\\\frac{4}{7}\end{pmatrix}$$

The point Q is

$$\begin{pmatrix}1+q\\3-2q\\-1+q\end{pmatrix}=\begin{pmatrix}1+\frac{5}{7}\\3-2\cdot\frac{5}{7}\\-1+\frac{5}{7}\end{pmatrix}=\begin{pmatrix}\frac{7}{7}+\frac{5}{7}\\3\cdot 77+\frac{-10}{7}\\-\frac{7}{7}+\frac{5}{7}\end{pmatrix}=\begin{pmatrix}\frac{7+5}{7}\\\frac{21}{7}+\frac{-10}{7}\\\frac{-7+5}{7}\end{pmatrix}=\begin{pmatrix}\frac{12}{7}\\\frac{21-10}{7}\\\frac{-2}{7}\end{pmatrix}=\begin{pmatrix}\frac{12}{7}\\\frac{11}{7}\\\frac{-2}{7}\end{pmatrix}$$

54) There are 3 points $A(0,2,0)$, $B(1,3,2)$, and $C(3,3,4)$. A line l goes through A and is perpendicular to the line AB and the line AC. Find a point P which satisfies $\angle APB = 30^o$ and is on the line l.

- Solution 1 Since $\boldsymbol{l}\perp\overrightarrow{AB}$ and $\boldsymbol{l}\perp\overrightarrow{AC}$, $\boldsymbol{l}$ is parallel to $\overrightarrow{AB}\times\overrightarrow{AC}$. Thus we find $\overrightarrow{AB}\times\overrightarrow{AC}$.

$$\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}=-\overrightarrow{OA}+\overrightarrow{OB}=-\begin{pmatrix}0\\2\\0\end{pmatrix}+\begin{pmatrix}1\\3\\2\end{pmatrix}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\begin{pmatrix}1\\3\\2\end{pmatrix}=\begin{pmatrix}0+1\\-2+3\\0+2\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}$$
$$\overrightarrow{AC}=\overrightarrow{AO}+\overrightarrow{OC}=-\overrightarrow{OA}+\overrightarrow{OC}=-\begin{pmatrix}0\\2\\0\end{pmatrix}+\begin{pmatrix}3\\3\\4\end{pmatrix}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\begin{pmatrix}3\\3\\4\end{pmatrix}=\begin{pmatrix}0+3\\-2+3\\0+4\end{pmatrix}=\begin{pmatrix}3\\1\\4\end{pmatrix}$$

Thus

$$\begin{pmatrix}\overrightarrow{AB} & \overrightarrow{AC}\end{pmatrix}=\begin{pmatrix}1&3\\1&1\\2&4\end{pmatrix}$$
$$\therefore \overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}1&1\\2&4\end{vmatrix}\boldsymbol{i}+\begin{vmatrix}2&4\\1&3\end{vmatrix}\boldsymbol{j}+\begin{vmatrix}1&3\\1&1\end{vmatrix}\boldsymbol{k}=2\boldsymbol{i}+2\boldsymbol{j}-2\boldsymbol{k}$$

Thus we can write $\overrightarrow{OP}$ using a variable s as

$$\overrightarrow{OP}=\overrightarrow{OA}+s\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}0\\2\\0\end{pmatrix}+s\begin{pmatrix}2\\2\\-2\end{pmatrix}=\begin{pmatrix}2s\\2+2s\\-2s\end{pmatrix}$$

$\angle APB = 30^o$ gives

$$\overrightarrow{PA}\cdot\overrightarrow{PB}=|\overrightarrow{PA}|\cdot|\overrightarrow{PB}|\cos(\frac{\pi}{6})=|\overrightarrow{PA}|\cdot|\overrightarrow{PB}|\cdot\frac{\sqrt{3}}{2}$$

Thus we need $\overrightarrow{PA}$, $\overrightarrow{PB}$, $|\overrightarrow{PA}|$, and $|\overrightarrow{PB}|$.

$$\begin{aligned}
\overrightarrow{PA} &= \overrightarrow{PO} + \overrightarrow{OA} \\
&= -\overrightarrow{OP} + \overrightarrow{OA} \\
&= -(\overrightarrow{OA} + s\overrightarrow{AB} \times \overrightarrow{AC}) + \overrightarrow{OA} \\
&= -\overrightarrow{OA} - s\overrightarrow{AB} \times \overrightarrow{AC} + \overrightarrow{OA} \\
&= -s\overrightarrow{AB} \times \overrightarrow{AC} \\
&= -s \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} -2s \\ -2s \\ 2s \end{pmatrix} \\
\therefore |\overrightarrow{PA}| &= \sqrt{(-2s)^2 + (-2s)^2 + (2s)^2} \\
&= \sqrt{12s^2}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{PB} &= \overrightarrow{PO} + \overrightarrow{OB} \\
&= -\overrightarrow{OP} + \overrightarrow{OB} \\
&= -\begin{pmatrix} 2s \\ 2+2s \\ -2s \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} -2s \\ -2-2s \\ 2s \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 1-2s \\ 3-2-2s \\ 2+2s \end{pmatrix} \\
&= \begin{pmatrix} 1-2s \\ 1-2s \\ 2+2s \end{pmatrix} \\
\therefore |\overrightarrow{PB}| &= \sqrt{(1-2s)^2 + (1-2s)^2 + (2+2s)^2} \\
&= \sqrt{2(4s^2 - 4s + 1) + 4s^2 + 8s + 4} \\
&= \sqrt{8s^2 - 8s + 2 + 4s^2 + 8s + 4} \\
&= \sqrt{12s^2 + 6}
\end{aligned}$$

Now that we know $\overrightarrow{PA}$, $\overrightarrow{PB}$, $|\overrightarrow{PA}|$, and $|\overrightarrow{PB}|$, we can calculate r as follows:

$$\begin{aligned}
&\overrightarrow{PA} \cdot \overrightarrow{PB} \\
&= |\overrightarrow{PA}| \cdot |\overrightarrow{PB}| \cdot \frac{\sqrt{3}}{2} \\
\therefore &\begin{pmatrix} -2s \\ -2s \\ 2s \end{pmatrix} \cdot \begin{pmatrix} 1-2s \\ 1-2s \\ 2+2s \end{pmatrix} \\
&= \sqrt{12s^2} \cdot \sqrt{12s^2+6} \cdot \frac{\sqrt{3}}{2} \\
\therefore -2s \cdot (1-2s) - 2s \cdot (1-2s) + 2s \cdot (2+2s) &= \sqrt{12s^2(12s^2+6)} \cdot \frac{\sqrt{3}}{2} \\
\therefore -4s + 8s^2 + 4s + 4s^2 &= \sqrt{12s^2(12s^2+6)} \cdot \frac{\sqrt{3}}{2} \\
\therefore 12s^2 &= \sqrt{12s^2(12s^2+6)} \cdot \frac{\sqrt{3}}{2} \\
\therefore 12^2 s^4 &= 12s^2(12s^2+6) \cdot \frac{3}{4} \\
\therefore 12^2 s^2 &= 9(12s^2+6) (\because s \neq 0) \\
\therefore 12^2 s^2 &= 108s^2 + 54 \\
\therefore 36s^2 &= 54 \\
\therefore s &= \pm\frac{3}{2}
\end{aligned}$$

Thus

$$\overrightarrow{OP} = \begin{pmatrix} 2s \\ 2+2s \\ -2s \end{pmatrix}$$
$$= \begin{pmatrix} \pm\sqrt{6} \\ 2\pm\sqrt{6} \\ \mp\sqrt{6} \end{pmatrix}$$

- Solution 2 We assume the line l is parallel to

$$\boldsymbol{l} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$l \perp \overrightarrow{AB}$ and $l \perp \overrightarrow{AC}$. Now we find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$
$$= -\overrightarrow{OA} + \overrightarrow{OB}$$
$$= -\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 0+1 \\ -2+3 \\ 0+2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$
$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$$
$$= -\overrightarrow{OA} + \overrightarrow{OC}$$
$$= -\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} 0+3 \\ -2+3 \\ 0+4 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

$$\boldsymbol{l} \perp \overrightarrow{AB}$$
$$\therefore \boldsymbol{l} \cdot \overrightarrow{AB} = 0$$
$$\therefore \begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 0$$
$$\therefore p \cdot 1 + q \cdot 1 + r \cdot 2 = 0$$
$$\therefore p + q + 2r = 0$$

$$\boldsymbol{l} \perp \overrightarrow{AC}$$
$$\therefore \boldsymbol{l} \cdot \overrightarrow{AC} = 0$$
$$\therefore \begin{pmatrix} p \\ q \\ r \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 0$$
$$\therefore p \cdot 3 + q \cdot 1 + r \cdot 4 = 0$$
$$\therefore 3p + q + 4r = 0$$

Using these 2 equations,

$$\begin{array}{rcc}
 & p+q+2r & = & 0 \\
-) & 3p+q+4r & = & 0 \\
\hline
 & p+2r-3p-4r & = & 0 \\
\therefore & -2p-2r & = & 0 \\
\therefore & p+r & = & 0 \\
\therefore & p & = & -r
\end{array}$$

Substituting $p=-r$ into $p+q+2r=0$, we obtain

$$\begin{aligned}
p+q+2r &= 0 \\
\therefore -r+q+2r &= 0 \\
\therefore q+r &= 0 \\
\therefore q &= -r
\end{aligned}$$

Using $p=q=-r$, we get

$$\begin{aligned}
\boldsymbol{l} &= \begin{pmatrix} p \\ q \\ r \end{pmatrix} \\
&= \begin{pmatrix} -r \\ -r \\ r \end{pmatrix}
\end{aligned}$$

where r is an arbitrary non-zero real number. Thus we can write $\overrightarrow{OP}$ as

$$\begin{aligned}
\overrightarrow{OP} &= \overrightarrow{OA} + \boldsymbol{l} \\
&= \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -r \\ -r \\ r \end{pmatrix} \\
&= \begin{pmatrix} -r \\ 2-r \\ r \end{pmatrix}
\end{aligned}$$

$\angle APB = 30^o$ gives

$$\begin{aligned}
\overrightarrow{PA}\cdot\overrightarrow{PB} &= |\overrightarrow{PA}|\cdot|\overrightarrow{PB}|\cos(\frac{\pi}{6}) \\
&= |\overrightarrow{PA}|\cdot|\overrightarrow{PB}|\cdot\frac{\sqrt{3}}{2}
\end{aligned}$$

Thus we need $\overrightarrow{PA}$, $\overrightarrow{PB}$, $|\overrightarrow{PA}|$, and $|\overrightarrow{PB}|$.

$$\begin{aligned}
\overrightarrow{PA} &= \overrightarrow{PO} + \overrightarrow{OA} \\
&= -\overrightarrow{OP} + \overrightarrow{OA} \\
&= -(\overrightarrow{OA} + \boldsymbol{l}) + \overrightarrow{OA} \\
&= -\overrightarrow{OA} - \boldsymbol{l} + \overrightarrow{OA} \\
&= -\boldsymbol{l} \\
&= -\begin{pmatrix} -r \\ -r \\ r \end{pmatrix} \\
&= \begin{pmatrix} r \\ r \\ -r \end{pmatrix} \\
\therefore |\overrightarrow{PA}| &= \sqrt{r^2+r^2+(-r)^2} \\
&= \sqrt{r^2+r^2+r^2} \\
&= \sqrt{3r^2}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{PB} &= \overrightarrow{PO} + \overrightarrow{OB} \\
&= -\overrightarrow{OP} + \overrightarrow{OB} \\
&= -\begin{pmatrix} -r \\ 2-r \\ r \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} r \\ -(2-r) \\ -r \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} r+1 \\ -2+r+3 \\ -r+2 \end{pmatrix}$$
$$= \begin{pmatrix} r+1 \\ r+1 \\ -r+2 \end{pmatrix}$$
$$\therefore |\overrightarrow{PB}| = \sqrt{(r+1)^2+(r+1)^2+(-r+2)^2}$$
$$= \sqrt{2(r^2+2r+1)+r^2-4r+4}$$
$$= \sqrt{2r^2+4r+2+r^2-4r+4}$$
$$= \sqrt{3r^2+6}$$

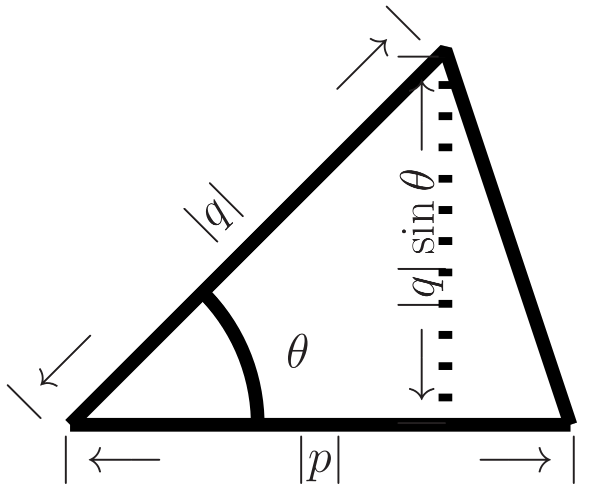
Now that we know $\overrightarrow{PA}$, $\overrightarrow{PB}$, $|\overrightarrow{PA}|$, and $|\overrightarrow{PB}|$, we can calculate r as follows:

$$\overrightarrow{PA} \cdot \overrightarrow{PB}$$
$$= |\overrightarrow{PA}| \cdot |\overrightarrow{PB}| \cdot \frac{\sqrt{3}}{2}$$
$$\therefore \begin{pmatrix} r \\ r \\ -r \end{pmatrix} \cdot \begin{pmatrix} r+1 \\ r+1 \\ -r+2 \end{pmatrix}$$
$$= \sqrt{3r^2} \cdot \sqrt{3r^2+6} \cdot \frac{\sqrt{3}}{2}$$
$$\therefore r \cdot (r+1) + r \cdot (r+1) + (-r) \cdot (-r+2)$$
$$= \sqrt{3r^2(3r^2+6)} \cdot \frac{\sqrt{3}}{2}$$
$$\therefore r^2 + r + r^2 + r + r^2 - 2r$$
$$= \sqrt{3 \cdot 3r^2(3r^2+6)} \cdot \frac{1}{2}$$
$$\therefore 3r^2 = \sqrt{9r^2(3r^2+6)} \cdot \frac{1}{2}$$
$$\therefore 6r^2 = \sqrt{9r^2(3r^2+6)}$$
$$\therefore \left(6r^2\right)^2 = 9r^2(3r^2+6)$$
$$\therefore 6^2 r^{2\times 2} = 9r^2(3r^2+6)$$
$$\therefore 36r^4 = 9r^2(3r^2+6)$$
$$\therefore 4r^2 = 3r^2+6 (\because r \neq 0)$$
$$\therefore r^2 = 6$$
$$\therefore r = \pm\sqrt{6}$$

Thus

$$\overrightarrow{OP} = \begin{pmatrix} -r \\ 2-r \\ r \end{pmatrix}$$
$$= \begin{pmatrix} \mp\sqrt{6} \\ 2\mp\sqrt{6} \\ \pm\sqrt{6} \end{pmatrix}$$

55) Find the area of the triangle with vertices at the points with coordinates $P(1,2,3)$, $Q(4,-3,2)$ and $R(8,1,1)$.



$$S = \frac{|p|(|q|\sin\theta)}{2} = \frac{|p \times q|}{2}$$

$$p \times q = |p||q|\sin\theta\hat{n}$$
$$|p \times q| = |p||q||\sin\theta|$$

In general, when a triangle is defined with two sides $|\boldsymbol{p}|$ and $|\boldsymbol{q}|$ and the angle between these two sides is θ, the area of the triangle is $\frac{1}{2}|\boldsymbol{p}| \cdot |\boldsymbol{q}| \cdot \sin\theta$. On the other hand, when we define

$$\boldsymbol{p} = \overrightarrow{RP}$$

and

$$\boldsymbol{q} = \overrightarrow{RQ},$$

$$\boldsymbol{p} \times \boldsymbol{q} = |\boldsymbol{p}| \cdot |\boldsymbol{q}| \sin\theta\hat{\boldsymbol{n}}$$

Thus the area of the triangle is $\frac{1}{2}|\boldsymbol{p} \times \boldsymbol{q}|$.

$$\begin{aligned}\boldsymbol{p} &= \overrightarrow{RP} \\ &= \overrightarrow{RO} + \overrightarrow{OP} \\ &= -\overrightarrow{OR} + \overrightarrow{OP} \\ &= -\begin{pmatrix} 8 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} -8+1 \\ -1+2 \\ -1+3 \end{pmatrix} \\ &= \begin{pmatrix} -7 \\ 1 \\ 2 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}&\overrightarrow{RQ} \\ &= \overrightarrow{RO} + \overrightarrow{OQ} \\ &= -\overrightarrow{OR} + \overrightarrow{OQ} \\ &= -\begin{pmatrix} 8 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -8+4 \\ -1-3 \\ -1+2 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ -4 \\ 1 \end{pmatrix}\end{aligned}$$

$$\therefore \begin{pmatrix} \boldsymbol{p} & \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} -7 & -4 \\ 1 & -4 \\ 2 & 1 \end{pmatrix}$$

Thus

$$\begin{aligned} & \boldsymbol{p} \times \boldsymbol{q} \\ = & \begin{vmatrix} 1 & -4 \\ 2 & 1 \end{vmatrix} \boldsymbol{i} \\ + & \begin{vmatrix} 2 & 1 \\ -7 & -4 \end{vmatrix} \boldsymbol{j} \\ + & \begin{vmatrix} -7 & -4 \\ 1 & -4 \end{vmatrix} \boldsymbol{k} \\ = & \{1 \cdot (1) - (-4) \cdot 2\}\boldsymbol{i} \\ + & \{2 \cdot (-4) - (1) \cdot (-7)\}\boldsymbol{j} \\ + & \{-7 \cdot (-4) - (-4) \cdot (1)\}\boldsymbol{k} \\ = & \{1 + 8\}\boldsymbol{i} \\ + & \{-8 + 7\}\boldsymbol{j} \\ + & \{28 + 4\}\boldsymbol{k} \\ = & 9\boldsymbol{i} - \boldsymbol{j} + 32\boldsymbol{k} \end{aligned}$$

The modulus of $9\boldsymbol{i} - \boldsymbol{j} + 32\boldsymbol{k}$ is

$$\sqrt{9^2 + 1^2 + 32^2} = \sqrt{1106}$$

Thus, the area of the triangle is $\frac{\sqrt{1106}}{2}$.

X. Exercises on Coordinate

coordinateall.tex

1) **DAY1**
2) Solve the following equation using the quadratic equation

$$2x^2 - 4x - 3 = 0$$

Therefore $a = 2, b - 4, c = -3$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ \therefore x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot (2) \cdot (-3)}}{2 \cdot 2} \\ \therefore x &= \frac{4 \pm \sqrt{16 - (-24)}}{4} \\ \therefore x &= \frac{4 \pm \sqrt{40}}{4} \\ \therefore x &= \frac{4 \pm \sqrt{4} \cdot \sqrt{10}}{4} \\ \therefore x &= \frac{4 \pm 2\sqrt{10}}{4} \\ \therefore x &= \frac{2 \pm \sqrt{10}}{2} \end{aligned}$$

3) Simplify

$$(a^2b^2)(a^4b^{-2})$$

$$\begin{aligned} &(a^2b^2)(a^4b^{-2}) \\ &= a^{2+4}b^{2-2} \\ &= a^6b^0 \\ &= a^6 \cdot 1 \\ &= a^6 \end{aligned}$$

4) Simplify

$$3(a + b + a^2) - 2(a - b)$$

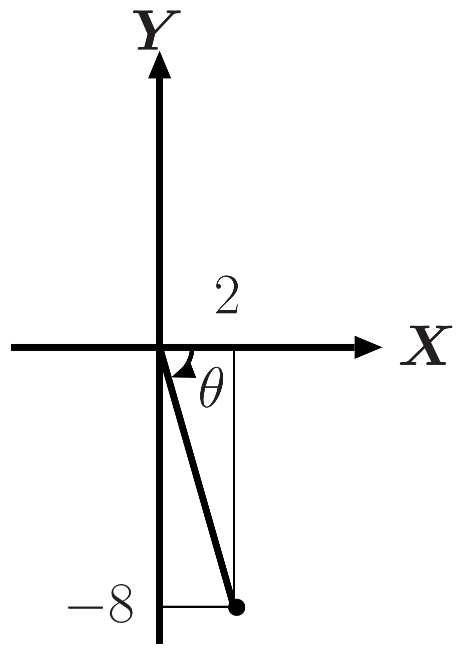
$$\begin{aligned} &3(a + b + a^2) - 2(a - b) \\ &= 3a + 3b + 3a^2 - 2a + 2b \\ &= a + 5b + 3a^2 \end{aligned}$$

5) Solve the following equation

$$\frac{x+1}{3} + \frac{1}{2} = \frac{5}{2}$$

$$\begin{aligned} \frac{x+1}{3} + \frac{1}{2} &= \frac{5}{2} \\ \therefore 3 \cdot \frac{x+1}{3} + 3 \cdot \frac{1}{2} &= 3 \cdot \frac{5}{2} \\ \therefore (x + 1) + \frac{3}{2} &= \frac{15}{2} \\ \therefore 2(x + 1) + 2 \cdot \frac{3}{2} &= 2 \cdot \frac{15}{2} \\ \therefore 2(x + 1) + 3 &= 15 \\ \therefore 2x + 2 + 3 &= 15 \\ \therefore 2x &= 15 - 2 - 3 \\ \therefore 2x &= 10 \\ \therefore x &= 5 \end{aligned}$$

6) Calculate the equivalent polar coordinates of the following Cartesian coordinates within $-\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$.
   a) $(2, -8)$



$$r=\sqrt{x^2+y^2}=\sqrt{2^2+(-8)^2}=\sqrt{4+64}=\sqrt{68}=\sqrt{4}\cdot\sqrt{17}=2\sqrt{17}$$

$$\tan\theta=\frac{y}{x}=\frac{-8}{2}=-4$$
$$\therefore\theta=\tan^{-1}(-4)=-1.33$$

Since this Cartesian coordinate is in the fourth quadrant the answer is valid.

b) $(-2,8)$

$$r=\sqrt{x^2+y^2}=\sqrt{(-2)^2+(8)^2}=\sqrt{4+64}=\sqrt{68}=\sqrt{4}\cdot\sqrt{17}=2\sqrt{17}$$

$$\tan\theta=\frac{y}{x}=\frac{8}{-2}=-4$$
$$\therefore\theta=\tan^{-1}(-4)=-1.33$$

Since this Cartesian coordinate is in the second quadrant whilst $\theta=-1.33$ is in the fourth quadrant the answer is not correct. The correct θ is $\theta=-1.33+3.14=1.81$ radian.

c) $(-2,-8)$

$$r=\sqrt{x^2+y^2}=\sqrt{(-2)^2+(-8)^2}=\sqrt{4+64}=\sqrt{68}=\sqrt{4}\cdot\sqrt{17}=2\sqrt{17}$$

$$\tan\theta=\frac{y}{x}=\frac{-8}{-2}=4$$
$$\therefore\theta=\tan^{-1}(4)=1.33$$

Since this Cartesian coordinate is in the third quadrant whilst $\theta=1.33$ is in the first quadrant the answer is not correct. The correct θ is $\theta=1.33+3.14=4.47$ radian.

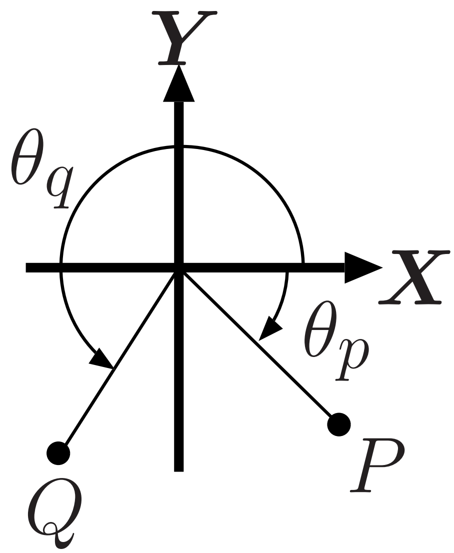
d) $(2,8)$

$$r=\sqrt{x^2+y^2}=\sqrt{(2)^2+(8)^2}=\sqrt{4+64}=\sqrt{68}=\sqrt{4}\cdot\sqrt{17}=2\sqrt{17}$$

$$\tan\theta=\frac{y}{x}=\frac{8}{2}=4$$
$$\therefore\theta=\tan^{-1}(4)=1.33$$

Since this Cartesian coordinate lies in the first quadrant whilst $\theta=1.33$ is in the first quadrant the answer is correct.

7) The Cartesian coordinates of P, Q are $(1,-1)$ and $(-1,-\sqrt{3})$. What are their equivalent polar coordinates within $-\frac{\pi}{2}\leq\theta\leq\frac{3\pi}{2}$?



We assume that $\overrightarrow{OP}$, $\overrightarrow{OQ}$ can be expressed as

$$\overrightarrow{OP} = \left[|\overrightarrow{OP}|, \theta_p\right]$$
$$\overrightarrow{OQ} = \left[|\overrightarrow{OQ}|, \theta_q\right]$$

In order to find the polar coordinates, we need to find out $|\overrightarrow{OP}|$ and $|\overrightarrow{OQ}|$ and their angle

$$|\overrightarrow{OP}| = \left|\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right| = \sqrt{1^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$
$$|\overrightarrow{OQ}| = \left|\begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix}\right| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

θ_p can be found by :

$$\tan\theta_p = \frac{-1}{1}$$
$$\therefore \theta_p = -\frac{\pi}{4}$$

Since this is in the fourth quadrant, $\theta_p = -\frac{\pi}{4}$ is valid.
θ_q can be found by :

$$\tan\theta_q = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$
$$\therefore \theta_q = \frac{\pi}{3}$$

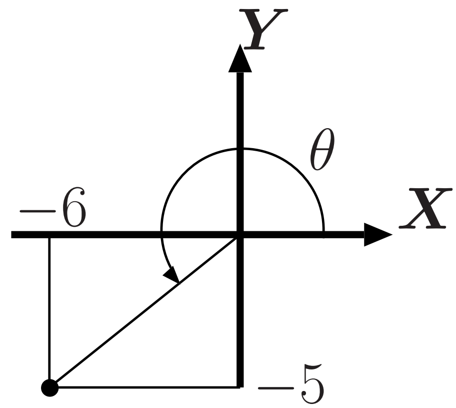
This θ_q is in the first quadrant, whilst $Q(-1, -\sqrt{3})$ is in the third quadrant. Therefore, the appropriate θ_q is calculated as

$$\theta_q = \frac{\pi}{3} + \pi = \frac{\pi}{3} + \pi\frac{3}{3} = \frac{\pi}{3} + \frac{3\pi}{3} = \frac{\pi + 3\pi}{3} = \frac{4\pi}{3}$$

Therefore

$$\overrightarrow{OP} = \left[\sqrt{2}, -\frac{\pi}{4}\right]$$
$$\overrightarrow{OQ} = \left[2, \frac{4\pi}{3}\right]$$

8) Calculate the equivalent polar coordinates of the following Cartesian coordinates within $-\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$.
   a) $(-6, -5)$



$$r=\sqrt{x^2+y^2}=\sqrt{(-6)^2+(-5)^2}=\sqrt{36+25}=\sqrt{61}$$

$$\tan\theta=\frac{y}{x}=\frac{-5}{-6}=\frac{5}{6}$$
$$\therefore\theta=\tan^{-1}\left(\frac{5}{6}\right)=0.69$$

Since this Cartesian coordinate lies in the third quadrant whilst $\theta=0.69$ is in the first quadrant the answer is not correct. The correct answer for θ is calculated as $\theta=0.69+3.14=3.83$ radian.

b) $(6,5)$

$$r=\sqrt{x^2+y^2}=\sqrt{(6)^2+(5)^2}=\sqrt{36+25}=\sqrt{61}$$

$$\tan\theta=\frac{y}{x}=\frac{5}{6}=\frac{5}{6}$$
$$\therefore\theta=\tan^{-1}\left(\frac{5}{6}\right)=0.69$$

Since this Cartesian coordinate lies in the first quadrant whilst $\theta=0.69$ is in the first quadrant the answer is correct.

c) $(-6,5)$

$$r=\sqrt{x^2+y^2}=\sqrt{(-6)^2+(5)^2}=\sqrt{36+25}=\sqrt{61}$$

$$\tan\theta=\frac{y}{x}=\frac{5}{-6}=-\frac{5}{6}$$
$$\therefore\theta=\tan^{-1}\left(-\frac{5}{6}\right)=-0.69$$

Since this Cartesian coordinate lies in the second quadrant whilst $\theta=-0.69$ is in the fourth quadrant the answer is not correct. The correct answer for θ is calculated as $\theta=-0.69+3.14=2.45$ radian.

d) $(6,-5)$

$$r=\sqrt{x^2+y^2}=\sqrt{(6)^2+(-5)^2}=\sqrt{36+25}=\sqrt{61}$$

$$\tan\theta=\frac{y}{x}=\frac{-5}{6}=-\frac{5}{6}$$
$$\therefore\theta=\tan^{-1}\left(-\frac{5}{6}\right)=-0.69$$

Since this Cartesian coordinate lies in the fourth quadrant whilst $\theta=-0.69$ is in the fourth quadrant the answer is correct.

9) Calculate the equivalent polar coordinates of the following Cartesian coordinates within $-\infty\le\theta\le\infty$.

a) $(1,0)$

$$r=\sqrt{x^2+y^2}=\sqrt{(1)^2+(0)^2}=1$$

$$\tan\theta=\frac{y}{x}=\frac{0}{1}=0$$
$$\therefore\theta=\tan^{-1}(0)=0$$

As $(1,0)$ is on the $+x$ axis, $\theta=0$ is correct. Since $-\infty\le\theta\le\infty$, $\theta=0+2n\pi=2n\pi$ where n is the interger and $-\infty\le n\le\infty$.

b) $(0,1)$

$$r=\sqrt{x^2+y^2}=\sqrt{(0)^2+(1)^2}=1$$

$$\tan\theta = \frac{1}{0} = +\infty$$
$$\therefore \theta = \tan^{-1}(+\infty) = \frac{\pi}{2}$$

Since $-\infty \leq \theta \leq \infty$, $\theta = \frac{\pi}{2} + 2n\pi$ where n is the interger and $-\infty \leq n \leq +\infty$

c) $(-1, 0)$

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (0)^2} = 1$$

$$\tan\theta = \frac{0}{-1} = -0$$
$$\therefore \theta = \tan^{-1}(-0) = 0$$

As $(-1, 0)$ is on the $-x$ axis, $\theta = 0$ is incorrect. The correct θ is obtained as $\theta = 0+\pi = \pi$. Since $-\infty \leq \theta \leq \infty$, $\theta = \pi+2n\pi = (2n+1)\pi$ where n is the interger and $-\infty \leq n \leq \infty$.

d) $(0, -1)$

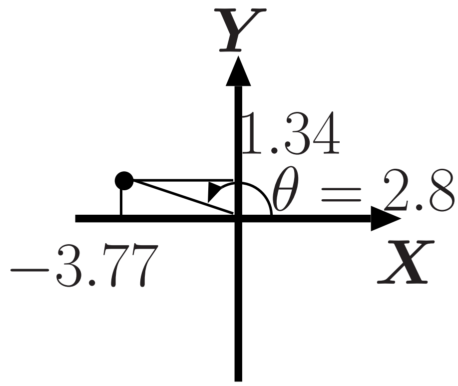
$$r = \sqrt{x^2 + y^2} = \sqrt{(0)^2 + (-1)^2} = 1$$

$$\tan\theta = \frac{y}{x} = \frac{-1}{0} = -\infty$$
$$\therefore \theta = \tan^{-1}(-\infty) = -\frac{\pi}{2}$$

Since $-\infty \leq \theta \leq \infty$, $\theta = -\frac{\pi}{2} + 2n\pi$ where n is the interger and $-\infty \leq n \leq +\infty$

DAY2

10) Calculate the equivalent Cartesian coordinates of the following polar coordinates.
   a) $r = 4, \theta = 2.8$



$$x = r \cdot \cos\theta = 4 \cdot \cos(2.8) = -3.77$$

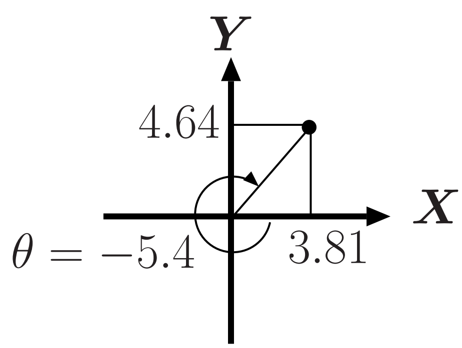
$$y = r \cdot \sin\theta = 4 \cdot \sin(2.8) = 1.34$$

11) Calculate the equivalent Cartesian coordinates of the following polar coordinates.
   a) $\left[3, \frac{7\pi}{4}\right]$

$$x = r \cdot \cos\theta = 3 \cdot \cos(\frac{7\pi}{4}) = \frac{3}{\sqrt{2}}$$

$$y = r \cdot \sin\theta = 3 \cdot \sin(\frac{7\pi}{4}) = -\frac{3}{\sqrt{2}}$$

12) Calculate the equivalent Cartesian coordinates of the following polar coordinates.
   a) $r = 6, \theta = -5.4$



$$\begin{aligned} x &= r \cdot \cos\theta \\ &= 6 \cdot \cos(-5.4) \\ &= 3.81 \end{aligned}$$

$$\begin{aligned} y &= r \cdot \sin\theta \\ &= 6 \cdot \sin(-5.4) \\ &= 4.64 \end{aligned}$$

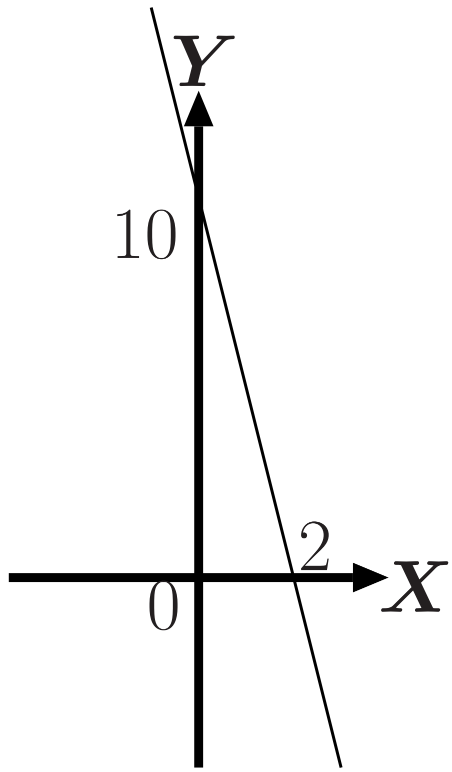
13) Calculate the equivalent Cartesian coordinates of the following polar coordinates.
   a) $\left[2, -\frac{3\pi}{4}\right]$

$$x = r \cdot \cos\theta = 2 \cdot \cos(-\frac{3\pi}{4}) = -\sqrt{2}$$

$$y = r \cdot \sin\theta = 2 \cdot \sin(-\frac{3\pi}{4}) = -\sqrt{2}$$

14) Find the Cartesian form of
   a)

$$r = \frac{10}{\sin\theta + 5 \cdot \cos\theta}$$

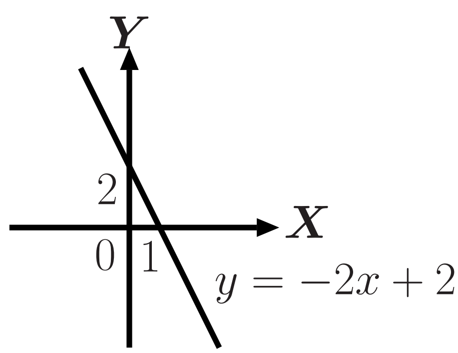


$$\therefore r(\sin\theta + 5\cdot\cos\theta) = 10$$
$$\therefore r\cdot\sin\theta + 5\cdot r\cdot\cos\theta = 10$$
$$\therefore y + 5x = 10 \because r\cdot\sin\theta = y \quad \text{and} \quad r\cdot\cos\theta = x$$

15) Find the Cartesian form of the following curves given in polar form

a)

$$r = \frac{2}{\sin\theta + 2\cos\theta}$$



Equation (16) gives:

$$\cos\theta = \frac{x}{r}$$
$$\sin\theta = \frac{y}{r}$$

Substituting these into $r = \frac{2}{\sin\theta + 2\cos\theta}$, we get

$$r = \frac{2}{\sin\theta + 2\cos\theta}$$
$$= \frac{2}{\frac{y}{r} + 2\frac{x}{r}}$$
$$\therefore r\left(\frac{y}{r} + 2\frac{x}{r}\right) = 2$$

$$\therefore \frac{y \cdot r}{r} + 2\frac{x \cdot r}{r} = 2$$
$$\therefore y + 2x = 2$$

Alternatively,

$$r = \frac{2}{\sin\theta + 2\cos\theta}$$
$$\therefore r(\sin\theta + 2\cos\theta) = 2$$
$$\therefore r\sin\theta + 2r\cos\theta = 2$$
$$\therefore y + 2x = 2$$

b)

$$r = 3\cos\theta$$

Equation (16) and Equation (17) give

$$\cos\theta = \frac{x}{r}$$
$$r^2 = x^2 + y^2$$

Substituting these into $r = 3\cos\theta$, we get

$$r = 3 \cdot \frac{x}{r}$$
$$\therefore r^2 = 3x$$
$$\therefore x^2 + y^2 = 3x$$
$$\therefore x^2 - 3x + y^2 = 0$$
$$\therefore (x - 3/2)^2 - (3/2)^2 + y^2 = 0$$
$$\therefore \left(x - \frac{3}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2$$

c)

$$r = \frac{\sqrt{6}}{2 + \sqrt{6}\cos\theta}$$

Equation (16) and Equation (17) give

$$\cos\theta = \frac{x}{r}$$
$$r^2 = x^2 + y^2$$

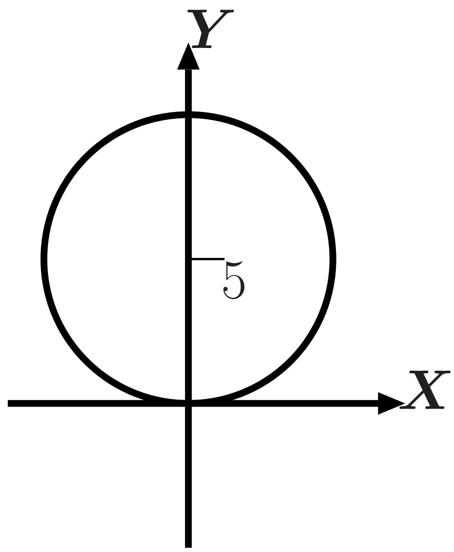
Substituting these into $r = \frac{\sqrt{6}}{2+\sqrt{6}\cos\theta}$, we get

$$r = \frac{\sqrt{6}}{2 + \sqrt{6}\frac{x}{r}}$$
$$\therefore r\left(2 + \sqrt{6}\frac{x}{r}\right) = \sqrt{6}$$
$$\therefore 2r + x\sqrt{6} = \sqrt{6}$$
$$\therefore 2r = \sqrt{6} - x\sqrt{6}$$
$$\therefore (2r)^2 = (\sqrt{6} - x\sqrt{6})^2$$
$$\therefore 4r^2 = 6(1 - x)^2$$
$$\therefore 4(x^2 + y^2) = 6(1 + x^2 - 2x)$$
$$\therefore 4x^2 + 4y^2 = 6 + 6x^2 - 12x$$
$$\therefore 4x^2 + 4y^2 - 6x^2 + 12x = 6$$
$$\therefore -2x^2 + 12x + 4y^2 = 6$$
$$\therefore -2(x^2 - 6x) + 4y^2 = 6$$
$$\therefore -2\left\{(x - 3)^2 - 9\right\} + 4y^2 = 6$$
$$\therefore -2(x - 3)^2 + 18 + 4y^2 = 6$$
$$\therefore -2(x - 3)^2 + 4y^2 = 6 - 18$$
$$\therefore -2(x - 3)^2 + 4y^2 = -12$$
$$\therefore \frac{-2(x - 3)^2 + 4y^2}{-12} = 1$$
$$\therefore \frac{(x - 3)^2 - 2y^2}{6} = 1$$

$$\therefore \frac{(x-3)^2}{6} - \frac{2y^2}{6} = 1$$
$$\therefore \frac{(x-3)^2}{6} - \frac{y^2}{3} = 1$$

16) Find the Cartesian form of
    a)

$$r = 10 \cdot \sin\theta$$



$$\therefore r = 10 \cdot \frac{y}{r} \because \sin\theta = \frac{y}{r}$$
$$\therefore r^2 = 10y$$
$$\therefore x^2 + y^2 = 10y \because r^2 = x^2 + y^2$$
$$\therefore x^2 + y^2 - 10y = 0$$

By completing the square,

$$x^2 + (y-5)^2 - 5^2 = 0$$
$$\therefore x^2 + (y-5)^2 = 5^2$$

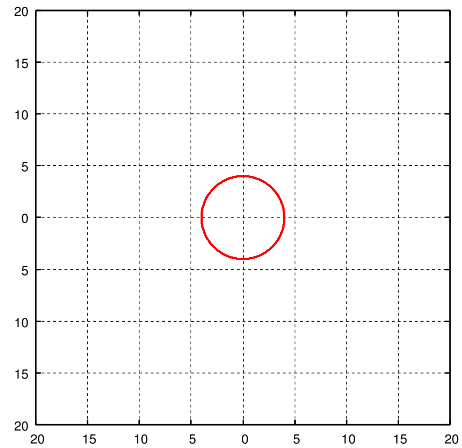
DAY3

17) Sketch the following curve

a)

$$x^2 + y^2 = 16$$

The equation of a circle is $x^2 + y^2 = r^2$. Therefore the sketch is a circle centred at $(0, 0)$ with a radius of $r = 4$ since $r > 0$.



18) Sketch the curve

$$\theta = \frac{\pi}{3}$$

Initially we obtain

$$x = r\cos\left(\frac{\pi}{3}\right)$$
$$y = r\sin\left(\frac{\pi}{3}\right)$$

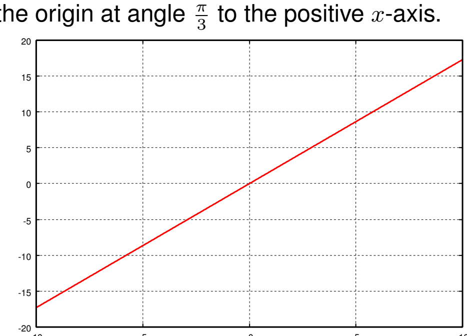
In order to remove r, we perform the division of these two equations as follows:

$$\frac{y = r\sin\left(\frac{\pi}{3}\right)}{x = r\cos\left(\frac{\pi}{3}\right)}$$

Thus

$$\frac{y}{x} = \frac{r\sin\left(\frac{\pi}{3}\right)}{r\cos\left(\frac{\pi}{3}\right)}$$
$$= \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)}$$
$$= \tan\left(\frac{\pi}{3}\right)$$
$$\therefore y = \tan\left(\frac{\pi}{3}\right) \cdot x$$
$$\therefore y = \sqrt{3}x$$

Therefore the curve is a line passing through the origin at angle $\frac{\pi}{3}$ to the positive x-axis.



19) Sketch the ellipse

$$r = \frac{4}{2 - \cos\theta}$$

Equation (17) and Equation (16) give

$$r^2 = x^2 + y^2$$
$$\cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

Substituting these into $r = \frac{4}{2-\cos\theta}$, we get

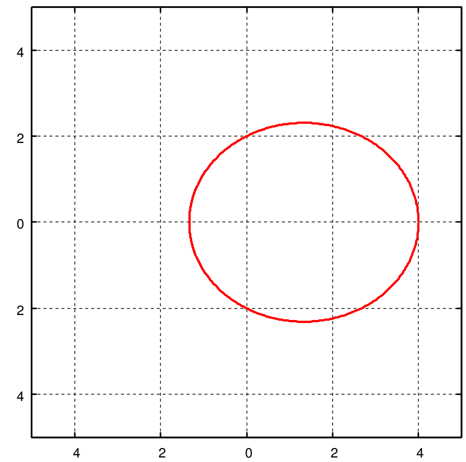
$$\begin{aligned}
r &= \frac{4}{2-\cos\theta} \\
\therefore r &= \frac{4}{2-\frac{x}{r}} \\
\therefore r &= \frac{4r}{2r-x} \\
\therefore 1 &= \frac{4}{2r-x} (\because r \neq 0) \\
\therefore 2r - x &= 4 \\
\therefore 2r &= 4 + x \\
\therefore (2r)^2 &= (4+x)^2 \\
\therefore 4(x^2+y^2) &= 16 + 8x + x^2 \\
\therefore 4x^2 + 4y^2 &= 16 + 8x + x^2 \\
\therefore 3x^2 - 8x + 4y^2 &= 16 \\
\therefore 3\left[\left(x-\frac{4}{3}\right)^2 - \left(\frac{4}{3}\right)^2\right] + 4y^2 &= 16 \\
\therefore 3\left(x-\frac{4}{3}\right)^2 - 3\left(\frac{4}{3}\right)^2 + 4y^2 &= 16 \\
\therefore 3\left(x-\frac{4}{3}\right)^2 + 4y^2 &= 16 + 3\left(\frac{4}{3}\right)^2 \\
\therefore 3\left(x-\frac{4}{3}\right)^2 + 4y^2 &= 16 \cdot \frac{3}{3} + \frac{16}{3} \\
\therefore 3\left(x-\frac{4}{3}\right)^2 + 4y^2 &= \frac{3 \cdot 16 + 16}{3} \\
\therefore 3\left(x-\frac{4}{3}\right)^2 + 4y^2 &= \frac{4 \cdot 16}{3} \\
\therefore 3\left(x-\frac{4}{3}\right)^2 + 4y^2 &= \frac{64}{3} \\
\therefore \frac{3\left(x-\frac{4}{3}\right)^2}{\frac{64}{3}} + \frac{4y^2}{\frac{64}{3}} &= 1 \\
\therefore \frac{\left(x-\frac{4}{3}\right)^2}{\frac{64}{3\cdot 3}} + \frac{y^2}{\frac{64}{3\cdot 4}} &= 1 \\
\therefore \frac{\left(x-\frac{4}{3}\right)^2}{\frac{8^2}{3^2}} + \frac{y^2}{\frac{16}{3}} &= 1 \\
\therefore \frac{\left(x-\frac{4}{3}\right)^2}{\frac{8^2}{3^2}} + \frac{y^2}{\frac{4^2}{(\sqrt{3})^2}} &= 1
\end{aligned}$$

Therefore the centre of the ellipse is $\left(\frac{4}{3}, 0\right)$ and the ellipse passes through

$$\begin{aligned}
&\left(\frac{4}{3} \pm \frac{8}{3}, 0\right) \\
&= \left(\frac{4 \pm 8}{3}, 0\right) \\
&= \left(\frac{12}{3}, 0\right), \left(\frac{-4}{3}, 0\right) \\
&= (4, 0), \left(\frac{-4}{3}, 0\right)
\end{aligned}$$

and

$$\left(\frac{4}{3}, 0 \pm \frac{4}{\sqrt{3}}\right)$$
$$= \left(\frac{4}{3}, \frac{4}{\sqrt{3}}\right), \left(\frac{4}{3}, -\frac{4}{\sqrt{3}}\right)$$

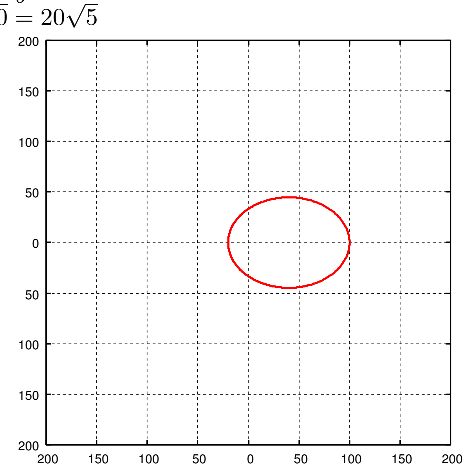


20) Sketch the following curve

a)

$$\frac{(x-40)^2}{3600} + \frac{y^2}{2000} = 1$$

The equation of an ellipse is $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$. Therefore the sketch is an ellipse centred at $(40, 0)$ with a semi-major axis of $\sqrt{3600} = 60$ and a semi-minor axis of $\sqrt{2000} = 20\sqrt{5}$



21) Sketch the following curve

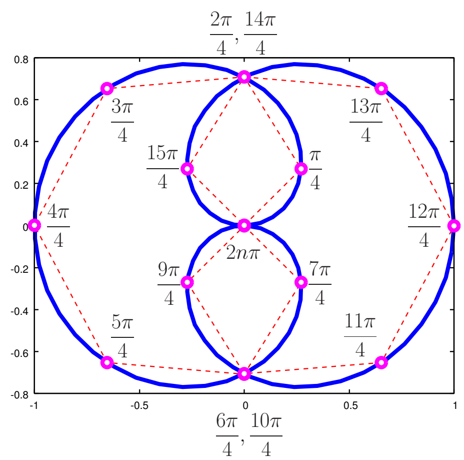
a)

$$r = \sin\left(\frac{\theta}{2}\right), 0 \leq \theta \leq 4\pi$$

When we scan θ every $\frac{\pi}{4}$ we obtain the following table

θ	r	$x = r\cos\theta$	$y = r\sin\theta$
$\frac{0\pi}{4}$	0	0	0
$\frac{1\pi}{4}$	0.383	0.271	0.271
$\frac{2\pi}{4}$	0.707	0	0.707
$\frac{3\pi}{4}$	0.924	-0.653	0.653
$\frac{4\pi}{4}$	1	-1	0
$\frac{5\pi}{4}$	0.924	-0.653	-0.653
$\frac{6\pi}{4}$	0.707	0	-0.707
$\frac{7\pi}{4}$	0.383	0.271	-0.271
$\frac{8\pi}{4}$	0	0	0
$\frac{9\pi}{4}$	-0.383	-0.271	-0.271
$\frac{10\pi}{4}$	-0.707	0	-0.707
$\frac{11\pi}{4}$	-0.924	0.653	-0.653
$\frac{12\pi}{4}$	-1	1	0
$\frac{13\pi}{4}$	-0.924	0.653	0.653
$\frac{14\pi}{4}$	-0.707	0	0.707
$\frac{15\pi}{4}$	-0.383	-0.271	0.271
$\frac{16\pi}{4}$	0	0	0

When we plot these points we obtain the pink circles. Then when we connect the points in the order presented in the table we obtain red dotted line. The real curve is in blue.



22) Sketch the following curve

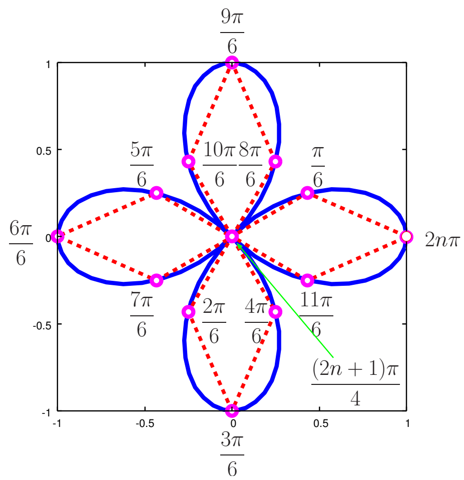
a)

$$r = \cos(2\theta), 0 \le \theta \le 2\pi$$

When we scan θ we obtain the following table

θ	r	$x = r\cos\theta$	$y = r\sin\theta$
$\frac{0\pi}{6}$	1	1	0
$\frac{1\pi}{6}$	0.5	0.433	0.25
$\frac{\pi}{4}$	0	0	0
$\frac{2\pi}{6}$	-0.5	-0.25	-0.433
$\frac{3\pi}{6}$	-1	0	-1
$\frac{4\pi}{6}$	-0.5	0.25	-0.433
$\frac{3\pi}{4}$	0	0	0
$\frac{5\pi}{6}$	0.5	-0.433	0.25
$\frac{6\pi}{6}$	1	-1	0
$\frac{7\pi}{6}$	0.5	-0.433	-0.25
$\frac{5\pi}{4}$	0	0	0
$\frac{8\pi}{6}$	-0.5	0.25	0.433
$\frac{9\pi}{6}$	-1	0	1
$\frac{10\pi}{6}$	-0.5	-0.25	0.433
$\frac{7\pi}{4}$	0	0	0
$\frac{11\pi}{6}$	0.5	0.433	-0.25
$\frac{12\pi}{6}$	1	1	0

When we plot these points we obtain the pink circles. Then when we connect the points in the order presented in the table we obtain red dotted line. The real curve is in blue.



23) Sketch the following curve

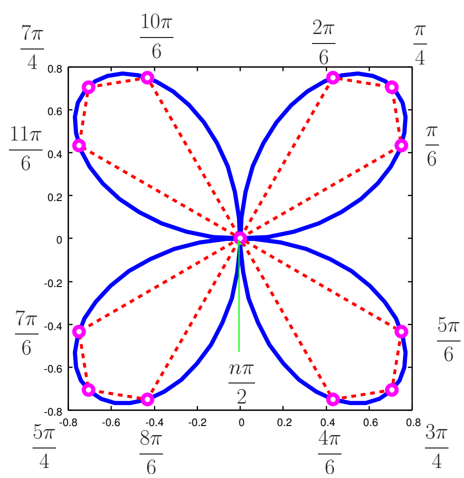
a)

$$r = \sin(2\theta), 0 \leq \theta \leq 2\pi$$

When we scan θ we obtain the following table

θ	$r = \sin(2\theta)$	$x = r\cos\theta$	$y = r\sin\theta$
$\frac{0\pi}{6}$	0	0	0
$\frac{1\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2} \cdot \frac{1}{2}$
$\frac{1\pi}{4}$	1	$1 \cdot \frac{1}{\sqrt{2}}$	$1 \cdot \frac{1}{\sqrt{2}}$
$\frac{2\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2} \cdot \frac{1}{2}$	$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$
$\frac{3\pi}{6}$	0	0	0
$\frac{4\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2} \cdot \frac{1}{2}$	$-\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	-1	$1 \cdot \frac{1}{\sqrt{2}}$	$-1 \cdot \frac{1}{\sqrt{2}}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2} \cdot \frac{1}{2}$
$\frac{6\pi}{6}$	0	0	0
$\frac{7\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2} \cdot \frac{1}{2}$
$\frac{5\pi}{4}$	1	$-1 \cdot \frac{1}{\sqrt{2}}$	$-1 \cdot \frac{1}{\sqrt{2}}$
$\frac{8\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2} \cdot \frac{1}{2}$	$-\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$
$\frac{9\pi}{6}$	0	0	0
$\frac{10\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2} \cdot \frac{1}{2}$	$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$
$\frac{7\pi}{4}$	-1	$-1 \cdot \frac{1}{\sqrt{2}}$	$1 \cdot \frac{1}{\sqrt{2}}$
$\frac{11\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2} \cdot \frac{1}{2}$
$\frac{12\pi}{6}$	0	0	0

When we plot these points we obtain the pink circles. Then when we connect the points in the order presented in the table we obtain red dotted line. The real curve is in blue.



DAY4

24) Find the polar form of

a)

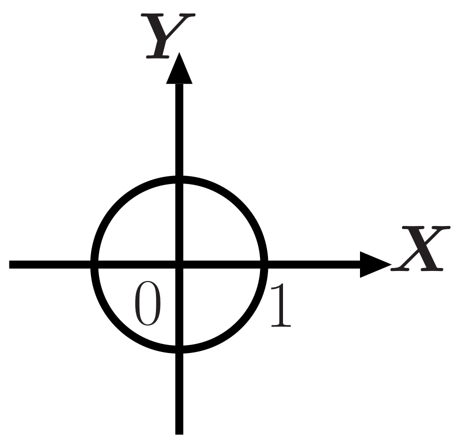
$$2xy = 1$$

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ \therefore 2\cdot r\cos\theta\cdot r\sin\theta &= 1 \\ \therefore 2r^2\sin\theta\cos\theta &= 1 \\ \therefore r^2\sin 2\theta = 1 \because 2\sin\theta\cos\theta &= \sin 2\theta \\ \therefore r^2 &= \frac{1}{\sin 2\theta} \end{aligned}$$

25) Find the polar coordinate form (remember $r > 0$) of

a) a circle

$$x^2 + y^2 = 1$$



We put $x = r\cos\theta$ and $y = r\sin\theta$ into the equation ($r > 0$):

$$\begin{aligned} (r\cos\theta)^2 + (r\sin\theta)^2 &= 1 \\ \therefore r^2(\cos^2\theta + \sin^2\theta) &= 1 \\ \therefore r^2 = 1 \because \cos^2\theta + \sin^2\theta &= 1 \\ \therefore r &= 1 \end{aligned}$$

b) a parabola

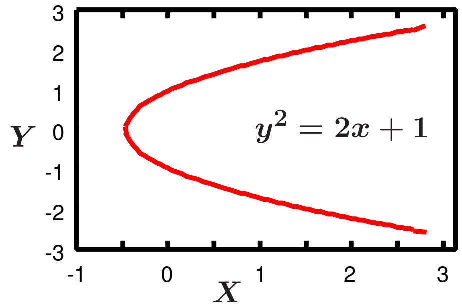
$$y = 2x^2$$

We put $x = r\cos\theta$ and $y = r\sin\theta$ into the equation ($r > 0$) :

$$\begin{aligned} r\sin\theta &= 2(r\cos\theta)^2 \\ \therefore r\sin\theta &= 2r^2\cos^2\theta \\ \therefore r\sin\theta - 2r^2\cos^2\theta &= 0 \\ \therefore r(\sin\theta - 2r\cos^2\theta) &= 0 \\ \therefore \sin\theta - 2r\cos^2\theta &= 0(\because r \neq 0) \\ \therefore \sin\theta &= 2r\cos^2\theta \\ \therefore \frac{\sin\theta}{2\cos^2\theta} &= r \\ \therefore \frac{\cos\theta\tan\theta}{2\cos^2\theta} = r \because \tan\theta &= \frac{\sin\theta}{\cos\theta} \quad \text{and} \quad \therefore \sin\theta = \cos\theta\tan\theta \\ \therefore \frac{\tan\theta}{2\cos\theta} &= r \end{aligned}$$

26) Find the polar form of the following curve given in Cartesian form

$$y^2 = 1 + 2x$$



Substituting

$$x = r\cos\theta$$
$$y = r\sin\theta$$

into $y^2 = 1 + 2x$, we get

$$(r\sin\theta)^2 = 1 + 2r\cos\theta$$
$$\therefore r^2\sin^2\theta = 1 + 2r\cos\theta$$
$$\therefore r^2\sin^2\theta - 2r\cos\theta - 1 = 0$$

The roots of

$$ar^2 + br^1 + c = 0$$

are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The equation in question is identical to

$$ar^2 + br^1 + c = 0$$

when

$$a = \sin^2\theta$$
$$b = -2\cos\theta$$
$$c = -1$$

Thus

$$r = \frac{2\cos\theta \pm \sqrt{(2\cos\theta)^2 + 4\sin^2\theta}}{2\sin^2\theta}$$
$$= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta + 4\sin^2\theta}}{2\sin^2\theta}$$
$$= \frac{2\cos\theta \pm \sqrt{4(\cos^2\theta + \sin^2\theta)}}{2\sin^2\theta}$$
$$= \frac{2\cos\theta \pm \sqrt{4}}{2\sin^2\theta}$$
$$= \frac{2\cos\theta \pm 2}{2\sin^2\theta}$$
$$= \frac{\cos\theta \pm 1}{\sin^2\theta}$$

Since $r > 0$, $\cos\theta - 1 (< 0)$ is not appropriate for r. Thus

$$r = \frac{\cos\theta + 1}{\sin^2\theta}$$
$$= \frac{\cos\theta + 1}{1 - \cos^2\theta}$$
$$= \frac{\cos\theta + 1}{(1 + \cos\theta)(1 - \cos\theta)}$$
$$= \frac{1}{1 - \cos\theta}$$

27) Find the polar form of

a)

$$(x - 6)^2 + y = 36$$

$$
\begin{aligned}
x &= r \cdot \cos\theta \\
y &= r \cdot \sin\theta \\
\therefore (r \cdot \cos\theta - 6)^2 + (r \cdot \sin\theta)^2 &= 36 \\
\therefore r^2 \cdot \cos^2\theta - 12 \cdot r \cdot \cos\theta + 36 + r^2 \cdot \sin^2\theta &= 36 \\
\therefore r^2(\cos^2\theta + \sin^2\theta) - 12 \cdot r \cdot \cos\theta + 36 &= 36 \\
\therefore r^2 - 12 \cdot r \cdot \cos\theta + 36 = 36 \because \cos^2\theta + \sin^2\theta &= 1 \\
\therefore r^2 &= 12 \cdot r \cdot \cos\theta - 36 + 36 \\
\therefore r^2 &= 12 \cdot r \cdot \cos\theta \\
\therefore r &= 12 \cdot \cos\theta
\end{aligned}
$$

28) Find point where the curve whose equation in 2D polar form is

$$r = 2\cos\theta$$

with $-\pi \le \theta \le \pi$ meet the line whose equation in 2D Cartesian coordinates is $y = x - 1$. Express your answer in Cartesian form.

- First solution

$$
\begin{aligned}
x = r \cdot \cos\theta; y &= r \cdot \sin\theta \\
\therefore r = 2\cos\theta &= 2\frac{x}{r} \\
\therefore r^2 &= 2x \\
\therefore x^2 + y^2 &= 2x \\
\therefore (x-1)^2 + y^2 &= 1 \qquad ①
\end{aligned}
$$

When we put $y = x - 1$ into ① we obtain

$$(y)^2 + y^2 = 1; 2y^2 = 1; y = \pm\frac{1}{\sqrt{2}} \qquad ②$$

When we put ② into $y = x - 1$, we obtain $x = y + 1 = \pm\frac{1}{\sqrt{2}} + 1$. Thus the answer is $(x, y) = (1 \pm \frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$

- Second solution The Cartesian coordiante of a point on the line $r = 2\cos(\theta)$ can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} 2\cos^2\theta \\ 2\cos\theta\sin\theta \end{pmatrix} \qquad ①$$

The Cartesian coordiante of a point on the line $y = x - 1$ can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t-1 \end{pmatrix} \qquad ②$$

If ① and ② meet, then there are θ and t which satisfy

$$\begin{pmatrix} 2\cos^2\theta \\ 2\cos\theta\sin\theta \end{pmatrix} = \begin{pmatrix} t \\ t-1 \end{pmatrix} \qquad ③$$

③ generates

$$
\begin{aligned}
2\cos^2\theta &= t \qquad ④ \\
\therefore 2\cos\theta\sin\theta &= t - 1 \qquad ⑤
\end{aligned}
$$

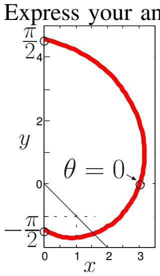
④-⑤ gives us

$$
\begin{aligned}
2\cos^2\theta - 2\cos\theta\sin\theta &= 1 \\
\therefore 2\cos^2\theta - 1 &= 2\cos\theta\sin\theta \\
\therefore \cos 2\theta &= \sin 2\theta \\
\therefore 1 = \frac{\sin 2\theta}{\cos 2\theta} &= \tan 2\theta \\
\therefore 2\theta = \tan^{-1}(1) &= \frac{\pi}{4} + n\pi
\end{aligned}
$$

where n is $-2, -1, 0, 1$. Thus the Cartesian coordinate of the intersection of these two lines is

$$
\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2\cos^2\theta \\ 2\cos\theta\sin\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos 2\theta + 1 \\ \sin 2\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\left(\frac{\pi}{4} + n\pi\right) + 1 \\ \sin\left(\frac{\pi}{4} + n\pi\right) \end{pmatrix} \\
&= \begin{pmatrix} \pm\frac{1}{\sqrt{2}} + 1 \\ \pm\frac{1}{\sqrt{2}} \end{pmatrix}
\end{aligned}
$$

29) Find a point where the curve whose equation in 2D polar coordinates is $r = 3 + \theta$ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ meets the line whose equation in 2D Cartesian coordinates is $y = -x$. Express your answer in Cartesian form.



The Cartesian coordiante of a point on the line $r = 3 + \theta$ can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} (3+\theta)\cos\theta \\ (3+\theta)\sin\theta \end{pmatrix} \quad ①$$

The Cartesian coordiante of a point on the line $y = -x$ can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -t \end{pmatrix} \quad ②$$

If ① and ② meet, then there are θ and t which satisfy

$$\begin{pmatrix} (3+\theta)\cos\theta \\ (3+\theta)\sin\theta \end{pmatrix} = \begin{pmatrix} t \\ -t \end{pmatrix} \quad ③$$

③ generates

$$(3+\theta)\cos\theta = t \quad ④$$
$$(3+\theta)\sin\theta = -t \quad ⑤$$

$\frac{⑤}{④}$ gives us

$$\frac{(3+\theta)\sin\theta}{(3+\theta)\cos\theta} = \frac{-t}{t}$$
$$\therefore \tan\theta = -1$$
$$\therefore \theta = \tan^{-1}(-1) = -\frac{\pi}{4}$$

Thus the Cartesian coordinate of the intersection of these two lines is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (3+\theta)\cos\theta \\ (3+\theta)\sin\theta \end{pmatrix} = \begin{pmatrix} (3-\frac{\pi}{4})\cos(-\frac{\pi}{4}) \\ (3-\frac{\pi}{4})\sin(-\frac{\pi}{4}) \end{pmatrix} = \begin{pmatrix} \frac{3-\frac{\pi}{4}}{\sqrt{2}} \\ -\frac{3-\frac{\pi}{4}}{\sqrt{2}} \end{pmatrix}$$

30) Simplify

$$18 - 4 \div 2$$

$$\begin{aligned} &18 - 4 \div 2 \\ &= 18 - 2 \\ &= 16 \end{aligned}$$

31) Simplify

$$(3+9)^0$$

$$\begin{aligned} &(3+9)^0 \\ &= 12^0 \\ &= 1 \end{aligned}$$

32) Simplify

$$(-3)^2$$

$$\begin{aligned} &(-3)^2 \\ &= 9 \end{aligned}$$

33) Evaluate the following expression

$$2|-2-(-6)| \div 4$$

$$\begin{aligned} 2|-2-(-6)| \div 4 \\ = 2|-2+6| \div 4 \\ = 2|4| \div 4 \\ = 8 \div 4 \\ = 2 \end{aligned}$$

34) Solve the following equation

$$(x\cos\theta)^2 + (x\sin\theta)^2 = 4$$

$$\begin{aligned} (x\cos\theta)^2 + (x\sin\theta)^2 = 4 \\ \therefore x^2(\cos^2\theta + \sin^2\theta) = 4 \\ \therefore x^2 = 4 \because \cos^2\theta + \sin^2\theta = 1 \\ \therefore x = \pm\sqrt{4} \\ = \pm 2 \end{aligned}$$

35) Solve the following equation by completing the square

$$x^2 + 6x - 7 = 0$$

$$\begin{aligned} x^2 + 6x - 7 = 0 \\ \therefore x^2 + 6x = 7 \\ \therefore (x+3)^2 - 9 = 7 \\ \therefore (x+3)^2 = 16 \\ \therefore x + 3 = \pm\sqrt{16} \\ \therefore x + 3 = \pm 4 \\ \therefore x = -3 \pm 4 \\ = 1, -7 \end{aligned}$$

DAY5

36) In Cartesian coordinates a point P is given by $x = -1$, $y = -\sqrt{3}$, $z = 2$. Give the position of P in spherical coordinates. We need to find out r, θ and ϕ which

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{1^2+(-\sqrt{3})^2+2^2} = \sqrt{1+3+4} = \sqrt{8} = 2\sqrt{2}$$
$$\cos\theta = \frac{z}{r} = \frac{2}{\sqrt{8}}$$
$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$
$$\tan\phi = \frac{y}{x} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$
$$\therefore \phi = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}, \frac{4\pi}{3}$$

However ϕ is in the 3rd quadrant. Thus $\phi = \frac{4\pi}{3}$.

37) In 3D Cylindrical polar coordinates point P is $(\rho, \phi, z) = (2, 2\pi/3, -\sqrt{12})$. Convert this to

a) 3D Cartesians coordinates

$$x = \rho\cos\phi = 2\cos(2\pi/3) = -1$$
$$y = \rho\sin\phi = 2\sin(2\pi/3) = \sqrt{3}$$
$$z = z = -\sqrt{12}$$

b) 3D Spherical polar coordinates

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2} = \sqrt{2^2+12} = \sqrt{16} = 4$$
$$\theta = \cos^{-1}(z/r) = \cos^{-1}(\frac{-\sqrt{12}}{\sqrt{16}}) = \frac{\pi}{6}\text{radians}$$
$$\phi = \frac{2\pi}{3}\text{radians}$$

38) A point Q lies on the sphere $r = 5$ in Spherical polar coordinates, and on the cylinder $\rho = 4$ in Cylindrical polar coordinates and on the plane $2x = z$ in Cartesian coordinates. (All coordinates are in 3D). Find Q in Cartesian coordinates, given that x, y and z are positive.
$r = 5$ in Spherical polar coordinates can be written as

$$x^2+y^2+z^2 = r^2 = 25 \qquad ①$$

$\rho = 4$ in Cylindrical polar coordinates can be written as

$$x^2+y^2 = \rho^2 = 16 \qquad ②$$

We also got

$$2x = z \qquad ③$$

①–② gives $z^2 = 9$, *i.e.,* $z = \pm 3$. Since the problem tells that z is positive, $z = 3$ is the part of the answer to the problem. By putting $z = 3$ into ③, we obtain $x = \frac{3}{2}$. By putting $x = \frac{3}{2}$ into ②,

$$y^2 = 16 - \left(\frac{3}{2}\right)^2 = 16 - \frac{9}{4} = 55/4$$
$$\therefore y = \sqrt{55/4}(\because y > 0)$$

39) In 3D Cartesian coordinates a point P is $(\sqrt{3}, \sqrt{3}, \sqrt{2})$.

a) Convert P to 3D Cylindrical coordinates. In cylindrical coordinates

$$\begin{pmatrix}\rho\\ \phi\\ z\end{pmatrix} = \begin{pmatrix}\sqrt{x^2+y^2}\\ \tan^{-1}\left(\frac{y}{x}\right)\\ z\end{pmatrix} = \begin{pmatrix}\sqrt{(\sqrt{3})^2+(\sqrt{3})^2}\\ \tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)\\ \sqrt{2}\end{pmatrix} = \begin{pmatrix}\sqrt{3+3}\\ \tan^{-1}(1)\\ \sqrt{2}\end{pmatrix} = \begin{pmatrix}\sqrt{6}\\ \frac{\pi}{4}\\ \sqrt{2}\end{pmatrix}$$

b) Convert P to 3D Spherical coordinates.
In spherical coordinates

$$\begin{pmatrix}r\\ \theta\\ \phi\end{pmatrix} = \begin{pmatrix}\sqrt{x^2+y^2+z^2}\\ \cos^{-1}\left(\frac{z}{r}\right)\\ \tan^{-1}\left(\frac{y}{x}\right)\end{pmatrix} = \begin{pmatrix}\sqrt{(\sqrt{3})^2+(\sqrt{3})^2+(\sqrt{2})^2}\\ \cos^{-1}\left(\frac{\sqrt{2}}{r}\right)\\ \tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)\end{pmatrix} = \begin{pmatrix}\sqrt{8}\\ \cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{8}}\right)\\ \tan^{-1}(1)\end{pmatrix}$$
$$= \begin{pmatrix}2\sqrt{2}\\ \cos^{-1}\left(\frac{1}{\sqrt{4}}\right)\\ \frac{\pi}{4}\end{pmatrix} = \begin{pmatrix}2\sqrt{2}\\ \cos^{-1}\left(\frac{1}{2}\right)\\ \frac{\pi}{4}\end{pmatrix} = \begin{pmatrix}2\sqrt{2}\\ \frac{\pi}{3}\\ \frac{\pi}{4}\end{pmatrix}$$

40) In Cartesian coordinates a point P is given by $x = 3$, $y = \sqrt{3}$, $z = 2$. Give the position of P in spherical coordinates. We need to find out r, θ and ϕ which

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{3^2 + (\sqrt{3})^2 + 2^2} = \sqrt{9 + 3 + 4} = \sqrt{16} = 4$$
$$\cos\theta = \frac{z}{r} = \frac{2}{4}$$
$$\therefore \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$
$$\tan\phi = \frac{y}{x} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$
$$\therefore \phi = \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$$

41) In 3D Cylindrical polar coordinates point P is $(\rho, \phi, z) = (\sqrt{2}, 3\pi/4, \sqrt{6})$. Convert this to

a) 3D Cartesians coordinates

$$x = \rho\cos\phi = \sqrt{2}\cos(3\pi/4) = \frac{\sqrt{2}}{-\sqrt{2}} = -1$$
$$y = \rho\sin\phi = \sqrt{2}\sin(3\pi/4) = \frac{\sqrt{2}}{\sqrt{2}} = 1$$
$$z = z = \sqrt{6}$$

b) 3D Spherical polar coordinates

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2} = \sqrt{(\sqrt{2})^2 + (\sqrt{6})^2} = \sqrt{8} = 2\sqrt{2}$$
$$\theta = \cos^{-1}(z/r) = \cos^{-1}(\frac{\sqrt{6}}{2\sqrt{2}}) = \cos^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{6}\text{radians}$$
$$\phi = \frac{3\pi}{4}\text{radians}$$

42) A point Q lies on the sphere $r = 10$ in Spherical polar coordinates, and on the cylinder $\rho = 8$ in Cylindrical polar coordinates and on the plan $\frac{\sqrt{3}}{2}x = z$ in Cartesian coordinates. (All coordinates are in 3D) Find Q in Cartesian coordinates, given that x, y and z are positive.
$r = 10$ in Spherical polar coordinates can be written as

$$x^2 + y^2 + z^2 = r^2 = 100 \qquad ①$$

$\rho = 8$ in Cylindrical polar coordinates can be written as

$$x^2 + y^2 = \rho^2 = 64 \qquad ②$$

We also got

$$\frac{\sqrt{3}}{2}x = z \qquad ③$$

①−② gives $z^2 = 36$, *i.e.*, $z = \pm 6$. Since the problem tells that z is positive, $z = 6$ is the part of the answer to the problem. By putting $z = 6$ into ③, we obtain $x = 4\sqrt{3}$. By putting $x = 4\sqrt{3}$ into ②,

$$y^2 = 64 - \left(4\sqrt{3}\right)^2 = 64 - 48 = 16$$
$$\therefore y = 4(\because y > 0)$$

43) In 3D Cartesian coordinates a point P is $(5, 0, 0)$.

a) Convert P to 3D Cylindrical coordinates. In cylindrical coordinates

$$\begin{pmatrix} \rho \\ \phi \\ z \end{pmatrix} = \begin{pmatrix} \sqrt{x^2 + y^2} \\ \tan^{-1}\left(\frac{y}{x}\right) \\ z \end{pmatrix} = \begin{pmatrix} \sqrt{5^2 + 0^2} \\ \tan^{-1}\left(\frac{0}{5}\right) \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{25} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}$$

b) Convert P to 3D Spherical coordinates.
In spherical coordinates

$$\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix} = \begin{pmatrix} \sqrt{x^2 + y^2 + z^2} \\ \cos^{-1}\left(\frac{z}{r}\right) \\ \tan^{-1}\left(\frac{y}{x}\right) \end{pmatrix} = \begin{pmatrix} \sqrt{5^2 + 0^2 + 0^2} \\ \cos^{-1}\left(\frac{0}{r}\right) \\ \tan^{-1}\left(\frac{0}{5}\right) \end{pmatrix} = \begin{pmatrix} \sqrt{25} \\ \cos^{-1}(0) \\ \tan^{-1}(0) \end{pmatrix} = \begin{pmatrix} 5 \\ \frac{\pi}{2} \\ 0 \end{pmatrix}$$

44) A point Q lies on the sphere $r = 5$ in Spherical polar coordinates, and on the cylinder $\rho = 1$ in Cylindrical polar coordinates and on the plane $z = 6x$ in Cartesian coordinates. (All coordinates are in 3D). Find Q in Cartesian coordinates, given that x, y and z are positive. $r = 5$ in Spherical polar coordinates can be written as

$$x^2 + y^2 + z^2 = r^2 = 25 \qquad ①$$

$\rho = 1$ in Cylindrical polar coordinates can be written as

$$x^2 + y^2 = \rho^2 = 1 \qquad ②$$

We also got

$$z = 6x \qquad ③$$

①−② gives $z^2 = 24$, *i.e.,* $z = \pm 2\sqrt{6}$. Since the problem tells that z is positive, $z = 2\sqrt{6}$ is the part of the answer to the problem. By putting $z = 2\sqrt{6}$ into ③, we obtain $x = \frac{\sqrt{6}}{3}$. By putting $x = \frac{\sqrt{6}}{3}$ into ②,

$$y^2 = 1 - \left(\frac{\sqrt{6}}{3}\right)^2 = 1 - \frac{6}{9} = 3/9 = 1/3$$
$$\therefore y = \sqrt{1/3} (\because y > 0)$$

45) Find the Cartesian form of

a)

$$\rho = \frac{z+1}{\sin\phi - \cos\phi}$$

We know that

$$x = \rho\cos\phi; y = \rho\sin\phi$$

Therefore we can manipulate the original equation as

$$\rho(\sin\phi - \cos\phi) = z + 1$$
$$\therefore \rho\sin\phi - \rho\cos\phi = z + 1$$
$$\therefore y - x = z + 1; \therefore x - y + z = -1$$

XI. Exercises on Complex numbers

complexnumberall.tex

1) **DAY1**
2) Make y the subject of $2\mathfrak{e}^{9y} = x + c$.

$$\begin{aligned}
2\mathfrak{e}^{9y} &= x + c \\
\mathfrak{e}^{9y} &= \frac{1}{2}(x + c) \\
\ln \mathfrak{e}^{9y} &= \ln |\frac{1}{2}(x + c)| \\
9y &= \ln |\frac{1}{2}(x + c)| \\
y &= \frac{1}{9} \ln |\frac{1}{2}(x + c)|
\end{aligned}$$

3) Solve the equation

$$\log_4(x) + \log_4(2x + 1) = 0$$

We work under the condition of $x > 0$ and $2x + 1 > 0$

$$\begin{aligned}
\log_4(x) + \log_4(2x + 1) &= 0 \\
\therefore \log_4 (x(2x + 1)) &= 0 \\
\therefore \log_4 (x(2x + 1)) &= \log_4 1 \\
\therefore x(2x + 1) &= 1 \\
\therefore 2x^2 + x - 1 &= 0 \\
\therefore x &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot 2 \cdot -1}}{2 \cdot 2} \\
\therefore x &= \frac{-1 \pm \sqrt{1 - (-8)}}{4} \\
\therefore x &= \frac{-1 \pm \sqrt{9}}{4} \\
\therefore x &= \frac{-1 \pm 3}{4} \\
\therefore x &= \frac{-1 + 3}{4} (\because x > 0) \\
\therefore x &= \frac{1}{2}
\end{aligned}$$

4) Evaluate the following expression

$$12 + \sqrt{-9 + 15 \times 5 \div 3}$$

$$\begin{aligned}
&12 + \sqrt{-9 + 15 \times 5 \div 3} \\
&= 12 + \sqrt{-9 + 75 \div 3} \\
&= 12 + \sqrt{-9 + 25} \\
&= 12 + \sqrt{16} \\
&= 12 + 4 \\
&= 16
\end{aligned}$$

5) Simplify the following expression

$$3x(-x + 1) - 5(x - 3) + 8x^2$$

$$\begin{aligned}
&3x(-x + 1) - 5(x - 3) + 8x^2 \\
&= -3x^2 + 3x - 5x + 15 + 8x^2 \\
&= 5x^2 - 2x + 15
\end{aligned}$$

6) Evaluate the following expression when $a = -1$.

$$(1 - 5a)^2 - 4a \cdot (-1 + 8a)$$

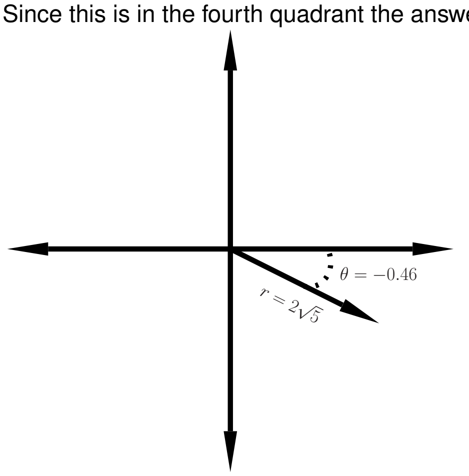
$$
\begin{aligned}
&(1-5a)^2 - 4a \cdot (-1+8a) \\
&= (1-5a)(1-5a) - 4a(-1+8a) \\
&= 1 - 10a + 25a^2 + 4a - 32a^2 \\
&= 1 - 10(-1) + 25(-1)^2 + 4(-1) - 32(-1)^2 \\
&= 1 - 10(-1) + 25(1) + 4(-1) - 32(1) \\
&= 1 + 10 + 25 - 4 - 32 \\
&= 0
\end{aligned}
$$

7) Calculate the equivalent modulus/argument form of $4 - \jmath 2$ and draw the argand diagram.

$$
\begin{aligned}
r &= \sqrt{a^2 + b^2} \\
&= \sqrt{4^2 + (-2)^2} \\
&= \sqrt{16 + 4} \\
&= \sqrt{20} \\
&= \sqrt{4} \cdot \sqrt{5} \\
&= 2\sqrt{5}
\end{aligned}
$$

$$
\begin{aligned}
\tan\theta &= \frac{b}{a} \\
\therefore \theta &= \tan^{-1}\frac{b}{a} \\
&= \tan^{-1}\frac{-2}{4} \\
&= \tan^{-1}\frac{-1}{2} \\
&= -0.46
\end{aligned}
$$

Since this is in the fourth quadrant the answer is valid.

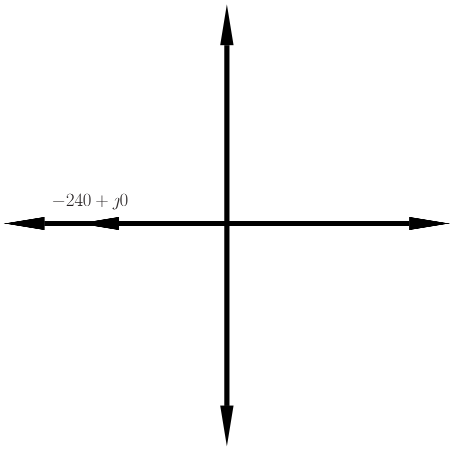


8) Calculate the standard form of $240\angle\pi$ and draw the argand diagram.

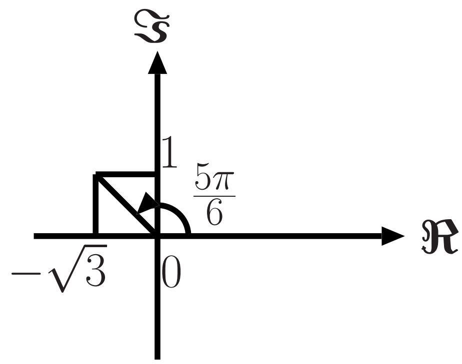
$x = r \cdot \cos\theta$
$y = r \cdot \sin\theta$

Therefore

$$
\begin{aligned}
x &= r \cdot \cos\theta \\
&= 240 \cdot \cos\pi \\
&= 240 \cdot (-1) \\
&= -240 \\
y &= r \cdot \sin\theta \\
&= 240 \cdot \sin\pi \\
&= 240 \cdot 0 \\
&= 0
\end{aligned}
$$



9) Find the modulus/argument form of $(\jmath - \sqrt{3})$



The modulus of $(\jmath - \sqrt{3})$ is

$$\begin{aligned} &\sqrt{1^2 + (-\sqrt{3})^2} \\ &= \sqrt{1+3} \\ &= \sqrt{4} \\ &= \sqrt{2^2} = 2 \end{aligned}$$

Now we find the phase (angle) in the Argand Diagram. Since the real part is $-\sqrt{3}$ and the imaginary part is 1,

$$\begin{aligned} \tan\theta &= \frac{1}{-\sqrt{3}} \\ \therefore \theta &= -\frac{\pi}{6} \end{aligned}$$

which is in the fourth quadrant. However, since $(\jmath - \sqrt{3})$ in Argand diagram is in the second quadrant, the angle has to be changed to the second quadrant equivalent by adding π as follows:

$$\begin{aligned} &-\frac{\pi}{6} + \pi \\ &= -\frac{\pi}{6} + \pi\frac{6}{6} \\ &= \frac{-\pi}{6} + \frac{6\pi}{6} \\ &= \frac{6\pi - \pi}{6} \\ &= \frac{5\pi}{6} \end{aligned}$$

Therefore

$$(\jmath - \sqrt{3}) = 2\mathfrak{e}^{\jmath\frac{5\pi}{6}}$$

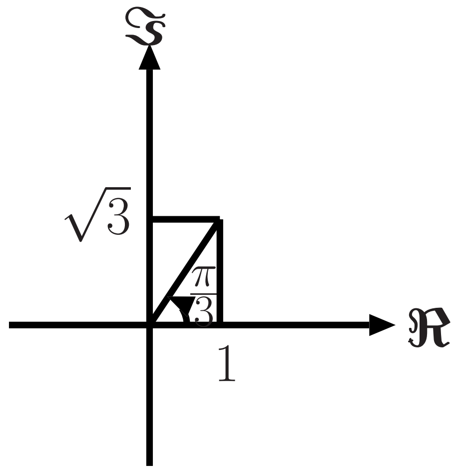
10) The point p is at $z = r\mathfrak{e}^{\jmath t}$. We rotate p about origin and the angle of the rotation is $\pi/3$. The new position of p is $\hat{p}$. Calculate $\hat{p}$ in standard form. $z = r\mathfrak{e}^{\jmath t}$ is rotated by $\pi/3$.
This can be expressed as

$$\begin{aligned} z &= r\mathfrak{e}^{\jmath(t+\pi/3)} \\ &= r(\cos(t+\pi/3) + \jmath\sin(t+\pi/3)) \end{aligned}$$

Thus the coordinate of $\hat{p}$ is

$$(r\cos(t+\pi/3), r\sin(t+\pi/3))$$

11) Express the complex number $1+\jmath\sqrt{3}$ in modulus/argument form.



$$r=\sqrt{a^2+b^2}$$
$$r=\sqrt{1^2+(\sqrt{3})^2}$$
$$r=\sqrt{1+3}$$
$$r=2$$

$$\tan\theta=\frac{b}{a}$$
$$\theta=\tan^{-1}(\frac{b}{a})$$
$$\theta=\tan^{-1}(\frac{\sqrt{3}}{1})$$
$$\theta=\frac{\pi}{3}$$

Since this is in the first quadrant the answer is valid

$$\therefore re^{\jmath\theta}=2e^{\jmath\frac{\pi}{3}}$$

12) Express the complex number $-\sqrt{3}-\jmath$ in modulus/argument form.

$$r=\sqrt{a^2+b^2}$$
$$=\sqrt{(-\sqrt{3})^2+(-1)^2}$$
$$=\sqrt{3+1}$$
$$\therefore r=2$$

$$\tan\theta=\frac{b}{a}$$
$$\therefore\theta=\tan^{-1}(\frac{-1}{-\sqrt{3}})$$
$$=\tan^{-1}(\frac{1}{\sqrt{3}})$$
$$=\frac{\pi}{6}$$

which is in the first quadrant. However, since $-\sqrt{3}-\jmath$ in Argand diagram is in the third quadrant, the angle has to be changed to the third quadrant equivalent by adding π as follows:

$$\frac{\pi}{6}+\pi=\frac{\pi}{6}+\frac{6\pi}{6}=\frac{7\pi}{6}$$

Therefore

$$\therefore re^{\jmath\theta} = 2e^{\jmath\frac{7\pi}{6}}$$

13) Express the complex number $-1-\jmath$ in modulus/argument form.

$$\begin{aligned} r &= \sqrt{a^2+b^2} \\ &= \sqrt{(-1)^2+(-1)^2} \\ &= \sqrt{1+1} \\ \therefore r &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \tan\theta &= \frac{b}{a} \\ \therefore \theta &= \tan^{-1}(\frac{-1}{-1}) \\ &= \tan^{-1}(\frac{1}{1}) \\ &= \frac{\pi}{4} \end{aligned}$$

which is in the first quadrant. However, since $-1-\jmath$ in Argand diagram is in the third quadrant, the angle has to be changed to the third quadrant equivalent by adding π as follows:

$$\frac{\pi}{4}+\pi = \frac{\pi}{4}+\frac{4\pi}{4} = \frac{5\pi}{4}$$

Therefore

$$\therefore re^{\jmath\theta} = \sqrt{2}e^{\jmath\frac{5\pi}{4}}$$

14) Express the complex number $\sqrt{3}-\jmath$ in modulus/argument form.

$$\begin{aligned} r &= \sqrt{a^2+b^2} \\ &= \sqrt{(\sqrt{3})^2+(-1)^2} \\ &= \sqrt{3+1} \\ \therefore r &= 2 \end{aligned}$$

$$\begin{aligned} \tan\theta &= \frac{b}{a} \\ \therefore \theta &= \tan^{-1}(\frac{-1}{\sqrt{3}}) \\ &= \tan^{-1}(-\frac{1}{\sqrt{3}}) \\ &= -\frac{\pi}{6} \end{aligned}$$

which is in the fourth quadrant. As $\sqrt{3}-\jmath$ is in the fourth quadrant, the answer is valid Therefore

$$\therefore re^{\jmath\theta} = 2e^{-\jmath\frac{\pi}{6}}$$

15) Express the complex number $-1+\jmath$ in modulus/argument form.

$$\begin{aligned} r &= \sqrt{a^2+b^2} \\ &= \sqrt{(-1)^2+(1)^2} \\ &= \sqrt{1+1} \\ \therefore r &= \sqrt{2} \end{aligned}$$

$$\tan\theta = \frac{b}{a}$$
$$\therefore \theta = \tan^{-1}(\frac{1}{-1})$$
$$= \tan^{-1}(-\frac{1}{1})$$
$$= -\frac{\pi}{4}$$

which is in the fourth quadrant. However, since $-1+\jmath$ in Argand diagram is in the second quadrant, the angle has to be changed to the third quadrant equivalent by adding π as follows:

$$-\frac{\pi}{4} + \pi = -\frac{\pi}{4} + \frac{4\pi}{4} = \frac{3\pi}{4}$$

Therefore

$$\therefore r\mathfrak{e}^{\jmath\theta} = \sqrt{2}\mathfrak{e}^{\jmath\frac{3\pi}{4}}$$

16) Express the complex number $1-\jmath$ in modulus/argument form.

$$r = \sqrt{a^2+b^2}$$
$$= \sqrt{(1)^2+(-1)^2}$$
$$= \sqrt{2}$$
$$\therefore r = \sqrt{2}$$

$$\tan\theta = \frac{b}{a}$$
$$\therefore \theta = \tan^{-1}(\frac{-1}{1})$$
$$= \tan^{-1}(-\frac{1}{1})$$
$$= -\frac{\pi}{4}$$

which is in the fourth quadrant. As $1-\jmath$ is in the fourth quadrant, the answer is valid. Therefore

$$\therefore r\mathfrak{e}^{\jmath\theta} = \sqrt{2}\mathfrak{e}^{-\jmath\frac{\pi}{4}}$$

17) Express the complex number 1 in modulus/argument form.

$$r = \sqrt{a^2+b^2}$$
$$= \sqrt{(1)^2+(0)^2}$$
$$= \sqrt{1}$$
$$\therefore r = 1$$

$$\tan\theta = \frac{b}{a}$$
$$\therefore \theta = \tan^{-1}(\frac{0}{1})$$
$$= \tan^{-1}(0)$$
$$= 0$$

As 1 is on the $+x$ axis, the answer is valid. Therefore

$$\therefore r\mathfrak{e}^{\jmath\theta} = 1\mathfrak{e}^{\jmath 0} = 1$$

18) Express the complex number -1 in modulus/argument form.

$$r = \sqrt{a^2+b^2}$$
$$= \sqrt{(-1)^2+(0)^2}$$
$$= \sqrt{1}$$
$$\therefore r = 1$$

$$\tan\theta = \frac{b}{a}$$
$$\therefore \theta = \tan^{-1}(\frac{0}{-1})$$
$$= \tan^{-1}(0)$$
$$= 0$$

As -1 is on the $-x$ axis, the answer is not correct. The angle has to be changed by adding π as $0 + \pi = \pi$ Therefore

$$\therefore r\mathfrak{e}^{\jmath\theta} = 1\mathfrak{e}^{\jmath\pi} = \mathfrak{e}^{\jmath\pi}$$

19) Express the complex number $3\jmath$ in modulus/argument form.
As there is no real value, we can use $\jmath = \mathfrak{e}^{\jmath\frac{\pi}{2}}$.

$$\jmath = \mathfrak{e}^{\jmath\frac{\pi}{2}}$$
$$\therefore 3\jmath = 3\mathfrak{e}^{\jmath\frac{\pi}{2}}$$

20) Express the complex number $-2\jmath$ in modulus/argument form.
As there is no real value, we can use $-\jmath = \mathfrak{e}^{-\jmath\frac{\pi}{2}}$.

$$-\jmath = \mathfrak{e}^{-\jmath\frac{\pi}{2}}$$
$$\therefore -2\jmath = 2\cdot(-\jmath) = 2\mathfrak{e}^{-\jmath\frac{\pi}{2}}$$

Please note that the value of r in the modulus/argument form is positive. So $-2\mathfrak{e}^{\jmath\frac{\pi}{2}}$ is mathematically identical to $-2\jmath$ but in the modulus/argument form $r = -2 < 0$ is not an ideal answer.

DAY2

21) What is the complex conjugate number of

$$z = \left(\frac{a+\jmath b}{a-\jmath b}\right)^2 + \left(\frac{a-\jmath b}{a+\jmath b}\right)^2$$

$z^* = \left(\frac{a-\jmath b}{a+\jmath b}\right)^2 + \left(\frac{a+\jmath b}{a-\jmath b}\right)^2$

22) a, b, c are real numbers in the polynomial

$$p(z) = 2z^4 + az^3 + bz^2 + cz + 3.$$

Find a such that the numbers 2 and $\jmath$ are roots of $p(z) = 0$. Note that the remaining roots are not given.
Since $z = 2$ and $z = \jmath$ are roots, $p(2) = 0$ and $p(\jmath) = 0$. Therefore

$$\begin{aligned} p(2) &= 2^5 + 2^3a + 2^2b + 2c + 3 = 0 \\ p(\jmath) &= 2(\jmath)^4 + a(\jmath)^3 + b(\jmath)^2 + c(\jmath) + 3 \\ &= 2(\jmath^2)^2 + a\jmath\cdot(\jmath)^2 + b\cdot(-1) + c\jmath + 3 \\ &= 2(-1)^2 + a\jmath\cdot(-1) - b + c\jmath + 3 \\ &= 2 - a\jmath - b + c\jmath + 3 \\ &= 5 - b + (c-a)\jmath = 0 \end{aligned}$$

By equating real and imaginary parts to zero (right handside is $0 + \jmath 0$),

$$\begin{aligned} 5 - b &= 0 \\ c - a &= 0 \end{aligned}$$

These can be manipulated as

$$\begin{aligned} 5 &= b \\ c &= a \end{aligned}$$

By putting these into $p(2) = 0$, we get

$$\begin{aligned} p(2) &= 2^5 + 2^3a + 2^2b + 2c + 3 \\ &= 2^5 + 2^3a + 2^2\cdot 5 + 2a + 3 = 0 \\ \therefore 2^5 + 5\cdot 2^2 + 3 + 2^3a + 2a &= 0 \\ \therefore 32 + 20 + 3 + 8a + 2a &= 0 \\ \therefore 55 + 10a &= 0 \\ \therefore 10a &= -55 \\ \therefore 2a &= -11 \\ \therefore a &= -\frac{11}{2} \end{aligned}$$

23) Let $\hat{z}$ the complex conjugate number of z. Now find z such that

$$z^2 + \hat{z}^2 = 0$$

We assume $z = a + \jmath b$ where a and b are any real numbers.
By putting $z = a + \jmath b$ into $z^2 + \hat{z}^2 = 0$, we get

$$\begin{aligned} &(a+\jmath b)^2 + (a-\jmath b)^2 \\ &= a^2 + 2\cdot a\cdot(\jmath b) + (\jmath b)^2 \\ &+ a^2 + 2\cdot a\cdot(-\jmath b) + (-\jmath b)^2 \\ &= a^2 + 2ab\jmath + (\jmath)^2b^2 \\ &+ a^2 - 2ab\jmath + (\jmath)^2(-b)^2 \\ &= a^2 + 2ab\jmath + (-1)\cdot b^2 \\ &+ a^2 - 2ab\jmath + (-1)\cdot b^2 \\ &= 2a^2 - 2b^2 \equiv 0 \\ &\therefore 2a^2 = 2b^2 \\ &\therefore a^2 = b^2 \\ &\therefore \pm a = b \end{aligned}$$

Thus z can be found out as

$$\begin{aligned} z &= a + \jmath b \\ &= a + \jmath \cdot (\pm a) \\ &= a \pm \jmath a \end{aligned}$$

where a is an arbitrary real number.

24) Solve

$$\jmath x^2 + (1 - 5\jmath)x - 1 + 8\jmath = 0$$

Please do not leave $\jmath$ inside $\sqrt{}$ and inside the denominator.
The roots of

$$ax^2 + bx^1 + c = 0$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The equation in question is identical to

$$ax^2 + bx^1 + c = 0$$

when

$$\begin{aligned} a &= \jmath \\ b &= 1 - 5\jmath \\ c &= -1 + 8\jmath \end{aligned}$$

First, we now calculate $b^2 - 4ac$ and $\sqrt{b^2 - 4ac}$.

$$\begin{aligned} &b^2 - 4ac \\ &= (1 - 5\jmath)^2 - 4 \cdot \jmath \cdot (-1 + 8\jmath) \\ &= 1^2 + 2 \cdot (-5\jmath) + (-5\jmath)^2 - 4\jmath(-1 + 8\jmath) \\ &= 1 - 10\jmath + 5^2(\jmath)^2 + 4\jmath - 4 \cdot 8(\jmath)^2 \\ &= 1 - 10\jmath + 25 \cdot (-1) + 4\jmath - 32 \cdot (-1) \\ &= 1 - 10\jmath - 25 + 4\jmath + 32 \\ &= 8 - 6\jmath \end{aligned}$$

In order to turn $\sqrt{8 - 6\jmath}$ into the form of $p + q\jmath$ where p and q are non-zero real, we set the following equation and manipulate it as follows:

$$\begin{aligned} p + q\jmath &= \sqrt{8 - 6\jmath} \\ \therefore (p + q\jmath)^2 &= \left(\sqrt{8 - 6\jmath}\right)^2 \\ \therefore p^2 + 2pq\jmath + (q\jmath)^2 &= 8 - 6\jmath \\ \therefore p^2 + 2pq\jmath + q^2(\jmath)^2 &= 8 - 6\jmath \\ \therefore p^2 + 2pq\jmath + q^2 \cdot (-1) &= 8 - 6\jmath \\ \therefore p^2 + 2pq\jmath - q^2 &= 8 - 6\jmath \\ \therefore p^2 - q^2 + 2pq\jmath &= 8 - 6\jmath \end{aligned}$$

By separating the real and the imaginary part of the equation, we obtain:

$$\begin{aligned} p^2 - q^2 &= 8 \\ 2pq &= -6 \end{aligned}$$

The second equation can be modified as :

$$\begin{aligned} 2pq &= -6 \\ \therefore pq &= -3 \\ \therefore q &= -3/p \end{aligned}$$

When we put $q = -3/p$ into $p^2 - q^2 = 8$, then we obtain

$$\begin{aligned} p^2 - q^2 &= 8 \\ \therefore p^2 - (-3/p)^2 &= 8 \\ \therefore p^2 - \frac{9}{p^2} &= 8 \\ \therefore p^2 \cdot p^2 - 9 &= 8p^2 \\ \therefore p^4 - 8p^2 - 9 &= 0 \\ \therefore (p^2 - 9)(p^2 + 1) &= 0 \end{aligned}$$

However since we assume that p is a real non-zero number, $p^2 > 0$. Thus

$$\begin{aligned}
p^2 - 9 &= 0 \\
\therefore p^2 &= 9 \\
\therefore p &= \pm 9^{\frac{1}{2}} \\
\therefore p &= \pm (3^2)^{\frac{1}{2}} \\
\therefore p &= \pm 3^{2 \times \frac{1}{2}} \\
\therefore p &= \pm 3
\end{aligned}$$

Using $q = -3/p$, we can get

$$\begin{aligned}
q &= -\frac{3}{p} \\
\therefore q &= -\frac{3}{\pm 3} \\
\therefore q &= \mp \frac{3}{3} \\
\therefore q &= \mp 1
\end{aligned}$$

Thus

$$(p, q) = (\pm 3, \mp 1).$$

This means

$$\pm(3 - \jmath) = \sqrt{8 - 6\jmath}.$$

Going back to the original equation, the answer is

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(1 - 5\jmath) \pm (\pm(3 - \jmath))}{2\jmath} \\
&= \frac{-(1 - 5\jmath) + (3 - \jmath)}{2\jmath}, \frac{-(1 - 5\jmath) - (3 - \jmath)}{2\jmath} \\
&= \frac{-1 + 5\jmath + 3 - \jmath}{2\jmath}, \frac{-1 + 5\jmath - 3 + \jmath}{2\jmath} \\
&= \frac{2 + 4\jmath}{2\jmath}, \frac{-4 + 6\jmath}{2\jmath} \\
&= \frac{1 + 2\jmath}{\jmath}, \frac{-2 + 3\jmath}{\jmath} \\
&= \frac{(1 + 2\jmath) \cdot \jmath}{(\jmath)^2}, \frac{(-2 + 3\jmath) \cdot \jmath}{(\jmath)^2} \\
&= \frac{\jmath + 2(\jmath)^2}{(\jmath)^2}, \frac{-2\jmath + 3(\jmath)^2}{(\jmath)^2} \\
&= \frac{\jmath + 2 \cdot (-1)}{(\jmath)^2}, \frac{-2\jmath + 3 \cdot (-1)}{(\jmath)^2} \\
&= \frac{\jmath - 2}{-1}, \frac{-2\jmath - 3}{-1} \\
&= -(\jmath - 2), -(-2\jmath - 3) \\
&= -\jmath + 2, 2\jmath + 3
\end{aligned}$$

25) Find out whether or not a polynomial equation $k^4 - 2k^3 + 4k^2 - k + 4 = 0$ has roots $k = \jmath b$. If it has, find all real values of b. We must first substitute the form $k = \jmath b$ into the polynomial.

$$\begin{aligned}
k^4 - 2k^3 + 4k^2 - k + 4 &= 0 \\
(\jmath b)^4 - 2(\jmath b)^3 + 4(\jmath b)^2 - (\jmath b) + 4 &= 0 \\
\jmath^4 b^4 - 2\jmath^3 b^3 + 4\jmath^2 b^2 - \jmath b + 4 &= 0 \\
\jmath^2 \cdot \jmath^2 b^4 - 2\jmath^2 \cdot \jmath b^3 + 4\jmath^2 b^2 - \jmath b + 4 &= 0 \\
(-1 \cdot -1) b^4 - 2 \times -1\jmath b^3 + 4 \times -1 b^2 - \jmath b + 4 &= 0 \\
1b^4 + 2\jmath b^3 - 4b^2 - \jmath b + 4 &= 0 \\
b^4 - 4b^2 + 4 - \jmath b + 2b^3 &= 0 \\
b^4 - 4b^2 + 4 + \jmath(-b + 2b^3) &= 0
\end{aligned}$$

This is now in the form of a complex number $\alpha \pm j\beta$. Therefore,

$$\begin{aligned} b^4 - 4b^2 + 4 &= 0 \\ \therefore (b^2 - 2)^2 &= 0 \\ 2b^3 - b &= 0 \\ \therefore b(2b^2 - 1) &= 0 \end{aligned}$$

Now we can find all the roots that satisfy the above equations.

$$\begin{aligned} (b^2 - 2)^2 &= 0 \\ b^2 - 2 &= 0 \\ b^2 &= 2 \\ b &= \pm\sqrt{2} \\ b(2b^2 - 1) &= 0 \\ b &= 0 \\ 2b^2 - 1 &= 0 \\ 2b^2 &= 1 \\ b^2 &= \frac{1}{2} \\ b &= \pm\sqrt{\frac{1}{2}} \\ b &= \pm\frac{1}{\sqrt{2}} \end{aligned}$$

Since the roots for the real and imaginary parts are not the same, there is no real b which satisfies the equation.

26) Calculate all solutions of $\mid z - 1 \mid \cdot \mid z - 1 \mid = 1$ assuming z is in the form of $x + \jmath y$ and show z in Argand diagram.

We put $z = x + \jmath y$ into $\mid z - 1 \mid^2 = 1$ as follows:

$$\begin{aligned} &\mid z - 1 \mid^2 \\ &= |x + \jmath y - 1|^2 \\ &= |x - 1 + \jmath y|^2 \\ &= (x - 1)^2 + y^2 = 1 \end{aligned}$$

Thus in Argand diagram, z is on the circle with centre $(1, 0)$ and radius 1.

27) Find the real value of m such that the equation

$$2z^2 - (3 + 8\jmath)z - (m + 4\jmath) = 0$$

has a real root. Then find the roots.

Since a real value of z satisfies

$$2z^2 - (3 + 8\jmath)z - (m + 4\jmath) = 0$$

it can be manipulated as

$$\begin{aligned} 2z^2 - (3z + 8z\jmath) - m - 4\jmath &= 0 \\ \therefore 2z^2 - 3z - 8z\jmath - m - 4\jmath &= 0 \\ \therefore 2z^2 - 3z - m + (-8z - 4)\jmath &= 0 \end{aligned}$$

Equating real and imaginary parts,

$$\begin{aligned} 2z^2 - 3z - m &= 0 \\ -8z - 4 &= 0 \end{aligned}$$

From the second equation,

$$\begin{aligned} -8z - 4 &= 0 \\ \therefore -4 &= 8z \\ \therefore -1 &= 2z \\ \therefore -\frac{1}{2} &= z \end{aligned}$$

Now putting $z=-\frac{1}{2}$ into the first equation,

$$\begin{aligned}
2z^2-3z-m&=0\\
\therefore 2z^2-3z&=m\\
\therefore 2(-\frac{1}{2})^2-3\cdot(-\frac{1}{2})&=m\\
\therefore 2\cdot\frac{1}{4}+\frac{3}{2}&=m\\
\therefore \frac{2}{4}+\frac{3}{2}&=m\\
\therefore \frac{1}{2}+\frac{3}{2}&=m\\
\therefore \frac{1+3}{2}&=m\\
\therefore \frac{4}{2}&=m\\
\therefore 2&=m
\end{aligned}$$

Thus the original equation becomes

$$2z^2-(3+8\jmath)z-(2+4\jmath)=0$$

We know that $z=-\frac{1}{2}$ is one of the roots of the original equation. It can be manipulated as

$$\begin{aligned}
z&=-\frac{1}{2}\\
\therefore 2z&=-1\\
\therefore 2z+1&=0
\end{aligned}$$

This means,

$$2z^2-(3+8\jmath)z-(2+4\jmath)=0$$

has a factor of $2z+1$. In order to find out the other factor, we perform a division:

$$\begin{array}{rrr}
 & z & -2-4\jmath \\
\hline
2z+1\,) & 2z^2\quad-(3+8\jmath)z & -(2+4\jmath) \\
-)\ & 2z^2\qquad\qquad +z & \\
\hline
 & -3z-\jmath 8z-z & -(2+4\jmath) \\
\therefore & 2z(-2-4\jmath) & -(2+4\jmath) \\
-) & 2z(-2-4\jmath) & -2-4\jmath \\
\hline
 & 0 & 0
\end{array}$$

Thus, we find out that the other factor is

$$z-2-4\jmath.$$

Therefore another root is

$$\begin{aligned}
z-2-4\jmath&=0\\
\therefore z&=2+4\jmath
\end{aligned}$$

28) A polynomial equation

$$z^4-2z^3+7z^2-4z+10=0$$

has (a) root(s) $z=a\cdot\jmath$ where a is real. Find the value of a and all roots of this equation.
When we put $z=a\jmath$ into $z^4-2z^3+7z^2-4z+10=0$,
we get

$$\begin{aligned}
z^4-2z^3+7z^2-4z+10&=0\\
\therefore (a\jmath)^4-2(a\jmath)^3+7(a\jmath)^2-4(a\jmath)+10&=0\\
\therefore a^4(\jmath)^4-2a^3(\jmath)^3&\\
+7a^2(\jmath)^2-4a\jmath+10&=0\\
\therefore a^4\left((\jmath)^2\right)^2-2a^3(\jmath)^2\cdot\jmath&\\
+7a^2\cdot(-1)-4a\jmath+10&=0\\
\therefore a^4\,(-1)^2-2a^3\cdot(-1)\cdot\jmath&\\
+7a^2\cdot(-1)-4a\jmath+10&=0\\
\therefore a^4+2a^3\jmath-7a^2-4a\jmath+10&=0\\
\therefore a^4-7a^2+10+2a^3\jmath-4a\jmath&=0\\
\therefore a^4-7a^2+10+(2a^3-4a)\jmath&=0
\end{aligned}$$

From this equation, we get

$$\begin{aligned} a^4 - 7a^2 + 10 &= 0 \\ 2a^3 - 4a &= 0 \end{aligned}$$

These can be manipulated as

$$\begin{aligned} a^4 - 7a^2 + 10 &= 0 \\ \therefore (a^2 - 2)(a^2 - 5) &= 0 \\ 2a^3 - 4a &= 0 \\ \therefore 2a(a^2 - 2) &= 0 \end{aligned}$$

Both equations have a factor of $a^2 - 2$ and

$$\begin{aligned} a^2 - 2 &= 0 \\ \therefore a^2 &= 2 \\ \therefore a &= \pm\sqrt{2} \end{aligned}$$

satisfy both equations. Thus $z = \pm\sqrt{2}\jmath$ are the roots of $z^4 - 2z^3 + 7z^2 - 4z + 10 = 0$.
This means $z^4 - 2z^3 + 7z^2 - 4z + 10 = 0$ has factors of $z - \jmath\sqrt{2}$ and $z + \jmath\sqrt{2}$. In other words, the original equation has a factor of

$$\begin{aligned} & (z - \jmath\sqrt{2})(z + \jmath\sqrt{2}) \\ &= z^2 - (\jmath\sqrt{2}) \\ &= z^2 - 2(\jmath)^2 \\ &= z^2 - 2 \cdot (-1) \\ &= z^2 + 2 \end{aligned}$$

We now find the rest of the factors of the original equation by doing the following division:

$$\begin{array}{rlll}
 & & z^2 & -2z \quad +5 \\ \cline{2-4}
z^2 + 2 \,) & z^4 & -2z^3 & +7z^2 - 4z + 10 \\
-) & z^4 & +2z^2 & \\ \hline
 & & -2z^3 - 2z^2 & +7^2 \\
\therefore & & -2z^3 & +5z^2 \\
-) & & -2z^3 & -4z \\ \hline
 & & & 5z^2 + 4z - 4z \\
-) & & & 5z^2 + 10 \\ \hline
 & & & -10 + 10 \\ \hline
 & & & 0
\end{array}$$

Thus

$$\frac{z^4 - 2z^3 + 7z^2 - 4z + 10}{z^2 + 2} = z^2 - 2z + 5.$$

The roots of $z^2 - 2z + 5 = 0$ are the rest of the roots for $z^4 - 2z^3 + 7z^2 - 4z + 10 = 0$. In general the roots of

$$az^2 + bz^1 + c = 0$$

are

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The equation $z^2 - 2z + 5 = 0$ is identical to

$$az^2 + bz^1 + c = 0$$

when

$$\begin{aligned} a &= 1 \\ b &= -2 \\ c &= 5 \end{aligned}$$

Therefore the roots of $z^2 - 2z + 5 = 0$ are

$$\begin{aligned} z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} \\ = \frac{2 \pm \sqrt{4 - 20}}{2} \\ = \frac{2 \pm \sqrt{-16}}{2} \\ = \frac{2 \pm \sqrt{-4^2}}{2} \\ = \frac{2 \pm \sqrt{-1 \cdot 4^2}}{2} \\ = \frac{2 \pm \sqrt{-1} \cdot \sqrt{4^2}}{2} \\ = \frac{2 \pm \jmath \cdot 4}{2} \\ = \frac{2}{2} \pm \frac{4\jmath}{2} \\ = 1 \pm 2\jmath \end{aligned}$$

29) u, v and w are the three roots of the equation $z^3 - 1 = 0$. Calculate $u \cdot v + v \cdot w + w \cdot u$ without calculating the 3 roots
When u,v and w are the three roots of a cubic equation, we can write:

$$\begin{aligned} (z-u)(z-v)(z-w) \\ = z^3 - (u+v+w)z^2 \\ +(uv+vw+wu)z - uvw \\ \equiv z^3 - 1 \end{aligned}$$

By equating the coefficient of z^3, z^2, z^1, and z^0 we get

$$\begin{aligned} -(u+v+w) = 0 \\ uv+vw+wu = 0 \\ -uvw = -1 \end{aligned}$$

From the second equation $uv + vw + wu = 0$.

30) The equation

$$z^3 - (n+\jmath)z + m + 2\jmath = 0$$

has three roots. n and m are real constants.

a) Calculate m such that the modulus of the product of the three roots is $\sqrt{5}$.
When u,v and w are the three roots of a cubic equation, we can write:

$$\begin{aligned} (z-u)(z-v)(z-w) \\ = z^3 - (u+v+w)z^2 \\ +(uv+vw+wu)z - uvw \\ \equiv z^3 - (n+\jmath)z + m + 2\jmath \end{aligned}$$

By equating the coefficient of z^3, z^2, z^1, and z^0 we get

$$\begin{aligned} -(u+v+w) = 0 \\ (uv+vw+wu) = -(n+\jmath) \\ -uvw = m + 2\jmath \end{aligned}$$

The product of the roots is uvw and from the third equation above,

$$\begin{aligned} -uvw = m + 2\jmath \\ \therefore uvw = -(m+2\jmath) \\ = -m - 2\jmath \end{aligned}$$

The modulus of $-m - 2\jmath$ is

$$\begin{aligned} |-m-2\jmath| \\ = \sqrt{(-m)^2 + (-2)^2} \\ = \sqrt{m^2 + 4} \end{aligned}$$

Since the modulus of the product of the roots is $\sqrt{5}$, we can say

$$\sqrt{m^2+4} = \sqrt{5}$$
$$\therefore m^2 + 4 = 5$$
$$\therefore m^2 = 5 - 4$$
$$\therefore m^2 = 1$$
$$\therefore m = \pm 1$$

b) Calculate the modulus of the sum of the roots.
The sum of the roots is $u + v + w$. Since$u + v + w = 0$, the modulus of the sum is 0.

31) Calculate the values of m, where m is real, such that the equation

$$z^2 - (3 + \jmath)z + m + 2\jmath = 0$$

has a real root. Calculate the second root.
The original equation can be manipulated as

$$z^2 - (3 + \jmath)z + m + 2\jmath = 0$$
$$\therefore z^2 - 3z - z\jmath + m + 2\jmath = 0$$
$$\therefore z^2 - 3z + m + (2 - z)\jmath = 0$$

Since z can be real and m is real, the equation above gives the following two simultaneous equations:

$$z^2 - 3z + m = 0$$
$$2 - z = 0$$

From the second equation, we get

$$2 - z = 0$$
$$\therefore 2 = z$$

Here we know that the first root is $z = 2$. By putting $z = 2$ into the first equation, we get

$$z^2 - 3z + m = 0$$
$$\therefore 2^2 - 3 \cdot 2 + m = 0$$
$$\therefore 4 - 6 + m = 0$$
$$\therefore -2 + m = 0$$
$$\therefore m = 2$$

When we assume another root is a, using the first root of 2, the original equation can be expressed as

$$z^2 - (3 + \jmath)z + m + 2\jmath$$
$$= z^2 - (3 + \jmath)z + 2 + 2\jmath$$
$$\equiv (z - a) \cdot (z - 2)$$
$$= z^2 + (-a - 2)z + 2a$$

By equating the coefficients of z^2,z^1,z^0, we obtain

$$-(3 + \jmath) = (-a - 2)$$
$$2 + 2\jmath = 2a$$

From the second equation, we get

$$1 + \jmath = a$$

Therefore another root is $1 + \jmath$.

DAY3

32) Solve the equation

$$-3\mathfrak{e}^{2x} + 5\mathfrak{e}^{x} = 2$$

$$\begin{aligned}
-3\mathfrak{e}^{2x} + 5\mathfrak{e}^{x} &= 2\\
\therefore 3\mathfrak{e}^{2x} - 5\mathfrak{e}^{x} + 2 &= 0\\
\therefore 3\left(\mathfrak{e}^{x}\right)^2 - 5\mathfrak{e}^{x} + 2 &= 0\\
\therefore \mathfrak{e}^{x} &= \frac{5 \pm \sqrt{5^2 - 4\cdot 3\cdot 2}}{2\cdot 3}\\
\therefore \mathfrak{e}^{x} &= \frac{5 \pm \sqrt{25 - 24}}{6}\\
\therefore \mathfrak{e}^{x} &= \frac{5 \pm \sqrt{1}}{6}\\
\therefore \mathfrak{e}^{x} &= \frac{5+1}{6}, \frac{5-1}{6}\\
\therefore \mathfrak{e}^{x} &= \frac{6}{6}, \frac{4}{6}\\
\therefore \mathfrak{e}^{x} &= 1, \frac{2}{3}\\
\therefore \ln(\mathfrak{e}^{x}) &= \ln(1), \ln(\frac{2}{3})\\
\therefore x\ln(\mathfrak{e}) &= 0, \ln(\frac{2}{3})\\
\therefore x\cdot 1 &= 0, \ln(\frac{2}{3})\\
\therefore x &= 0, \ln(\frac{2}{3})
\end{aligned}$$

33) Solve

$$2^x - 6\left(2^{-x}\right) = 6$$

$$\begin{aligned}
2^x - 6\left(2^{-x}\right) &= 6\\
\therefore 2^x(2^x) - 6\left(2^{-x}\right)\left(2^x\right) &= 6(2^x)\\
\therefore (2^x)^2 - 6\left(2^{-x+x}\right) &= 6(2^x)\\
\therefore (2^x)^2 - 6\left(2^{0}\right) &= 6(2^x)\\
\therefore (2^x)^2 - 6(1) &= 6(2^x)\\
\therefore (2^x)^2 - 6(2^x) - 6 &= 0\\
\therefore 2^x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4\cdot 1\cdot(-6)}}{2\cdot 1}\\
\therefore 2^x &= \frac{6 \pm \sqrt{36 - (-24)}}{2}\\
\therefore 2^x &= \frac{6 \pm \sqrt{60}}{2}\\
\therefore 2^x &= \frac{6 \pm \sqrt{4}\sqrt{15}}{2}\\
\therefore 2^x &= \frac{6 \pm 2\sqrt{15}}{2}\\
\therefore 2^x &= 3 + \sqrt{15} (\because 2^x > 0)\\
\therefore \log_2(2^x) &= \log_2(3 + \sqrt{15})\\
\therefore x\log_2 2 &= \log_2(3 + \sqrt{15})\\
\therefore x &= \log_2(3 + \sqrt{15})
\end{aligned}$$

34) Simplify the following expression

$$\frac{54(x^7y^3)^2}{(-3x^5y^{-4})^3}$$

$$\begin{aligned}
&\frac{54(x^7y^3)^2}{(-3x^5y^{-4})^3}\\
&=\frac{54x^{7\times 2}y^{3\times 2}}{(-3)^3x^{5\times 3}y^{-4\times 3}}\\
&=\frac{54x^{14}y^6}{-27x^{15}y^{-12}}\\
&=-2x^{14-15}y^{6+12}\\
&=-2x^{-1}y^{18}\\
&=-2y^{18}/x
\end{aligned}$$

35) Solve the following equation

$$\frac{2x}{3}=-6-\frac{x-4}{5}$$

$$\begin{aligned}
\frac{2x}{3}&=-6-\frac{x-4}{5}\\
\therefore 5\left(\frac{2x}{3}\right)&=5(-6)-5\left(\frac{x-4}{5}\right)\\
\therefore 3\left(\frac{10x}{3}\right)&=3(-30)-3(x-4)\\
\therefore 10x&=-90-3x+12\\
\therefore 10x+3x&=-78\\
\therefore 13x&=-78\\
\therefore x&=\frac{-78}{13}\\
\therefore x&=-6
\end{aligned}$$

36) Solve the following inequality

$$12x-2<7x-17$$

$$\begin{aligned}
12x-2&<7x-17\\
\therefore 12x-7x&<-17+2\\
\therefore 5x&<-15\\
\therefore x&<\frac{-15}{5}\\
\therefore x&<-3
\end{aligned}$$

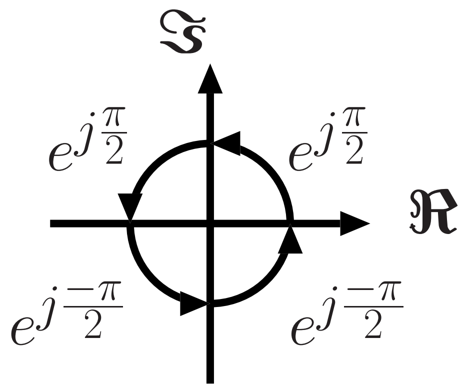
37)

$$\frac{1+\jmath}{1-\jmath}=$$

$$\begin{aligned}
&\frac{1+\jmath}{1-\jmath}\\
&=\frac{(1+\jmath)(1+\jmath)}{(1-\jmath)(1+\jmath)}\\
&=\frac{1+2\jmath+\jmath^2}{1-\jmath^2}\\
&=\frac{1+2\jmath-1}{1-(-1)}\\
&=\frac{2\jmath}{2}\\
&=\jmath
\end{aligned}$$

38)

$$\jmath\cdot\jmath\cdot\jmath\cdot\jmath=$$



Using

$$\jmath = \mathfrak{e}^{\jmath\frac{\pi}{2}},$$

we get

$$\begin{aligned}&\jmath^4\\ &= \mathfrak{e}^{\jmath\frac{\pi}{2}\times 4}\\ &= \mathfrak{e}^{\jmath\frac{4\pi}{2}}\\ &= \mathfrak{e}^{\jmath\cdot(2\pi)} = 1\end{aligned}$$

Alternatively, using $\jmath^2 = -1$

$$\begin{aligned}\jmath\cdot\jmath\cdot\jmath\cdot\jmath &= \jmath^4\\ \therefore \jmath^4 = (\jmath^2)^2 = (-1)^2 &= 1\end{aligned}$$

39) Calculate the real values of x and y or find the relationship between x and y such that $(x+\jmath y)^3$ is real and the modulus of $x+\jmath y$ is higher than 8.
We re-write $x+\jmath y$ as follows:

$$\begin{aligned}x+\jmath y &= r\mathfrak{e}^{\jmath\theta}\\ &= r\cos\theta + \jmath r\sin\theta\\ \therefore x = r\cos\theta, y &= r\sin\theta\end{aligned}$$

where r is real and positive and θ is real. Then the problem can be re-phrased as
$(r\mathfrak{e}^{\jmath\theta})^3 = r^3\mathfrak{e}^{\jmath 3\theta}$ is real and

$$|r\mathfrak{e}^{\jmath\theta}| = r > 8.$$

The fact that $\mathfrak{e}^{\jmath 3\theta}$ is real means

$$\begin{aligned}3\theta &= n\pi\\ \therefore \theta &= \frac{n\pi}{3}\end{aligned}$$

where n is an integer. Thus we can write

$$\begin{aligned}x &= r\cos\theta\\ \therefore x &= r\cos\left(\frac{n\pi}{3}\right)\\ y &= r\sin\theta\\ \therefore y &= r\sin\left(\frac{n\pi}{3}\right)\end{aligned}$$

Now that r and n are the variables which are not in the problem statement, we need to remove r and n from the equations above to obtain the relation between x and y. In the following, we assume m is an integer.

a) When $n = 0+6m$,

$$\begin{aligned}x &= r\cos\left(\frac{n\pi}{3}\right)\\ &= r\cos\left(\frac{6m\pi}{3}\right)\\ &= r\cos(2m\pi)\\ &= r > 8\\ y &= r\sin\left(\frac{n\pi}{3}\right)\\ &= r\sin\left(\frac{6m\pi}{3}\right)\\ &= r\sin(2m\pi)\\ &= 0\end{aligned}$$

Thus $x = r > 8$, and $y = 0$ are the values of x, y.

b) When $n = 1 + 6m$,

$$\begin{aligned} x &= r\cos\left(\frac{n\pi}{3}\right) \\ &= r\cos\left(\frac{(1+6m)\pi}{3}\right) \\ &= r\cos\left(\frac{\pi+6m\pi}{3}\right) \\ &= r\cos\left(\frac{\pi}{3}+\frac{6m\pi}{3}\right) \\ &= r\cos\left(\frac{\pi}{3}+2m\pi\right) \\ &= r\cos\left(\frac{\pi}{3}\right) \\ &= r\cos\left(\frac{\pi}{3}\right) \\ &= r\cdot\frac{1}{2} \\ &= \frac{r}{2} \\ y &= r\sin\left(\frac{n\pi}{3}\right) \\ &= r\sin\left(\frac{(1+6m)\pi}{3}\right) \\ &= r\sin\left(\frac{\pi+6m\pi}{3}\right) \\ &= r\sin\left(\frac{\pi}{3}+\frac{6m\pi}{3}\right) \\ &= r\sin\left(\frac{\pi}{3}+2m\pi\right) \\ &= r\sin\left(\frac{\pi}{3}\right) \\ &= \frac{r\sqrt{3}}{2} \end{aligned}$$

In order to remove r from these two equations, we perform the division of these two equations as follows:

$$\frac{y \quad = \quad \frac{r\sqrt{3}}{2}}{x \quad = \quad \frac{r}{2}}$$

$$\begin{aligned} \frac{y}{x} &= \frac{\frac{r\sqrt{3}}{2}}{\frac{r}{2}} \\ &= \frac{\sqrt{3}r}{r} \\ &= \sqrt{3} \\ \therefore y &= \sqrt{3}x \end{aligned}$$

$r > 8$ can be applied to $x = \frac{r}{2}$ as follows:

$$\begin{aligned} r &> 8 \\ \therefore \frac{r}{2} &> 4 \\ \therefore x = \frac{r}{2} &> 4 \end{aligned}$$

Thus $y = \sqrt{3}x$ and $x > 4$ are the relation of x and y.

c) When $n = 2 + 6m$

$$\begin{aligned} x &= r\cos\left(\frac{n\pi}{3}\right) \\ &= r\cos\left(\frac{(2+6m)\pi}{3}\right) \\ &= r\cos\left(\frac{2\pi+6m\pi}{3}\right) \\ &= r\cos\left(\frac{2\pi}{3}+\frac{6m\pi}{3}\right) \end{aligned}$$

$$
\begin{aligned}
&= r\cos\left(\frac{2\pi}{3} + 2m\pi\right) \\
&= r\cos\left(\frac{2\pi}{3}\right) \\
&= r\cos\left(\frac{2\pi}{3}\right) \\
&= r\cdot\left(-\frac{1}{2}\right) \\
&= -\frac{r}{2} \\
y &= r\sin\left(\frac{n\pi}{3}\right) \\
&= r\sin\left(\frac{(2+6m)\pi}{3}\right) \\
&= r\sin\left(\frac{2\pi + 6m\pi}{3}\right) \\
&= r\sin\left(\frac{2\pi}{3} + \frac{6m\pi}{3}\right) \\
&= r\sin\left(\frac{2\pi}{3} + 2m\pi\right) \\
&= r\sin\left(\frac{2\pi}{3}\right) \\
&= \frac{r\sqrt{3}}{2}
\end{aligned}
$$

In order to remove r from these two equations, we perform the division of these two equations as follows:

$$
\frac{y \quad = \quad \frac{r\sqrt{3}}{2}}{x \quad = \quad -\frac{r}{2}}
$$

$$
\begin{aligned}
\frac{y}{x} &= \frac{\frac{r\sqrt{3}}{2}}{-\frac{r}{2}} \\
&= \frac{\sqrt{3}r}{-r} \\
&= -\sqrt{3} \\
\therefore y &= -\sqrt{3}x
\end{aligned}
$$

$r > 8$ can be applied to $x = -\frac{r}{2}$ as follows:

$$
\begin{aligned}
r &> 8 \\
\therefore \frac{r}{2} &> 4 \\
\therefore -\frac{r}{2} &< -4 \\
\therefore x = -\frac{r}{2} &< -4
\end{aligned}
$$

Thus $y = -\sqrt{3}x$ and $x < -4$ are the relation of x and y.

d) When $n = 3 + 6m$,

$$
\begin{aligned}
x &= r\cos\left(\frac{n\pi}{3}\right) \\
&= r\cos\left(\frac{(3+6m)\pi}{3}\right) \\
&= r\cos\left(\frac{3\pi + 6m\pi}{3}\right) \\
&= r\cos\left(\frac{3\pi}{3} + \frac{6m\pi}{3}\right) \\
&= r\cos\left(\pi + 2m\pi\right) \\
&= r\cos\left(\pi\right) \\
&= -r \\
y &= r\sin\left(\frac{n\pi}{3}\right) \\
&= r\sin\left(\frac{(3+6m)\pi}{3}\right)
\end{aligned}
$$

$$= r\sin\left(\frac{3\pi + 6m\pi}{3}\right)$$
$$= r\sin\left(\frac{3\pi}{3} + \frac{6m\pi}{3}\right)$$
$$= r\sin(\pi + 2m\pi)$$
$$= r\sin(\pi)$$
$$= 0$$

$r > 8$ can be applied to $x = -r$ as follows:

$$r > 8$$
$$\therefore -r < -8$$
$$\therefore x = -r < -8$$

Thus the values of x and y are: $x < -8$, and $y = 0$.

e) When $n = 4 + 6m$,

$$x = r\cos\left(\frac{n\pi}{3}\right)$$
$$= r\cos\left(\frac{(4+6m)\pi}{3}\right)$$
$$= r\cos\left(\frac{4\pi + 6m\pi}{3}\right)$$
$$= r\cos\left(\frac{4\pi}{3} + \frac{6m\pi}{3}\right)$$
$$= r\cos\left(\frac{4\pi}{3} + 2m\pi\right)$$
$$= r\cos\left(\frac{4\pi}{3}\right)$$
$$= r\cdot\left(-\frac{1}{2}\right)$$
$$= -\frac{r}{2}$$
$$y = r\sin\left(\frac{n\pi}{3}\right)$$
$$= r\sin\left(\frac{(4+6m)\pi}{3}\right)$$
$$= r\sin\left(\frac{4\pi + 6m\pi}{3}\right)$$
$$= r\sin\left(\frac{4\pi}{3} + \frac{6m\pi}{3}\right)$$
$$= r\sin\left(\frac{4\pi}{3} + 2m\pi\right)$$
$$= r\sin\left(\frac{4\pi}{3}\right)$$
$$= -\frac{r\sqrt{3}}{2}$$

In order to remove r from these two equations, we perform the division of these two equations as follows:

$$\frac{y \quad = \quad -\frac{r\sqrt{3}}{2}}{x \quad = \quad -\frac{r}{2}}$$

$$\frac{y}{x} = \frac{-\frac{r\sqrt{3}}{2}}{-\frac{r}{2}}$$
$$= \frac{\sqrt{3}r}{r}$$
$$= \sqrt{3}$$
$$\therefore y = \sqrt{3}x$$

$r > 8$ can be applied to $x = -\frac{r}{2}$ as follows:

$$\begin{aligned} r &> 8 \\ \therefore \frac{r}{2} &> 4 \\ \therefore -\frac{r}{2} &< -4 \\ \therefore x = -\frac{r}{2} &< -4 \end{aligned}$$

Thus $y = \sqrt{3}x$ and $x < -4$ are the relation of x and y.

f) When $n = 5 + 6m$,

$$\begin{aligned} x &= r\cos\left(\frac{n\pi}{3}\right) \\ &= r\cos\left(\frac{(5+6m)\pi}{3}\right) \\ &= r\cos\left(\frac{5\pi + 6m\pi}{3}\right) \\ &= r\cos\left(\frac{5\pi}{3} + \frac{6m\pi}{3}\right) \\ &= r\cos\left(\frac{5\pi}{3} + 2m\pi\right) \\ &= r\cos\left(\frac{5\pi}{3}\right) \\ &= r\cos\left(\frac{5\pi}{3}\right) \\ &= r \cdot \frac{1}{2} \\ &= \frac{r}{2} \\ y &= r\sin\left(\frac{n\pi}{3}\right) \\ &= r\sin\left(\frac{(5+6m)\pi}{3}\right) \\ &= r\sin\left(\frac{5\pi + 6m\pi}{3}\right) \\ &= r\sin\left(\frac{5\pi}{3} + \frac{6m\pi}{3}\right) \\ &= r\sin\left(\frac{5\pi}{3} + 2m\pi\right) \\ &= r\sin\left(\frac{5\pi}{3}\right) \\ &= -\frac{r\sqrt{3}}{2} \end{aligned}$$

In order to remove r from these two equations, we perform the division of these two equations as follows:

$$\frac{\begin{array}{ccc} y & = & -\frac{r\sqrt{3}}{2} \end{array}}{\begin{array}{ccc} x & = & \frac{r}{2} \end{array}}$$

$$\begin{aligned} \frac{y}{x} &= \frac{-\frac{r\sqrt{3}}{2}}{\frac{r}{2}} \\ &= \frac{-\sqrt{3}r}{r} \\ &= -\sqrt{3} \\ \therefore y &= -\sqrt{3}x \end{aligned}$$

$r > 8$ can be applied to $x = \frac{r}{2}$ as follows:

$$\begin{aligned} r &> 8 \\ \therefore \frac{r}{2} &> 4 \\ \therefore x = \frac{r}{2} &> 4 \end{aligned}$$

Thus $y = -\sqrt{3}x$ and $x > 4$ are the relation of x and y.

40) Rationalise $\frac{12\jmath}{1-2\jmath}$. Note that the complex conjugate of $1-2\jmath$ is $1+2\jmath$.

$$\begin{aligned}\frac{12\jmath}{1-2\jmath} &= \frac{12\jmath}{1-2\jmath}\cdot\frac{1+2\jmath}{1+2\jmath} \\ &= \frac{12\jmath(1+2\jmath)}{(1-2\jmath)(1+2\jmath)} \\ &= \frac{12\jmath+24\jmath^2}{1-2\jmath+2\jmath-4\jmath^2} \\ &= \frac{12\jmath+24\jmath^2}{1-4\jmath^2} \\ &= \frac{12\jmath-24}{1+4}(\because \jmath^2=-1) \\ &= \frac{12\jmath-24}{5}\end{aligned}$$

41) Rationalise $\frac{1}{3+\jmath}$.

$$\begin{aligned}\frac{1}{3+\jmath} &= \frac{1}{3+\jmath}\cdot\frac{3-\jmath}{3-\jmath} \\ &= \frac{3-\jmath}{(3+\jmath)(3-\jmath)} \\ &= \frac{3-\jmath}{9-(\jmath)^2} \\ &= \frac{3-\jmath}{9-(-1)} \\ &= \frac{3-\jmath}{10}\end{aligned}$$

42) Given that

$$\begin{aligned}z &= 2\mathfrak{e}^{\jmath\frac{\pi}{3}} \\ \cos\frac{\pi}{12} &= \frac{\sqrt{6}+\sqrt{2}}{4} \\ \sin\frac{\pi}{12} &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

find the following in standard form

a) z

$$z = 2\cos\frac{\pi}{3} + 2\sin\frac{\pi}{3}\jmath = 1+\sqrt{3}\jmath$$

b) z^6

$$z^6 = (2\mathfrak{e}^{\jmath\frac{\pi}{3}})^6 = 2^6\mathfrak{e}^{2\pi\jmath} = 64\times(1+0\jmath) = 64$$

c) $z-\frac{8}{z}$

$$\begin{aligned}z-\frac{8}{z} &= (1+\sqrt{3}\jmath) - 4(\mathfrak{e}^{-\frac{\pi}{3}\jmath}) = (1+\sqrt{3}\jmath) - 4\left(\cos(-\frac{\pi}{3}) + \sin(-\frac{\pi}{3})\jmath\right) \\ &= 1+\sqrt{3}\jmath - 2 + 2\sqrt{3}\jmath = -1+3\sqrt{3}\jmath\end{aligned}$$

d) $|z-1|$

$$|z-1| = |1+\sqrt{3}\jmath-1| = |\sqrt{3}\jmath| = \sqrt{3}$$

e) $z-\sqrt{2}\mathfrak{e}^{-\frac{\pi}{4}\jmath}$

$$z-\sqrt{2}\mathfrak{e}^{-\frac{\pi}{4}\jmath} = 1+\sqrt{3}\jmath - \sqrt{2}(\cos(-\frac{\pi}{4}) + \sin(-\frac{\pi}{4})\jmath) = 1+\sqrt{3}\jmath - 1 + \jmath = (\sqrt{3}+1)\jmath$$

f) $\frac{1+\jmath}{z}$

$$\frac{1+\jmath}{z} = \frac{\sqrt{2}\mathfrak{e}^{\jmath\frac{\pi}{4}}}{2\mathfrak{e}^{\jmath\frac{\pi}{3}}} = \frac{\sqrt{2}}{2}\mathfrak{e}^{-\frac{\pi}{12}\jmath} = \frac{\sqrt{2}}{2}(\cos(-\frac{\pi}{12}) + \sin(-\frac{\pi}{12})\jmath) = \frac{1+\sqrt{3}}{4} + \frac{1-\sqrt{3}}{4}\jmath$$

43) Given that

$$z = -1 + \jmath$$

find the following in standard form

a) $\overline{z}$

$$\overline{z} = -1 - \jmath$$

b) $2z^{-1}$

$$2z^{-1} = \frac{2}{-1+\jmath} = \frac{2(-1-\jmath)}{(-1+\jmath)(-1-\jmath)} = -1 - \jmath$$

c) $\jmath z$

$$\jmath z = -\jmath - 1$$

d) $|z|$

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

e) $\sqrt[3]{z}$

$$\sqrt[3]{z} = (\sqrt{2}\mathfrak{e}^{\jmath\frac{3\pi}{4}})^{\frac{1}{3}} = 2^{\frac{1}{6}}\mathfrak{e}^{\jmath\frac{\pi}{4}} = 2^{\frac{1}{6}}(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\jmath) = 2^{-\frac{1}{3}} + 2^{-\frac{1}{3}}\jmath$$

f) $(\overline{z})^4$

$$(\overline{z})^4 = (-1-\jmath)^4 = (\sqrt{2}\mathfrak{e}^{\jmath\frac{5\pi}{4}})^4 = (\sqrt{2})^4\mathfrak{e}^{5\pi\jmath} = 4 \times (-1) = -4$$

44) Given that

$$z = 2\mathfrak{e}^{\jmath\frac{5\pi}{6}}$$

find the following in standard form

a) z

$$z = 2(\cos\frac{5\pi}{6} + \sin\frac{5\pi}{6}\jmath) = 2(-\frac{\sqrt{3}}{2} + \frac{1}{2}\jmath) - \sqrt{3} + \jmath$$

b) z^6

$$z^6 = 2^6\mathfrak{e}^{\jmath\frac{5\pi}{6}\times 6} = 64\mathfrak{e}^{5\pi\jmath} = -64$$

c) $z - \frac{4}{z}$

$$\begin{aligned} z - \frac{4}{z} &= -\sqrt{3} + \jmath - \frac{4}{-\sqrt{3}+\jmath} = -\sqrt{3} + \jmath - \frac{4(-\sqrt{3}-\jmath)}{(-\sqrt{3}+\jmath)(-\sqrt{3}-\jmath)} \\ &= -\sqrt{3} + \jmath - (-\sqrt{3} - \jmath) = -\sqrt{3} + \jmath + \sqrt{3} + \jmath = 2\jmath \end{aligned}$$

d) $|z + \sqrt{3}|$

$$|z + \sqrt{3}| = |-\sqrt{3} + \jmath + \sqrt{3}| = |\jmath| = 1$$

e) $z - 2\sqrt{3}\mathfrak{e}^{-\jmath\frac{2\pi}{3}}$

$$z - 2\sqrt{3}\mathfrak{e}^{-\jmath\frac{2\pi}{3}} = -\sqrt{3} + \jmath - 2\sqrt{3}(\cos(-\frac{2\pi}{3}) + \sin(-\frac{2\pi}{3})\jmath)$$
$$= -\sqrt{3} + \jmath - 2\sqrt{3}(-\frac{1}{2} - \frac{\sqrt{3}}{2}\jmath) = -\sqrt{3} + \jmath + \sqrt{3} + 3\jmath = 4\jmath$$

f) $\frac{1-\jmath}{z}$

$$\frac{1-\jmath}{z} = \frac{\sqrt{2}\mathfrak{e}^{-\jmath\frac{\pi}{4}}}{2\mathfrak{e}^{\jmath\frac{5\pi}{6}}} = \frac{1}{\sqrt{2}}\mathfrak{e}^{\jmath(-\frac{\pi}{4}-\frac{5\pi}{6})} = \frac{1}{\sqrt{2}}\mathfrak{e}^{-\jmath\frac{13\pi}{12}} = \frac{1}{\sqrt{2}}\mathfrak{e}^{\jmath\frac{11\pi}{12}} = \frac{1}{\sqrt{2}}(\cos(\frac{11\pi}{12}) + \jmath\sin(\frac{11\pi}{12}))$$

g) $\sqrt[5]{z}$

$$\sqrt[5]{z} = 2^{\frac{1}{5}}\mathfrak{e}^{\jmath\frac{5\pi}{6}\times\frac{1}{5}} = 2^{\frac{1}{5}}\mathfrak{e}^{\jmath\frac{\pi}{6}} = 2^{\frac{1}{5}}(\cos\frac{\pi}{6} + \jmath\sin\frac{\pi}{6}) = 2^{\frac{1}{5}}(\frac{\sqrt{3}}{2} + \jmath\frac{1}{2}) = \sqrt{3}\cdot 2^{-\frac{4}{5}} + \jmath 2^{-\frac{4}{5}}$$

45) Given that

$$z = \sqrt{3} - \jmath$$

find the following in standard form

a) $\overline{z}$

$$\overline{z} = \sqrt{3} + \jmath$$

b) $8z^{-1}$

$$8z^{-1} = \frac{8}{\sqrt{3}-\jmath} = \frac{8(\sqrt{3}+\jmath)}{4} = 2\sqrt{3} + 2\jmath$$

c) $-\jmath z$

$$-\jmath z = -\jmath\sqrt{3} + 1$$

d) $|z|$

$$|z| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$$

e) z^3

$$z^3 = (2\mathfrak{e}^{-\jmath\frac{\pi}{6}})^3 = 2^3\mathfrak{e}^{-\jmath\frac{\pi}{6}\times 3} = 2^3\mathfrak{e}^{-\jmath\frac{\pi}{2}} = -8\jmath$$

f) $\overline{z}^6$

$$\overline{z}^6 = (2\mathfrak{e}^{\jmath\frac{\pi}{6}})^6 = 2^6\mathfrak{e}^{\jmath\frac{\pi}{6}\times 6} = 64\mathfrak{e}^{\pi\jmath} = -64$$

46) Given that

$$z = -1 - \sqrt{3}\jmath$$

find the following in standard form

a) $\overline{z}$

$$\overline{z} = -1 + \sqrt{3}\jmath$$

b) $4z^{-1}$

$$4z^{-1} = \frac{4}{-1-\sqrt{3}\jmath} = \frac{4(-1+\sqrt{3}\jmath)}{(-1-\sqrt{3}\jmath)(-1+\sqrt{3}\jmath)} = -1 + \sqrt{3}\jmath$$

c) $\jmath z$

$$\jmath z = -\jmath - \sqrt{3}$$

d) $|z|$

$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

e) $\sqrt{z}$

$$\sqrt{z} = \sqrt{2\mathfrak{e}^{-\frac{2\pi}{3}\jmath}} = 2^{\frac{1}{2}}\mathfrak{e}^{-\frac{\pi}{3}\jmath} = 2^{\frac{1}{2}}(\cos(-\frac{\pi}{3}) + \sin(-\frac{\pi}{3})\jmath) = 2^{\frac{1}{2}}(\frac{1}{2} - \frac{\sqrt{3}}{2}\jmath) = \frac{1}{\sqrt{2}} - \sqrt{\frac{3}{2}}\jmath$$

f) $\overline{z}^3$

$$\overline{z}^3 = (2\mathfrak{e}^{\frac{2\pi}{3}\jmath})^3 = 2^3\mathfrak{e}^{\frac{2\pi\times 3}{3}\jmath} = 2^3\mathfrak{e}^{2\pi\jmath} = 8$$

47) Given that

$$z = 2\mathfrak{e}^{\jmath\frac{5\pi}{3}}$$

find the following in standard form

a) z

$$z = 2\mathfrak{e}^{\jmath\frac{5\pi}{3}} = 2(\cos\frac{5\pi}{3} + \sin\frac{5\pi}{3}\jmath) = 2(\frac{1}{2} - \frac{\sqrt{3}}{2}\jmath) = 1 - \sqrt{3}\jmath$$

b) z^3

$$z^3 = 2^3\mathfrak{e}^{\jmath\frac{5\pi\times 3}{3}} = 8\mathfrak{e}^{5\pi\jmath} = -8$$

c) $z + \dfrac{12}{z}$

$$\begin{aligned} z + \frac{12}{z} &= 2\mathfrak{e}^{\jmath\frac{5\pi}{3}} + 6\mathfrak{e}^{-\jmath\frac{5\pi}{3}} = 1 - \sqrt{3}\jmath + 6(\cos\frac{-5\pi}{3} + \sin\frac{-5\pi}{3}\jmath) \\ &= 1 - \sqrt{3}\jmath + 6(\frac{1}{2} + \frac{\sqrt{3}}{2}\jmath) = 1 - \sqrt{3}\jmath + 3 + 3\sqrt{3}\jmath = 4 + 2\sqrt{3}\jmath \end{aligned}$$

d) $|1 - z|$

$$|1 - z| = |1 - (1 - \sqrt{3}\jmath)| = |\sqrt{3}\jmath| = \sqrt{3}$$

e) $z + 2\mathfrak{e}^{\frac{\pi}{4}\jmath}$

$$z + 2\mathfrak{e}^{\frac{\pi}{4}\jmath} = 1 - \sqrt{3}\jmath + 2(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\jmath) = 1 - \sqrt{3}\jmath + \sqrt{2} + \sqrt{2}\jmath = 1 + \sqrt{2} + (\sqrt{2} - \sqrt{3})\jmath$$

f) $\dfrac{2 - 3\jmath}{z - 1}$

$$\begin{aligned} \frac{2 - 3\jmath}{z - 1} &= \frac{2 - 3\jmath}{-\sqrt{3}\jmath} \\ &= \frac{(2 - 3\jmath)\jmath}{\sqrt{3}} \\ &= \frac{3 + 2\jmath}{\sqrt{3}} \end{aligned}$$

g) $\sqrt[5]{z}$

$$\sqrt[5]{z} = (2\mathfrak{e}^{\jmath\frac{5\pi}{3}})^{\frac{1}{5}} = 2^{\frac{1}{5}}\mathfrak{e}^{\jmath\frac{\pi}{3}} = 2^{\frac{1}{5}}(\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\jmath) = 2^{\frac{1}{5}}(\frac{1}{2} + \frac{\sqrt{3}}{2}\jmath) = 2^{\frac{-4}{5}} + \sqrt{3}\cdot 2^{\frac{-4}{5}}\jmath$$

DAY4

48) Solve the equation

$$\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2} = 3$$

$$\begin{aligned}
\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2} &= 3 \\
\therefore \mathfrak{e}^x - \mathfrak{e}^{-x} &= 2(3) \\
\therefore \mathfrak{e}^x - \mathfrak{e}^{-x} &= 6 \\
\therefore \mathfrak{e}^x(\mathfrak{e}^x) - \mathfrak{e}^{-x}(\mathfrak{e}^x) &= 6(\mathfrak{e}^x) \\
\therefore (\mathfrak{e}^x)^2 - \mathfrak{e}^{-x+x} &= 6(\mathfrak{e}^x) \\
\therefore (\mathfrak{e}^x)^2 - \mathfrak{e}^0 &= 6\mathfrak{e}^x \\
\therefore (\mathfrak{e}^x)^2 - 1 &= 6\mathfrak{e}^x \\
\therefore (\mathfrak{e}^x)^2 - 6\mathfrak{e}^x - 1 &= 0 \\
\therefore \mathfrak{e}^x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot -1}}{2 \cdot 1} \\
\therefore \mathfrak{e}^x &= \frac{6 \pm \sqrt{36 - (-4)}}{2} \\
\therefore \mathfrak{e}^x &= \frac{6 \pm \sqrt{36 + 4}}{2} \\
\therefore \mathfrak{e}^x &= \frac{6 \pm \sqrt{40}}{2} \\
\therefore \mathfrak{e}^x &= \frac{6 \pm \sqrt{4}\sqrt{10}}{2} \\
\therefore \mathfrak{e}^x &= \frac{6 \pm 2\sqrt{10}}{2} \\
\therefore \mathfrak{e}^x &= 3 + \sqrt{10} (\because \mathfrak{e}^x > 0) \\
\therefore \ln(\mathfrak{e}^x) &= \ln(3 + \sqrt{10}) \\
\therefore x \ln \mathfrak{e} &= \ln(3 + \sqrt{10}) \\
\therefore x &= \ln(3 + \sqrt{10})
\end{aligned}$$

49) Solve the equation

$$2\ln(x+3) - \ln(x+1) = 3\ln 2$$

We work under the condition of $x + 3 > 0$ and $x + 1 > 0$.

$$\begin{aligned}
2\ln(x+3) - \ln(x+1) &= 3\ln 2 \\
\therefore \ln(x+3)^2 - \ln(x+1) &= \ln 2^3 \\
\therefore \ln \frac{(x+3)^2}{x+1} &= \ln 2^3 \\
\therefore \frac{(x+3)^2}{x+1} &= 2^3 \\
\therefore (x+3)^2 &= 2^3(x+1) \\
\therefore x^2 + 6x + 9 &= 8x + 8 \\
\therefore x^2 + 6x - 8x + 9 - 8 &= 0 \\
\therefore x^2 - 2x + 1 &= 0 \\
\therefore (x-1)^2 &= 0 \\
\therefore x &= 1
\end{aligned}$$

50) Solve the equation

$$\left(\frac{1}{2}\right)^{6-x} = 32$$

$$\begin{aligned}
\left(\frac{1}{2}\right)^{6-x} &= 32\\
\therefore \log_2\left(\frac{1}{2}\right)^{6-x} &= \log_2 32\\
\therefore (6-x)\log_2\left(\frac{1}{2}\right) &= \log_2 2^5\\
\therefore (6-x)\log_2 2^{-1} &= 5\log_2 2\\
\therefore -(6-x)\log_2 2 &= 5\log_2 2\\
\therefore -(6-x) &= 5\\
\therefore -6+x &= 5\\
\therefore x &= 5+6 = 11
\end{aligned}$$

51) Solve the equation

$$8^{2x}\left(\frac{1}{4}\right)^{x-2} = 4^{-x}\left(\frac{1}{2}\right)^{2-x}$$

$$\begin{aligned}
8^{2x}\left(\frac{1}{4}\right)^{x-2} &= 4^{-x}\left(\frac{1}{2}\right)^{2-x}\\
\therefore \left(2^3\right)^{2x}\left(2^{-2}\right)^{x-2} &= \left(2^2\right)^{-x}\left(2^{-1}\right)^{2-x}\\
\therefore \left(2^{3\times 2x}\right)\left(2^{-2\times(x-2)}\right) &= \left(2^{2\times(-x)}\right)\left(2^{-1\times(2-x)}\right)\\
\therefore \left(2^{6x}\right)\left(2^{-2x+4}\right) &= \left(2^{-2x}\right)\left(2^{x-2}\right)\\
\therefore 2^{6x-2x+4} &= 2^{-2x+x-2}\\
\therefore 2^{4x+4} &= 2^{-x-2}\\
\therefore 4x+4 &= -x-2\\
\therefore 4x+x &= -2-4\\
\therefore 5x &= -6\\
\therefore x &= -6/5
\end{aligned}$$

52) Solve the equation

$$10^{2\log(x^{\frac{1}{3}})} = x$$

$$\begin{aligned}
10^{2\log(x^{\frac{1}{3}})} &= x\\
\therefore \log 10^{2\log(x^{\frac{1}{3}})} &= \log x\\
\therefore 2\log(x^{\frac{1}{3}})\log 10 &= \log x\\
\therefore \log(x^{\frac{1}{3}})^2\cdot 1 &= \log x\\
\therefore (x^{\frac{1}{3}})^2 &= x\\
\therefore x^{\frac{2}{3}} &= x\\
\therefore x^{\frac{2}{3}} - x &= 0\\
\therefore x^{\frac{2}{3}} - x^{\frac{3}{3}} &= 0\\
\therefore x^{\frac{2}{3}}(1-x^{\frac{1}{3}}) &= 0\\
\therefore x &= 0, 1
\end{aligned}$$

However, since $x^{\frac{1}{3}}$ can not be zero, the answer is $x = 1$.

53) Find all z so that $z^4 = -8(\jmath - \sqrt{3})$

We assume we did the problem 9 prior to this problem. The answer of the problem 9 was

$$(\jmath - \sqrt{3}) = 2\mathfrak{e}^{\jmath\frac{5\pi}{6}}.$$

Using this, the original equation can be written as

$$\begin{aligned}z^4 &= -8(\jmath - \sqrt{3})\\ &= -8 \cdot 2\mathfrak{e}^{\jmath\frac{5\pi}{6}}\\ &= -2^4\mathfrak{e}^{\jmath\frac{5\pi}{6}}\\ &= \mathfrak{e}^{-\jmath\pi} \cdot 2^4\mathfrak{e}^{\jmath\frac{5\pi}{6}}\\ (\because -1 &= \mathfrak{e}^{\jmath\pi} = \mathfrak{e}^{-\jmath\pi})\\ &= 2^4\mathfrak{e}^{\jmath\frac{5\pi}{6}-\jmath\pi}\\ &= 2^4\mathfrak{e}^{\jmath\frac{5\pi}{6}-\jmath\pi\cdot\frac{6}{6}}\\ &= 2^4\mathfrak{e}^{\jmath\frac{5\pi}{6}-\jmath\frac{6\pi}{6}}\\ &= 2^4\mathfrak{e}^{\jmath\frac{5\pi-6\pi}{6}}\\ &= 2^4\mathfrak{e}^{-\jmath\frac{\pi}{6}}\end{aligned}$$

When we assume $z = r\mathfrak{e}^{\jmath\theta}$, where r is real and positive and θ is real,

$$\begin{aligned}(r\mathfrak{e}^{\jmath\theta})^4 &= 2^4\mathfrak{e}^{-\jmath\frac{\pi}{6}}\\ \therefore r^4\mathfrak{e}^{4\jmath\theta} &= 2^4\mathfrak{e}^{-\jmath\frac{\pi}{6}}\end{aligned}$$

By equating the coefficient of the exponential and the power of the exponential we obtain

$$\begin{aligned}r^4 &= 2^4\\ 4\theta &= -\frac{\pi}{6} + 2\pi n\end{aligned}$$

where n is an integer. Therefore

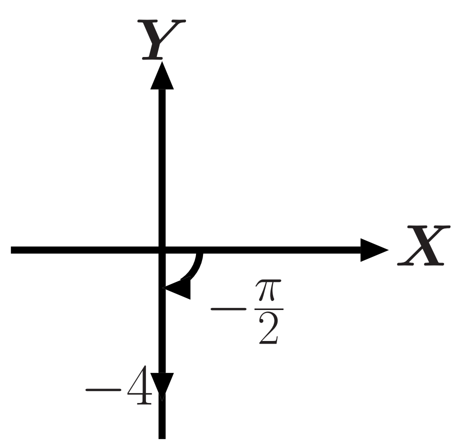
$$\begin{aligned}r &= 2\\ \theta &= -\frac{\pi}{6 \cdot 4} + \frac{2\pi n}{4}\\ &= -\frac{\pi}{24} + \frac{\pi n}{2}\end{aligned}$$

In the end,

$$z = 2\mathfrak{e}^{\jmath(-\frac{\pi}{24}+\frac{n\pi}{2})}$$

54) Find a complex number z which satisfies

$$z^2 = -4\jmath.$$



Using

$$-\jmath = \mathfrak{e}^{-\jmath\frac{\pi}{2}}$$

, the equation

$$z^2 = -4\jmath$$

can be re-written as

$$\begin{aligned}z^2 &= -4\jmath\\ \therefore z^2 &= 4\cdot(-\jmath)\\ &= 4\cdot\mathfrak{e}^{-\jmath\frac{\pi}{2}}\\ &= 4\mathfrak{e}^{-\jmath\frac{\pi}{2}}\end{aligned}$$

When we define

$$z \triangleq r\mathfrak{e}^{\jmath\theta}$$

where r is real and positive and θ is real,

$$z^2 = 4\mathfrak{e}^{-\jmath\frac{\pi}{2}}$$

can be furthermore re-written as

$$\begin{aligned}z^2 &= 4\mathfrak{e}^{-\jmath\frac{\pi}{2}}\\ \therefore \left(r\mathfrak{e}^{\jmath\theta}\right)^2 &= 4\mathfrak{e}^{-\jmath\frac{\pi}{2}}\\ \therefore r^2\mathfrak{e}^{\jmath\theta\times 2} &= 4\mathfrak{e}^{-\jmath\frac{\pi}{2}}\\ \therefore r^2\mathfrak{e}^{2\jmath\theta} &= 4\mathfrak{e}^{-\jmath\frac{\pi}{2}}\end{aligned}$$

By equating the coefficient of the exponential and the power of the exponential we obtain

$$\begin{aligned}r^2 &= 4\\ 2\jmath\theta &= -\jmath\frac{\pi}{2}\\ &= \jmath(-\frac{\pi}{2}+2\pi n)\end{aligned}$$

where n is an integer . Therefore we obtain

$$\begin{aligned}r^2 &= 4\\ r &= \pm 2 = 2(\because r > 0)\\ 2\jmath\theta &= \jmath(-\frac{\pi}{2}+2\pi n)\\ \therefore 2\theta &= -\frac{\pi}{2}+2\pi n\\ \therefore \theta &= -\frac{\pi}{2\times 2}+\frac{2\pi n}{2}\\ \therefore \theta &= -\frac{\pi}{4}+\pi n\end{aligned}$$

This gives us the answer of

$$\begin{aligned}z &= 2\mathfrak{e}^{-\jmath(\frac{\pi}{4}+n\pi)}\\ &= 2\mathfrak{e}^{-\jmath\frac{\pi}{4}-\jmath n\pi}\\ &= 2\mathfrak{e}^{-\jmath\frac{\pi}{4}}\cdot\mathfrak{e}^{-\jmath n\pi}\end{aligned}$$

When n is even, $\mathfrak{e}^{-\jmath n\pi} = 1$. When n is odd, $\mathfrak{e}^{-\jmath n\pi} = -1$. Therefore, for a general integer n, we can say $\mathfrak{e}^{-\jmath n\pi} = \pm 1$. Thus

$$\begin{aligned}z &= 2\mathfrak{e}^{-\jmath\frac{\pi}{4}}\cdot(\pm 1)\\ \therefore z &= \pm 2\mathfrak{e}^{-\jmath(\frac{\pi}{4})}\end{aligned}$$

55) Find all those values of z which satisfy

$$z^4+1=0$$

Write your values in standard form.

$$\begin{aligned}z^4+1 &= 0\\ \therefore z^4 &= -1\\ \therefore z^4 &= \mathfrak{e}^{\jmath\pi}\\ \therefore z^4 &= \mathfrak{e}^{\jmath(\pi+2\pi n)}\\ \therefore z^{4\cdot\frac{1}{4}} &= \mathfrak{e}^{\jmath(\pi+2\pi n)\cdot\frac{1}{4}}\\ \therefore z &= \mathfrak{e}^{\jmath\frac{\pi+2\pi n}{4}}\\ \therefore z &= \mathfrak{e}^{\jmath\pi\frac{1+2n}{4}}\\ \therefore z = \cos\left(\frac{1+2n}{4}\pi\right) &+ \jmath\sin\left(\frac{1+2n}{4}\pi\right)\end{aligned}$$

a) When $n = 4m$, where m is an integer,

$$z = \cos\left(\frac{1+2n}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{1+2n}{4}\pi\right)$$
$$= \cos\left(\frac{1+2\cdot 4m}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{1+2\cdot 4m}{4}\pi\right)$$
$$= \cos\left(\frac{1+8m}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{1+8m}{4}\pi\right)$$
$$= \cos\left(\frac{1}{4}\pi + 2m\pi\right)$$
$$+\jmath \sin\left(\frac{1}{4}\pi + 2m\pi\right)$$
$$= \cos\left(\frac{1}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{1}{4}\pi\right)$$
$$= \frac{1}{\sqrt{2}} + \jmath\frac{1}{\sqrt{2}}$$

b) When $n = 4m + 1$

$$z = \cos\left(\frac{1+2n}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{1+2n}{4}\pi\right)$$
$$= \cos\left(\frac{1+2\cdot(4m+1)}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{1+2\cdot(4m+1)}{4}\pi\right)$$
$$= \cos\left(\frac{1+8m+2}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{1+8m+2}{4}\pi\right)$$
$$= \cos\left(\frac{3+8m}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{3+8m}{4}\pi\right)$$
$$= \cos\left(\frac{3}{4}\pi + 2m\pi\right)$$
$$+\jmath \sin\left(\frac{3}{4}\pi + 2m\pi\right)$$
$$= \cos\left(\frac{3}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{3}{4}\pi\right)$$
$$= -\frac{1}{\sqrt{2}} + \jmath\frac{1}{\sqrt{2}}$$

c) When $n = 4m + 2$

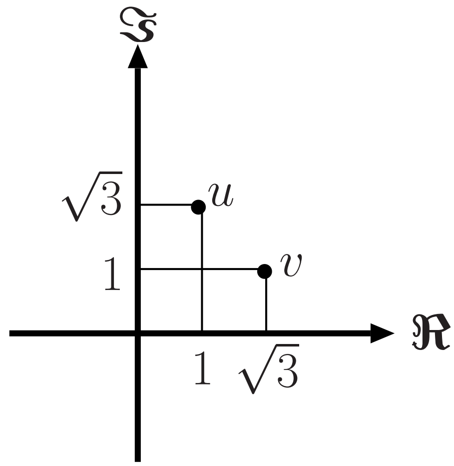
$$z = \cos\left(\frac{1+2n}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{1+2n}{4}\pi\right)$$
$$= \cos\left(\frac{1+2\cdot(4m+2)}{4}\pi\right)$$

$$+\jmath \sin\left(\frac{1+2\cdot(4m+2)}{4}\pi\right)$$
$$= \cos\left(\frac{1+8m+4}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{1+8m+4}{4}\pi\right)$$
$$= \cos\left(\frac{5+8m}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{5+8m}{4}\pi\right)$$
$$= \cos\left(\frac{5}{4}\pi+2m\pi\right)$$
$$+\jmath \sin\left(\frac{5}{4}\pi+2m\pi\right)$$
$$= \cos\left(\frac{5}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{5}{4}\pi\right)$$
$$= -\frac{1}{\sqrt{2}} - \jmath\frac{1}{\sqrt{2}}$$

d) When $n = 4m+3$

$$z = \cos\left(\frac{1+2n}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{1+2n}{4}\pi\right)$$
$$= \cos\left(\frac{1+2\cdot(4m+3)}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{1+2\cdot(4m+3)}{4}\pi\right)$$
$$= \cos\left(\frac{1+8m+6}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{1+8m+6}{4}\pi\right)$$
$$= \cos\left(\frac{7+8m}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{7+8m}{4}\pi\right)$$
$$= \cos\left(\frac{7}{4}\pi+2m\pi\right)$$
$$+\jmath \sin\left(\frac{7}{4}\pi+2m\pi\right)$$
$$= \cos\left(\frac{7}{4}\pi\right)$$
$$+\jmath \sin\left(\frac{7}{4}\pi\right)$$
$$= \frac{1}{\sqrt{2}} - \jmath\frac{1}{\sqrt{2}}$$

56) Given $u = 1+\jmath\sqrt{3}$ and $v = \sqrt{3}+\jmath$ calculate u^{12}/v^{16}



$$
\begin{aligned}
|u| &= \sqrt{1^2 + (\sqrt{3})^2} \\
&= \sqrt{1+3} \\
&= \sqrt{4} \\
&= 2 \\
|v| &= \sqrt{(\sqrt{3})^2 + 1^2} \\
&= 2
\end{aligned}
$$

u and v are both in the first quadrant in Argand diagram. The phase of u is

$$
\begin{aligned}
\tan\theta &= \sqrt{3} \\
\therefore \theta &= \frac{\pi}{3}
\end{aligned}
$$

The phase of v is

$$
\begin{aligned}
\tan\theta &= \frac{1}{\sqrt{3}} \\
\therefore \theta &= \frac{\pi}{6}
\end{aligned}
$$

Therefore in the modulus/argument form, $u = 1 + \jmath\sqrt{3} = 2\mathfrak{e}^{\jmath\pi/3}$ and $v = \sqrt{3} + \jmath = 2\mathfrak{e}^{\jmath\pi/6}$. Thus

$$
\begin{aligned}
u^{12} &= (2\mathfrak{e}^{\jmath\pi/3})^{12} \\
&= 2^{12}\mathfrak{e}^{\jmath\frac{\pi}{3}\times 12} \\
&= 2^{12}\mathfrak{e}^{\jmath\frac{12\pi}{3}} \\
&= 2^{12}\mathfrak{e}^{\jmath 4\pi} \\
v^{16} &= (2\mathfrak{e}^{\jmath\pi/6})^{16} \\
&= 2^{16}\mathfrak{e}^{\jmath\frac{\pi}{6}\times 16} \\
&= 2^{16}\mathfrak{e}^{\jmath\frac{16\pi}{6}} \\
&= 2^{16}\mathfrak{e}^{\jmath\frac{8\pi}{3}}
\end{aligned}
$$

Using these,

$$
\begin{aligned}
\frac{u^{12}}{v^{16}} &= \frac{2^{12}\mathfrak{e}^{\jmath 4\pi}}{2^{16}\mathfrak{e}^{\jmath\frac{8\pi}{3}}} \\
&= \frac{\mathfrak{e}^{\jmath 4\pi - \jmath\frac{8\pi}{3}}}{2^{16-12}} \\
&= \frac{\mathfrak{e}^{\jmath(4\pi\cdot\frac{3}{3} - \frac{8\pi}{3})}}{2^{16-12}} \\
&= \frac{\mathfrak{e}^{\jmath(\frac{12\pi}{3} - \frac{8\pi}{3})}}{2^4} \\
&= \frac{\mathfrak{e}^{\jmath\frac{12\pi-8\pi}{3}}}{2^4} \\
&= \frac{\mathfrak{e}^{\jmath\frac{4\pi}{3}}}{2^4}
\end{aligned}
$$

57) The equation

$$z^3 - \jmath 4\sqrt{3} = 4$$

has a root $z_1 = 2(\cos(\pi/9) + \jmath \sin(\pi/9))$ Find the other roots z_2 and z_3. Please do not leave $\jmath$ inside $\sqrt{\ }$.
Hint: You can get z_1, z_2 and z_3 without the information of z_1.

$$z^3 - \jmath 4\sqrt{3} = 4$$

can be re-written as

$$\begin{aligned} z^3 - \jmath 4\sqrt{3} &= 4 \\ \therefore z^3 &= 4 + \jmath 4\sqrt{3} \\ &= 4(1 + \sqrt{3}\jmath) \end{aligned}$$

The answer of the problem 56 gives

$$1 + \jmath\sqrt{3} = 2\mathfrak{e}^{\jmath\pi/3}.$$

Using this,

$$\begin{aligned} z^3 &= 4(1 + \sqrt{3}\jmath) \\ &= 4 \cdot 2\mathfrak{e}^{\jmath\pi/3} \\ &= 8\mathfrak{e}^{\jmath\pi/3} \\ &= 2^3\mathfrak{e}^{\jmath\pi/3} \end{aligned}$$

We assume $z = r\mathfrak{e}^{\jmath\theta}$ where r is real and positive and θ is real. When we put $z = r\mathfrak{e}^{\jmath\theta}$ into the equation above,

$$\begin{aligned} z^3 &= 2^3\mathfrak{e}^{\jmath\pi/3} \\ \therefore (r\mathfrak{e}^{\jmath\theta})^3 &= 2^3\mathfrak{e}^{\jmath\pi/3} \\ \therefore r^3\mathfrak{e}^{\jmath 3\theta} &= 2^3\mathfrak{e}^{\jmath\pi/3} \end{aligned}$$

By equating the modulus and the exponent part, we obtain the following:

$$\begin{aligned} r^3 &= 2^3 \\ \therefore r &= 2 \\ \mathfrak{e}^{\jmath 3\theta} &= \mathfrak{e}^{\jmath\pi/3} \\ \therefore 3\theta &= \pi/3 + 2\pi n \\ \therefore \theta &= \frac{\pi}{3 \cdot 3} + \frac{2\pi n}{3} \\ \therefore \theta &= \frac{\pi}{9} + \frac{2\pi n}{3} \end{aligned}$$

where n is an arbitrary integer.
When $n = 3m$, where m is an arbitrary integer,

$$\begin{aligned} z &= r\mathfrak{e}^{\jmath\theta} \\ &= 2\mathfrak{e}^{\jmath\left(\frac{\pi}{9} + \frac{2\pi n}{3}\right)} \\ &= 2\mathfrak{e}^{\jmath\left(\frac{\pi}{9} + \frac{2\pi \cdot 3m}{3}\right)} \\ &= 2\mathfrak{e}^{\jmath\left(\frac{\pi}{9} + 2m\pi\right)} \\ &= 2\mathfrak{e}^{\frac{\jmath\pi}{9} + 2m\pi\jmath} \\ &= 2\mathfrak{e}^{\frac{\jmath\pi}{9}}\mathfrak{e}^{2m\pi\jmath} \\ &= 2\mathfrak{e}^{\frac{\jmath\pi}{9}} \equiv z_1 \end{aligned}$$

When $n = 3m + 1$,

$$\begin{aligned} z &= r\mathfrak{e}^{\jmath\theta} \\ &= 2\mathfrak{e}^{\jmath\left(\frac{\pi}{9} + \frac{2\pi n}{3}\right)} \\ &= 2\mathfrak{e}^{\jmath\left(\frac{\pi}{9} + \frac{2\pi \cdot (3m+1)}{3}\right)} \\ &= 2\mathfrak{e}^{\jmath\left(\frac{\pi}{9} + \frac{3 \cdot 2\pi \cdot (3m+1)}{9}\right)} \\ &= 2\mathfrak{e}^{\jmath\frac{\pi + 6\pi(3m+1)}{9}} \\ &= 2\mathfrak{e}^{\jmath\frac{\pi + 18m\pi + 6\pi}{9}} \\ &= 2\mathfrak{e}^{\jmath\frac{7\pi + 18m\pi}{9}} \\ &= 2\mathfrak{e}^{\frac{7\pi\jmath + 18m\pi\jmath}{9}} \\ &= 2\mathfrak{e}^{\frac{7\pi\jmath}{9}}\mathfrak{e}^{\frac{18m\pi\jmath}{9}} \\ &= 2\mathfrak{e}^{\frac{7\pi\jmath}{9}}\mathfrak{e}^{2m\pi\jmath} \\ &= 2\mathfrak{e}^{\frac{7\pi\jmath}{9}} \equiv z_2 \end{aligned}$$

When $n = 3m + 2$,

$$\begin{aligned}
z &= r\mathfrak{e}^{\jmath\theta} \\
&= 2\mathfrak{e}^{\jmath\left(\frac{\pi}{9}+\frac{2\pi n}{3}\right)} \\
&= 2\mathfrak{e}^{\jmath\left(\frac{\pi}{9}+\frac{2\pi\cdot(3m+2)}{3}\right)} \\
&= 2\mathfrak{e}^{\jmath\left(\frac{\pi}{9}+\frac{6\pi\cdot(3m+2)}{9}\right)} \\
&= 2\mathfrak{e}^{\jmath\frac{\pi+6\pi(3m+2)}{9}} \\
&= 2\mathfrak{e}^{\jmath\frac{\pi+18m\pi+12\pi}{9}} \\
&= 2\mathfrak{e}^{\jmath\frac{13\pi+18m\pi}{9}} \\
&= 2\mathfrak{e}^{\frac{13\pi\jmath+18m\pi\jmath}{9}} \\
&= 2\mathfrak{e}^{\frac{13\pi\jmath}{9}}\mathfrak{e}^{\frac{18m\pi\jmath}{9}} \\
&= 2\mathfrak{e}^{\frac{13\pi\jmath}{9}}\mathfrak{e}^{2m\pi\jmath} \\
&= 2\mathfrak{e}^{\frac{13\pi\jmath}{9}} \equiv z_3
\end{aligned}$$

DAY5

58) Solve

$$\log(x-2)(2x-3) = \log x^2$$

We work under the condition of $(x-2)(2x-3) > 0$ and $x \neq 0$

$$\begin{aligned} \log(x-2)(2x-3) &= \log x^2 \\ \therefore (x-2)(2x-3) &= x^2 \\ \therefore 2x^2 - 3x - 4x + 6 &= x^2 \\ \therefore x^2 - 7x + 6 &= 0 \\ \therefore (x-1)(x-6) &= 0 \\ \therefore x &= 1, 6 \end{aligned}$$

Since both answers satisfy $(x-2)(2x-3) > 0$, $x = 1, 6$ are the answers.

59) Solve the following equation

$$z^2 - 4x + 6 = 0$$

$$\begin{aligned} z^2 - 4x + 6 &= 0 \\ z = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 6}}{2 \cdot 1} \\ &= \frac{4 \pm \sqrt{16-24}}{2} \\ &= \frac{4 \pm \sqrt{-8}}{2} \\ &= \frac{4 \pm \sqrt{4}\sqrt{-2}}{2} \\ &= \frac{4 \pm 2\sqrt{2}\sqrt{-1}}{2} \\ &= \frac{4 \pm 2\sqrt{2}\jmath}{2} \\ &= \frac{4}{2} \pm \frac{2\sqrt{2}\jmath}{2} \\ &= 2 \pm \sqrt{2}\jmath \end{aligned}$$

60) Perform the following polynomial division

$$x^2 + 5x + 12 \;\overline{)5x^3 + 29x^2 + 80x + 48}$$

$$\begin{array}{rl} & 5x \;\; +4 \\ x^2+5x+12 & \overline{)5x^3+29x^2+80x+48} \\ -) & 5x^3+25x^2+60x \\ \hline & 4x^2+20x+48 \\ -) & 4x^2+20x+48 \\ \hline & 0 \quad 0 \quad 0 \end{array}$$

Thus, we find the quotient to be $5x + 4$

61) Perform the following polynomial division

$$x + 3 \;\overline{)x^3 + x^2 + x + 2}$$

$$\begin{array}{rl} & x^2 - 2x + 7 \\ x+3 & \overline{)x^3 + x^2 + x + 2} \\ -) & x^3 + 3x^2 \\ \hline & -2x^2 + x + 2 \\ -) & -2x^2 - 6x \\ \hline & 7x + 2 \\ -) & 7x + 21 \\ \hline & 0 - 19 \end{array}$$

Thus, we find the quotient to be $x^2 - 2x + 7$ with a remainder of -19.

62) Perform the following polynomial division

$$x^2 + 3x + 3 \;\overline{)3x^3 - 2x^2 + 4x - 3}$$

$$
\begin{array}{rrrrr}
 & & & & 3x-11 \\ \cline{2-5}
x^2+3x+3 \,) & 3x^3 & -2x^2 & +4x & -3 \\
-) & 3x^3 & +9x^2 & +9x & \\ \hline
 & & -11x^2 & -5x & -3 \\
-) & & -11x^2 & -33x & -33 \\ \hline
 & & & & 0+28x+30
\end{array}
$$

Thus, we find the quotient to be $3x-11$ with a remainder of $28x+30$.

63) Using De Moivre's theorem, write $\left(\sqrt{3}+\jmath\sqrt{3}\right)^4$ in the form $\alpha \pm \jmath\beta$.

The strategy to tackle this problem is

a) change $\sqrt{3}+\jmath\sqrt{3}$ to the modulus/argument form
b) use Equation (28) to get the form of $r(\cos\theta+\jmath\sin\theta)$
c) Apply Equation (34)

First let's find r.

$$r=\sqrt{(\sqrt{3})^2+(\sqrt{3})^2}=\sqrt{3+3}=\sqrt{6}$$

Now to work out θ. We know that $\tan\theta=\dfrac{\sqrt{3}}{\sqrt{3}}=1$.

$$\theta=\tan^{-1}1=\frac{1}{4}\pi$$

Now using De Moivre's theorem.

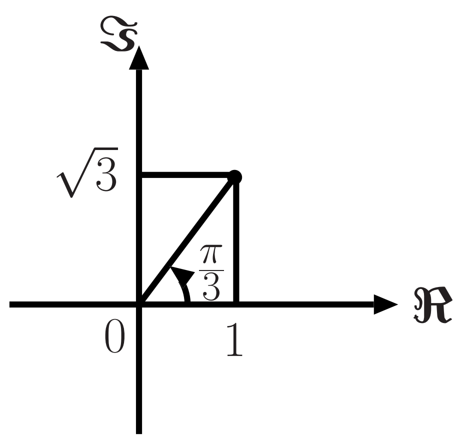
$$
\begin{aligned}
\left(\sqrt{3}-\jmath\sqrt{3}\right)^4 &= \left[\sqrt{6}(\cos(\frac{1}{4}\pi)+\jmath\sin(\frac{1}{4}\pi))\right]^4 \\
&= (\sqrt{6})^4(\cos(\frac{1}{4}\pi)+\jmath\sin(\frac{1}{4}\pi))^4 \\
&= (\sqrt{6})^4(\cos(4\cdot\frac{1}{4}\pi)+\jmath\sin(4\cdot\frac{1}{4}\pi)) \\
&= (\sqrt{6})^2\cdot(\sqrt{6})^2(\cos(4\cdot\frac{1}{4}\pi)+\jmath\sin(4\cdot\frac{1}{4}\pi)) \\
&= 6\cdot 6(\cos(4\cdot\frac{1}{4}\pi)+\jmath\sin(4\cdot\frac{1}{4}\pi)) \\
&= 36(\cos(\pi)+\jmath\sin(\pi)) \\
&= 36(-1+\jmath 0) \\
&= -36+\jmath 0
\end{aligned}
$$

Therefore $\alpha=-36$ and $\beta=0$.

64) Calculate integers n such that

$$z_n=(1+\jmath\sqrt{3})^n$$

is a real number.



We assume the problem 56 is handled prior to this question. From the answer in the problem 56 , we know that

$$1+\jmath\sqrt{3}=2e^{\jmath\frac{\pi}{3}}.$$

Thus

$$
\begin{aligned}
(1+\jmath\sqrt{3})^n &= (2\mathfrak{e}^{\jmath\frac{\pi}{3}})^n\\
&= 2^n\mathfrak{e}^{\jmath\frac{\pi}{3}\times n}\\
&= 2^n\mathfrak{e}^{\jmath\frac{n\pi}{3}}
\end{aligned}
$$

In order to make $\mathfrak{e}^{\jmath\frac{n\pi}{3}}$ real, using an integer m, we need

$$
\begin{aligned}
\frac{n\pi}{3} &= m\pi\\
\therefore n\pi &= 3m\pi\\
\therefore n &= 3m
\end{aligned}
$$

Thus $n = 3m$, *i.e.,* a multiple of 3.

65) Obtain the real and imaginary parts of $\cosh(1-\jmath\frac{\pi}{6})$ in standard form

$$
\begin{aligned}
\cosh(a+\jmath b) &= \frac{\mathfrak{e}^{a+\jmath b}+\mathfrak{e}^{-(a+\jmath b)}}{2} = \frac{\mathfrak{e}^{a+\jmath b}+\mathfrak{e}^{-a-\jmath b}}{2} = \frac{\mathfrak{e}^{a}\mathfrak{e}^{\jmath b}+\mathfrak{e}^{-a}\mathfrak{e}^{-\jmath b}}{2}\\
&= \frac{\mathfrak{e}^{a}(\cos(b)+\jmath\sin(b))+\mathfrak{e}^{-a}(\cos(-b)+\jmath\sin(-b))}{2} = \frac{\mathfrak{e}^{a}(\cos(b)+\jmath\sin(b))+\mathfrak{e}^{-a}(\cos(b)-\jmath\sin(b))}{2}\\
&= \frac{\mathfrak{e}^{a}\cos(b)+\jmath\mathfrak{e}^{a}\sin(b)+\mathfrak{e}^{-a}\cos(b)-\jmath\mathfrak{e}^{-a}\sin(b)}{2} = \frac{\mathfrak{e}^{a}\cos(b)+\mathfrak{e}^{-a}\cos(b)-\jmath\mathfrak{e}^{-a}\sin(b)+\jmath\mathfrak{e}^{a}\sin(b)}{2}\\
&= \frac{\mathfrak{e}^{a}\cos(b)+\mathfrak{e}^{-a}\cos(b)+\jmath(-\mathfrak{e}^{-a}\sin(b)+\mathfrak{e}^{a}\sin(b))}{2} = \frac{\mathfrak{e}^{a}\cos(b)+\mathfrak{e}^{-a}\cos(b)}{2}+\frac{\jmath(-\mathfrak{e}^{-a}\sin(b))+\mathfrak{e}^{a}\sin(b))}{2}\\
&= \cos(b)\frac{\mathfrak{e}^{a}+\mathfrak{e}^{-a}}{2}+\jmath\sin(b)\frac{-\mathfrak{e}^{-a}+\mathfrak{e}^{a}}{2}
\end{aligned}
$$

When we substitute $a=1$ and $b=-\frac{\pi}{6}$ we obtain

$$
\begin{aligned}
\cosh(1+\jmath(-\frac{\pi}{6})) &= \cos(-\frac{\pi}{6})\frac{\mathfrak{e}^{1}+\mathfrak{e}^{-1}}{2}+\jmath\sin(-\frac{\pi}{6})\frac{-\mathfrak{e}^{-1}+\mathfrak{e}^{1}}{2}\\
&= \frac{\sqrt{3}}{2}\frac{\mathfrak{e}+\mathfrak{e}^{-1}}{2}-\jmath\frac{1}{2}\frac{-\mathfrak{e}^{-1}+\mathfrak{e}}{2} = \frac{\sqrt{3}(\mathfrak{e}+\mathfrak{e}^{-1})}{4}-\jmath\frac{-\mathfrak{e}^{-1}+\mathfrak{e}}{4}
\end{aligned}
$$

66) Find the possible values of $\mathfrak{e}^z$ when $\cosh(z)=2$

$$
\cosh(z) = \frac{\mathfrak{e}^{z}+\mathfrak{e}^{-z}}{2} = 2 \;;\quad \therefore \mathfrak{e}^{z}+\mathfrak{e}^{-z} = 2\cdot 2 \;;\quad \therefore \mathfrak{e}^{2z}+1 = 2\cdot 2\cdot\mathfrak{e}^{z} \;;\quad \therefore \mathfrak{e}^{2z}-2\cdot 2\cdot\mathfrak{e}^{z}+1 = 0
$$
$$
\therefore \mathfrak{e}^{z} = 2\pm\sqrt{2^2-1}
$$

67) Find the possible values of $\mathfrak{e}^z$ when $\sinh(z)=2$

$$
\sinh(z) = \frac{\mathfrak{e}^{z}-\mathfrak{e}^{-z}}{2} = 2 \;;\quad \therefore \mathfrak{e}^{z}-\mathfrak{e}^{-z} = 2\cdot 2 \;;\quad \therefore \mathfrak{e}^{2z}-1 = 2\cdot 2\cdot\mathfrak{e}^{z} \;;\quad \therefore \mathfrak{e}^{2z}-2\cdot 2\cdot\mathfrak{e}^{z}-1 = 0
$$
$$
\therefore \mathfrak{e}^{z} = 2\pm\sqrt{2^2+1}
$$

68) Find the value of $\cosh(z)$ when $\mathfrak{e}^z=2$

$$
\cosh(z) = \frac{\mathfrak{e}^{z}+\mathfrak{e}^{-z}}{2} = \frac{2+2^{-1}}{2} = \frac{2^2+1}{2\cdot 2}
$$

69) Find the value of $\sinh(z)$ when $\mathfrak{e}^z=2$

$$
\sinh(z) = \frac{\mathfrak{e}^{z}-\mathfrak{e}^{-z}}{2} = \frac{2-2^{-1}}{2} = \frac{2^2-1}{2\cdot 2}
$$

70) Find the complex number z when $\cos(z)=2$
We assume $z=a+\jmath b$ where a and b are real and $b\neq 0$.

$$
\cos(a\pm\jmath b) = \cos(a)\cosh(b)\mp\jmath\sin(a)\sinh(b) \equiv 2+\jmath 0
$$

(here you need to copy/write down the proof from the keynote)
Thus we obtain

$$
\begin{aligned}
\cos(a)\cosh(b) &= 2 \qquad ①\\
\sin(a)\sinh(b) &= 0 \qquad ②
\end{aligned}
$$

② gives us

$$
\begin{aligned}
\sin(a) &= 0 \qquad ③\\
\sinh(b) &= 0 \qquad ④
\end{aligned}
$$

From ④ we obtain

$$2\sinh(b) = \mathfrak{e}^{b} - \mathfrak{e}^{-b} = 0 \;\;;\;\; \therefore \mathfrak{e}^{2b} - 1 = 0 \;\;;\;\; \therefore \mathfrak{e}^{2b} = 1 \;\;;\;\; \therefore 2b = 0 \;\;;\;\; \therefore b = 0$$

This is not acceptable because b is defined as $b \neq 0$. From ③ we obtain

$$\sin(a) = 0 \;\;;\;\; \therefore a = 2n\pi, (2n+1)\pi$$

where n is integer. When $a = (2n+1)\pi$, $\cos(a) = -1$. Therefore from ① we obtain

$$\cosh(b) = -2$$

which is not possible because $\cosh(b) > 1$ for any real b. When $a = 2n\pi$, $\cos(a) = 1$. Therefore from ① we obtain

$$\cosh(b) = 2 \;\;;\;\; \therefore \frac{\mathfrak{e}^{b} + \mathfrak{e}^{-b}}{2} = 2 \;\;;\;\; \therefore \mathfrak{e}^{b} + \mathfrak{e}^{-b} = 2 \cdot 2 \;\;;\;\; \therefore \mathfrak{e}^{2b} + 1 = 2 \cdot 2 \cdot \mathfrak{e}^{b} \;\;;\;\; \therefore \mathfrak{e}^{2b} - 2 \cdot 2 \cdot \mathfrak{e}^{b} + 1 = 0$$

$$\therefore \mathfrak{e}^{b} = 2 \pm \sqrt{2^2 - 1} \;\;;\;\; \therefore b = \log_{\mathfrak{e}}(2 \pm \sqrt{2^2 - 1})$$

Thus $z = 2n\pi + \jmath \log_{\mathfrak{e}}(2 \pm \sqrt{2^2 - 1})$

71) Find the complex number z when $\sin(z) = 2$

We assume $z = a + \jmath b$ where a and b are real and $b \neq 0$.

$$\sin(a \pm \jmath b) = \sin(a)\cosh(b) \pm \jmath \cos(a)\sinh(b) \equiv 2 + \jmath 0$$

(here you need to copy/write down the proof from the keynote)
Thus we obtain

$$\sin(a)\cosh(b) = 2 \qquad ①$$
$$\cos(a)\sinh(b) = 0 \qquad ②$$

② gives us

$$\cos(a) = 0 \qquad ③$$
$$\sinh(b) = 0 \qquad ④$$

From ④ we obtain

$$2\sinh(b) = \mathfrak{e}^{b} - \mathfrak{e}^{-b} = 0 \;\;;\;\; \therefore \mathfrak{e}^{2b} - 1 = 0 \;\;;\;\; \therefore \mathfrak{e}^{2b} = 1 \;\;;\;\; \therefore 2b = 0 \;\;;\;\; \therefore b = 0$$

This is not acceptable because b is defined as $b \neq 0$. From ③ we obtain

$$\cos(a) = 0 \;\;;\;\; \therefore a = 2n\pi + \frac{\pi}{2}, 2n\pi - \frac{\pi}{2}$$

where n is integer. When $a = 2n\pi - \frac{\pi}{2}$, $\sin(a) = -1$. Therefore from ① we obtain

$$\cosh(b) = -2$$

which is not possible because $\cosh(b) > 1$ for any real b. When $a = 2n\pi + \frac{\pi}{2}$, $\sin(a) = 1$. Therefore from ① we obtain

$$\cosh(b) = 2 \;\;;\;\; \therefore \frac{\mathfrak{e}^{b} + \mathfrak{e}^{-b}}{2} = 2 \;\;;\;\; \therefore \mathfrak{e}^{b} + \mathfrak{e}^{-b} = 2 \cdot 2 \;\;;\;\; \therefore \mathfrak{e}^{2b} + 1 = 2 \cdot 2 \cdot \mathfrak{e}^{b} \;\;;\;\; \therefore \mathfrak{e}^{2b} - 2 \cdot 2 \cdot \mathfrak{e}^{b} + 1 = 0$$

$$\therefore \mathfrak{e}^{b} = 2 \pm \sqrt{2^2 - 1} \;\;;\;\; \therefore b = \log_{\mathfrak{e}}(2 \pm \sqrt{2^2 - 1})$$

Thus $z = \left(2n + \frac{1}{2}\right)\pi + \jmath \log_{\mathfrak{e}}(2 \pm \sqrt{2^2 - 1})$

72) Find the possible values of $\mathfrak{e}^{z}$ when $\cosh(z) = 3$

$$\cosh(z) = \frac{\mathfrak{e}^{z} + \mathfrak{e}^{-z}}{2} = 3 \;\;;\;\; \therefore \mathfrak{e}^{z} + \mathfrak{e}^{-z} = 2 \cdot 3 \;\;;\;\; \therefore \mathfrak{e}^{2z} + 1 = 2 \cdot 3 \cdot \mathfrak{e}^{z} \;\;;\;\; \therefore \mathfrak{e}^{2z} - 2 \cdot 3 \cdot \mathfrak{e}^{z} + 1 = 0$$

$$\therefore \mathfrak{e}^{z} = 3 \pm \sqrt{3^2 - 1}$$

73) Find the possible values of $\mathfrak{e}^{z}$ when $\sinh(z) = 3$

$$\sinh(z) = \frac{\mathfrak{e}^{z} - \mathfrak{e}^{-z}}{2} = 3 \;\;;\;\; \therefore \mathfrak{e}^{z} - \mathfrak{e}^{-z} = 2 \cdot 3 \;\;;\;\; \therefore \mathfrak{e}^{2z} - 1 = 2 \cdot 3 \cdot \mathfrak{e}^{z} \;\;;\;\; \therefore \mathfrak{e}^{2z} - 2 \cdot 3 \cdot \mathfrak{e}^{z} - 1 = 0$$

$$\therefore \mathfrak{e}^{z} = 3 \pm \sqrt{3^2 + 1}$$

74) Find the value of $\cosh(z)$ when $\mathfrak{e}^{z} = 3$

$$\cosh(z) = \frac{\mathfrak{e}^{z} + \mathfrak{e}^{-z}}{2} = \frac{3 + 3^{-1}}{2} = \frac{3^2 + 1}{2 \cdot 3}$$

75) Find the value of $\sinh(z)$ when $\mathfrak{e}^{z} = 3$

$$\sinh(z) = \frac{\mathfrak{e}^{z} - \mathfrak{e}^{-z}}{2} = \frac{3 - 3^{-1}}{2} = \frac{3^2 - 1}{2 \cdot 3}$$

76) Find the complex number z when $\cos(z) = 3$
We assume $z = a + \jmath b$ where a and b are real and $b \neq 0$.

$$\cos(a \pm \jmath b) = \cos(a)\cosh(b) \mp \jmath \sin(a)\sinh(b) \equiv 3 + \jmath 0$$

(here you need to copy/write down the proof from the keynote)
Thus we obtain

$$\begin{aligned} \cos(a)\cosh(b) &= 3 \qquad ① \\ \sin(a)\sinh(b) &= 0 \qquad ② \end{aligned}$$

② gives us

$$\begin{aligned} \sin(a) &= 0 \qquad ③ \\ \sinh(b) &= 0 \qquad ④ \end{aligned}$$

From ④ we obtain

$$2\sinh(b) = \mathfrak{e}^b - \mathfrak{e}^{-b} = 0 \;\; ; \;\; \therefore \mathfrak{e}^{2b} - 1 = 0 \;\; ; \;\; \therefore \mathfrak{e}^{2b} = 1 \;\; ; \;\; \therefore 2b = 0 \;\; ; \;\; \therefore b = 0$$

This is not acceptable because b is defined as $b \neq 0$. From ③ we obtain

$$\sin(a) = 0 \;\; ; \;\; \therefore a = 2n\pi, (2n+1)\pi$$

where n is integer. When $a = (2n+1)\pi$, $\cos(a) = -1$. Therefore from ① we obtain

$$\cosh(b) = -3$$

which is not possible because $\cosh(b) > 1$ for any real b. When $a = 2n\pi$, $\cos(a) = 1$. Therefore from ① we obtain

$$\cosh(b) = 3 \;\; ; \;\; \therefore \frac{\mathfrak{e}^b + \mathfrak{e}^{-b}}{2} = 3 \;\; ; \;\; \therefore \mathfrak{e}^b + \mathfrak{e}^{-b} = 2 \cdot 3 \;\; ; \;\; \therefore \mathfrak{e}^{2b} + 1 = 2 \cdot 3 \cdot \mathfrak{e}^b \;\; ; \;\; \therefore \mathfrak{e}^{2b} - 2 \cdot 3 \cdot \mathfrak{e}^b + 1 = 0$$
$$\therefore \mathfrak{e}^b = 3 \pm \sqrt{3^2 - 1} \;\; ; \;\; \therefore b = \log_{\mathfrak{e}}(3 \pm \sqrt{3^2 - 1})$$

Thus $z = 2n\pi + \jmath \log_{\mathfrak{e}}(3 \pm \sqrt{3^2 - 1})$

77) Find the complex number z when $\sin(z) = 3$
We assume $z = a + \jmath b$ where a and b are real and $b \neq 0$.

$$\sin(a \pm \jmath b) = \sin(a)\cosh(b) \pm \jmath \cos(a)\sinh(b) \equiv 3 + \jmath 0$$

(here you need to copy/write down the proof from the keynote)
Thus we obtain

$$\begin{aligned} \sin(a)\cosh(b) &= 3 \qquad ① \\ \cos(a)\sinh(b) &= 0 \qquad ② \end{aligned}$$

② gives us

$$\begin{aligned} \cos(a) &= 0 \qquad ③ \\ \sinh(b) &= 0 \qquad ④ \end{aligned}$$

From ④ we obtain

$$2\sinh(b) = \mathfrak{e}^b - \mathfrak{e}^{-b} = 0 \;\; ; \;\; \therefore \mathfrak{e}^{2b} - 1 = 0 \;\; ; \;\; \therefore \mathfrak{e}^{2b} = 1 \;\; ; \;\; \therefore 2b = 0 \;\; ; \;\; \therefore b = 0$$

This is not acceptable because b is defined as $b \neq 0$. From ③ we obtain

$$\cos(a) = 0 \;\; ; \;\; \therefore a = 2n\pi + \frac{\pi}{2}, 2n\pi - \frac{\pi}{2}$$

where n is integer. When $a = 2n\pi - \frac{\pi}{2}$, $\sin(a) = -1$. Therefore from ① we obtain

$$\cosh(b) = -3$$

which is not possible because $\cosh(b) > 1$ for any real b. When $a = 2n\pi + \frac{\pi}{2}$, $\sin(a) = 1$. Therefore from ① we obtain

$$\cosh(b) = 3 \;\; ; \;\; \therefore \frac{\mathfrak{e}^b + \mathfrak{e}^{-b}}{2} = 3 \;\; ; \;\; \therefore \mathfrak{e}^b + \mathfrak{e}^{-b} = 2 \cdot 3 \;\; ; \;\; \therefore \mathfrak{e}^{2b} + 1 = 2 \cdot 3 \cdot \mathfrak{e}^b \;\; ; \;\; \therefore \mathfrak{e}^{2b} - 2 \cdot 3 \cdot \mathfrak{e}^b + 1 = 0$$
$$\therefore \mathfrak{e}^b = 3 \pm \sqrt{3^2 - 1} \;\; ; \;\; \therefore b = \log_{\mathfrak{e}}(3 \pm \sqrt{3^2 - 1})$$

Thus $z = \left(2n + \frac{1}{2}\right)\pi + \jmath \log_{\mathfrak{e}}(3 \pm \sqrt{3^2 - 1})$

78) Find the possible values of $\mathfrak{e}^z$ when $\cosh(z) = 4$

$$\cosh(z) = \frac{\mathfrak{e}^z + \mathfrak{e}^{-z}}{2} = 4 \;\; ; \;\; \therefore \mathfrak{e}^z + \mathfrak{e}^{-z} = 2 \cdot 4 \;\; ; \;\; \therefore \mathfrak{e}^{2z} + 1 = 2 \cdot 4 \cdot \mathfrak{e}^z \;\; ; \;\; \therefore \mathfrak{e}^{2z} - 2 \cdot 4 \cdot \mathfrak{e}^z + 1 = 0$$
$$\therefore \mathfrak{e}^z = 4 \pm \sqrt{4^2 - 1}$$

79) Find the possible values of $\mathfrak{e}^z$ when $\sinh(z) = 4$

$$\sinh(z) = \frac{\mathfrak{e}^z - \mathfrak{e}^{-z}}{2} = 4 \;\; ; \;\; \therefore \mathfrak{e}^z - \mathfrak{e}^{-z} = 2 \cdot 4 \;\; ; \;\; \therefore \mathfrak{e}^{2z} - 1 = 2 \cdot 4 \cdot \mathfrak{e}^z \;\; ; \;\; \therefore \mathfrak{e}^{2z} - 2 \cdot 4 \cdot \mathfrak{e}^z - 1 = 0$$
$$\therefore \mathfrak{e}^z = 4 \pm \sqrt{4^2 + 1}$$

80) Find the value of $\cosh(z)$ when $\mathfrak{e}^z = 4$

$$\cosh(z) = \frac{\mathfrak{e}^z + \mathfrak{e}^{-z}}{2} = \frac{4 + 4^{-1}}{2} = \frac{4^2 + 1}{2 \cdot 4}$$

81) Find the value of $\sinh(z)$ when $\mathfrak{e}^z = 4$

$$\sinh(z) = \frac{\mathfrak{e}^z - \mathfrak{e}^{-z}}{2} = \frac{4 - 4^{-1}}{2} = \frac{4^2 - 1}{2 \cdot 4}$$

82) Find the complex number z when $\cos(z) = 4$
We assume $z = a + \jmath b$ where a and b are real and $b \neq 0$.

$$\cos(a \pm \jmath b) = \cos(a)\cosh(b) \mp \jmath \sin(a)\sinh(b) \equiv 4 + \jmath 0$$

(here you need to copy/write down the proof from the keynote)
Thus we obtain

$$\cos(a)\cosh(b) = 4 \qquad ①$$
$$\sin(a)\sinh(b) = 0 \qquad ②$$

② gives us

$$\sin(a) = 0 \qquad ③$$
$$\sinh(b) = 0 \qquad ④$$

From ④ we obtain

$$2\sinh(b) = \mathfrak{e}^b - \mathfrak{e}^{-b} = 0 \;\; ; \quad \therefore \mathfrak{e}^{2b} - 1 = 0 \;\; ; \quad \therefore \mathfrak{e}^{2b} = 1 \;\; ; \quad \therefore 2b = 0 \;\; ; \quad \therefore b = 0$$

This is not acceptable because b is defined as $b \neq 0$. From ③ we obtain

$$\sin(a) = 0 \;\; ; \quad \therefore a = 2n\pi, (2n+1)\pi$$

where n is integer. When $a = (2n+1)\pi$, $\cos(a) = -1$. Therefore from ① we obtain

$$\cosh(b) = -4$$

which is not possible because $\cosh(b) > 1$ for any real b. When $a = 2n\pi$, $\cos(a) = 1$. Therefore from ① we obtain

$$\cosh(b) = 4 \;\; ; \quad \therefore \frac{\mathfrak{e}^b + \mathfrak{e}^{-b}}{2} = 4 \;\; ; \quad \therefore \mathfrak{e}^b + \mathfrak{e}^{-b} = 2 \cdot 4 \;\; ; \quad \therefore \mathfrak{e}^{2b} + 1 = 2 \cdot 4 \cdot \mathfrak{e}^b \;\; ; \quad \therefore \mathfrak{e}^{2b} - 2 \cdot 4 \cdot \mathfrak{e}^b + 1 = 0$$
$$\therefore \mathfrak{e}^b = 4 \pm \sqrt{4^2 - 1} \;\; ; \quad \therefore b = \log_{\mathfrak{e}}(4 \pm \sqrt{4^2 - 1})$$

Thus $z = 2n\pi + \jmath \log_{\mathfrak{e}}(4 \pm \sqrt{4^2 - 1})$

83) Find the complex number z when $\sin(z) = 4$
We assume $z = a + \jmath b$ where a and b are real and $b \neq 0$.

$$\sin(a \pm \jmath b) = \sin(a)\cosh(b) \pm \jmath \cos(a)\sinh(b) \equiv 4 + \jmath 0$$

(here you need to copy/write down the proof from the keynote)
Thus we obtain

$$\sin(a)\cosh(b) = 4 \qquad ①$$
$$\cos(a)\sinh(b) = 0 \qquad ②$$

② gives us

$$\cos(a) = 0 \qquad ③$$
$$\sinh(b) = 0 \qquad ④$$

From ④ we obtain

$$2\sinh(b) = \mathfrak{e}^b - \mathfrak{e}^{-b} = 0 \;\; ; \quad \therefore \mathfrak{e}^{2b} - 1 = 0 \;\; ; \quad \therefore \mathfrak{e}^{2b} = 1 \;\; ; \quad \therefore 2b = 0 \;\; ; \quad \therefore b = 0$$

This is not acceptable because b is defined as $b \neq 0$. From ③ we obtain

$$\cos(a) = 0 \;\; ; \quad \therefore a = 2n\pi + \frac{\pi}{2}, 2n\pi - \frac{\pi}{2}$$

where n is integer. When $a = 2n\pi - \frac{\pi}{2}$, $\sin(a) = -1$. Therefore from ① we obtain

$$\cosh(b) = -4$$

which is not possible because $\cosh(b) > 1$ for any real b. When $a = 2n\pi + \frac{\pi}{2}$, $\sin(a) = 1$. Therefore from ① we obtain

$$\cosh(b) = 4 \;\; ; \quad \therefore \frac{\mathfrak{e}^b + \mathfrak{e}^{-b}}{2} = 4 \;\; ; \quad \therefore \mathfrak{e}^b + \mathfrak{e}^{-b} = 2 \cdot 4 \;\; ; \quad \therefore \mathfrak{e}^{2b} + 1 = 2 \cdot 4 \cdot \mathfrak{e}^b \;\; ; \quad \therefore \mathfrak{e}^{2b} - 2 \cdot 4 \cdot \mathfrak{e}^b + 1 = 0$$
$$\therefore \mathfrak{e}^b = 4 \pm \sqrt{4^2 - 1} \;\; ; \quad \therefore b = \log_{\mathfrak{e}}(4 \pm \sqrt{4^2 - 1})$$

Thus $z = \left(2n + \frac{1}{2}\right)\pi + \jmath \log_{\mathfrak{e}}(4 \pm \sqrt{4^2 - 1})$

84) Find the possible values of $\mathfrak{e}^z$ when $\cosh(z) = 5$

$$\cosh(z) = \frac{\mathfrak{e}^z + \mathfrak{e}^{-z}}{2} = 5 \ ; \quad \therefore \mathfrak{e}^z + \mathfrak{e}^{-z} = 2 \cdot 5 \ ; \quad \therefore \mathfrak{e}^{2z} + 1 = 2 \cdot 5 \cdot \mathfrak{e}^z \ ; \quad \therefore \mathfrak{e}^{2z} - 2 \cdot 5 \cdot \mathfrak{e}^z + 1 = 0$$

$$\therefore \mathfrak{e}^z = 5 \pm \sqrt{5^2 - 1}$$

85) Find the possible values of $\mathfrak{e}^z$ when $\sinh(z) = 5$

$$\sinh(z) = \frac{\mathfrak{e}^z - \mathfrak{e}^{-z}}{2} = 5 \ ; \quad \therefore \mathfrak{e}^z - \mathfrak{e}^{-z} = 2 \cdot 5 \ ; \quad \therefore \mathfrak{e}^{2z} - 1 = 2 \cdot 5 \cdot \mathfrak{e}^z \ ; \quad \therefore \mathfrak{e}^{2z} - 2 \cdot 5 \cdot \mathfrak{e}^z - 1 = 0$$

$$\therefore \mathfrak{e}^z = 5 \pm \sqrt{5^2 + 1}$$

86) Find the value of $\cosh(z)$ when $\mathfrak{e}^z = 5$

$$\cosh(z) = \frac{\mathfrak{e}^z + \mathfrak{e}^{-z}}{2} = \frac{5 + 5^{-1}}{2} = \frac{5^2 + 1}{2 \cdot 5}$$

87) Find the value of $\sinh(z)$ when $\mathfrak{e}^z = 5$

$$\sinh(z) = \frac{\mathfrak{e}^z - \mathfrak{e}^{-z}}{2} = \frac{5 - 5^{-1}}{2} = \frac{5^2 - 1}{2 \cdot 5}$$

88) Find the complex number z when $\cos(z) = 5$
We assume $z = a + \jmath b$ where a and b are real and $b \neq 0$.

$$\cos(a \pm \jmath b) = \cos(a)\cosh(b) \mp \jmath \sin(a)\sinh(b) \equiv 5 + \jmath 0$$

(here you need to copy/write down the proof from the keynote)
Thus we obtain

$$\cos(a)\cosh(b) = 5 \qquad ①$$
$$\sin(a)\sinh(b) = 0 \qquad ②$$

② gives us

$$\sin(a) = 0 \qquad ③$$
$$\sinh(b) = 0 \qquad ④$$

From ④ we obtain

$$2\sinh(b) = \mathfrak{e}^b - \mathfrak{e}^{-b} = 0 \ ; \quad \therefore \mathfrak{e}^{2b} - 1 = 0 \ ; \quad \therefore \mathfrak{e}^{2b} = 1 \ ; \quad \therefore 2b = 0 \ ; \quad \therefore b = 0$$

This is not acceptable because b is defined as $b \neq 0$. From ③ we obtain

$$\sin(a) = 0 \ ; \quad \therefore a = 2n\pi, (2n+1)\pi$$

where n is integer. When $a = (2n+1)\pi$, $\cos(a) = -1$. Therefore from ① we obtain

$$\cosh(b) = -5$$

which is not possible because $\cosh(b) > 1$ for any real b. When $a = 2n\pi$, $\cos(a) = 1$. Therefore from ① we obtain

$$\cosh(b) = 5 \ ; \quad \therefore \frac{\mathfrak{e}^b + \mathfrak{e}^{-b}}{2} = 5 \ ; \quad \therefore \mathfrak{e}^b + \mathfrak{e}^{-b} = 2 \cdot 5 \ ; \quad \therefore \mathfrak{e}^{2b} + 1 = 2 \cdot 5 \cdot \mathfrak{e}^b \ ; \quad \therefore \mathfrak{e}^{2b} - 2 \cdot 5 \cdot \mathfrak{e}^b + 1 = 0$$

$$\therefore \mathfrak{e}^b = 5 \pm \sqrt{5^2 - 1} \ ; \quad \therefore b = \log_{\mathfrak{e}}(5 \pm \sqrt{5^2 - 1})$$

Thus $z = 2n\pi + \jmath \log_{\mathfrak{e}}(5 \pm \sqrt{5^2 - 1})$

89) Find the complex number z when $\sin(z) = 5$
We assume $z = a + \jmath b$ where a and b are real and $b \neq 0$.

$$\sin(a \pm \jmath b) = \sin(a)\cosh(b) \pm \jmath \cos(a)\sinh(b) \equiv 5 + \jmath 0$$

(here you need to copy/write down the proof from the keynote)
Thus we obtain

$$\sin(a)\cosh(b) = 5 \qquad ①$$
$$\cos(a)\sinh(b) = 0 \qquad ②$$

② gives us

$$\cos(a) = 0 \qquad ③$$
$$\sinh(b) = 0 \qquad ④$$

From ④ we obtain

$$2\sinh(b) = \mathfrak{e}^b - \mathfrak{e}^{-b} = 0 \ ; \quad \therefore \mathfrak{e}^{2b} - 1 = 0 \ ; \quad \therefore \mathfrak{e}^{2b} = 1 \ ; \quad \therefore 2b = 0 \ ; \quad \therefore b = 0$$

This is not acceptable because b is defined as $b \neq 0$. From ③ we obtain

$$\cos(a) = 0 \ ; \quad \therefore a = 2n\pi + \frac{\pi}{2}, 2n\pi - \frac{\pi}{2}$$

where n is integer. When $a = 2n\pi - \frac{\pi}{2}$, $\sin(a) = -1$. Therefore from ① we obtain

$$\cosh(b) = -5$$

which is not possible because $\cosh(b) > 1$ for any real b. When $a = 2n\pi + \frac{\pi}{2}$, $\sin(a) = 1$. Therefore from ① we obtain

$$\cosh(b) = 5 \ ; \quad \therefore \frac{\mathfrak{e}^b + \mathfrak{e}^{-b}}{2} = 5 \ ; \quad \therefore \mathfrak{e}^b + \mathfrak{e}^{-b} = 2 \cdot 5 \ ; \quad \therefore \mathfrak{e}^{2b} + 1 = 2 \cdot 5 \cdot \mathfrak{e}^b \ ; \quad \therefore \mathfrak{e}^{2b} - 2 \cdot 5 \cdot \mathfrak{e}^b + 1 = 0$$

$$\therefore \mathfrak{e}^b = 5 \pm \sqrt{5^2 - 1} \ ; \quad \therefore b = \log_{\mathfrak{e}}(5 \pm \sqrt{5^2 - 1})$$

Thus $z = \left(2n + \frac{1}{2}\right)\pi + \jmath \log_{\mathfrak{e}}(5 \pm \sqrt{5^2 - 1})$

DAY6

90) Find the complex number z in $\cosh z = \cos(1+\jmath)$

$$\cosh z = \cos(1+\jmath)$$

$$\therefore \frac{e^z + e^{-z}}{2} = \frac{e^{\jmath(1+\jmath)} + e^{-\jmath(1+\jmath)}}{2} = \frac{e^{\jmath-1} + e^{-\jmath+1}}{2} = \frac{e^{\jmath-1} + e^{-(\jmath-1)}}{2} \quad ; \quad \therefore z = -1+\jmath$$

91) Find the complex number z in $\cosh z = \cos(1-\jmath)$

$$\cosh z = \cos(1-\jmath)$$

$$\therefore \frac{e^z + e^{-z}}{2} = \frac{e^{\jmath(1-\jmath)} + e^{-\jmath(1-\jmath)}}{2} = \frac{e^{\jmath+1} + e^{-\jmath-1}}{2} = \frac{e^{\jmath+1} + e^{-(\jmath+1)}}{2} \quad ; \quad \therefore z = 1+\jmath$$

92) Find the complex number z in $\cosh z = \cos(-1+\jmath)$

$$\cosh z = \cos(-1+\jmath)$$

$$\therefore \frac{e^z + e^{-z}}{2} = \frac{e^{\jmath(-1+\jmath)} + e^{-\jmath(-1+\jmath)}}{2} = \frac{e^{-\jmath-1} + e^{\jmath+1}}{2} = \frac{e^{-\jmath-1} + e^{-(-\jmath-1)}}{2} \quad ; \quad \therefore z = -1-\jmath$$

93) Find the complex number z in $\cosh z = \cos(-1-\jmath)$

$$\cosh z = \cos(-1-\jmath)$$

$$\therefore \frac{e^z + e^{-z}}{2} = \frac{e^{\jmath(-1-\jmath)} + e^{-\jmath(-1-\jmath)}}{2} = \frac{e^{-\jmath+1} + e^{\jmath-1}}{2} = \frac{e^{-\jmath+1} + e^{-(-\jmath+1)}}{2} \quad ; \quad \therefore z = 1-\jmath$$

94) Find the complex number z in $\cos z = \cosh(1+\jmath)$

$$\frac{e^{z\jmath} + e^{-z\jmath}}{2} = \frac{e^{1+\jmath} + e^{1+\jmath}}{2} \quad ; \quad \therefore z\jmath = 1+\jmath \quad ; \quad \therefore z = -\jmath+1$$

95) Find the complex number z in $\cos z = \cosh(1-\jmath)$

$$\frac{e^{z\jmath} + e^{-z\jmath}}{2} = \frac{e^{1-\jmath} + e^{1-\jmath}}{2} \quad ; \quad \therefore z\jmath = 1-\jmath \quad ; \quad \therefore z = -1-\jmath$$

96) Find the complex number z in $\cos z = \cosh(-1+\jmath)$

$$\frac{e^{z\jmath} + e^{-z\jmath}}{2} = \frac{e^{-1+\jmath} + e^{-1+\jmath}}{2} \quad ; \quad \therefore z\jmath = -1+\jmath \quad ; \quad \therefore z = 1+\jmath$$

97) Find the complex number z in $\cos z = \cosh(-1-\jmath)$

$$\frac{e^{z\jmath} + e^{-z\jmath}}{2} = \frac{e^{-1-\jmath} + e^{-1-\jmath}}{2} \quad ; \quad \therefore z\jmath = -1-\jmath \quad ; \quad \therefore z = -1+\jmath$$

98) Find the complex number z in $\sin z = \cosh(1+\jmath)$

$$\frac{e^{z\jmath} - e^{-z\jmath}}{2\jmath} = \frac{e^{1+\jmath} + e^{-(1+\jmath)}}{2} \quad ; \quad \therefore e^{z\jmath} - e^{-z\jmath} = \jmath\left(e^{1+\jmath} + e^{-(1+\jmath)}\right) = e^{\jmath\frac{\pi}{2}} \cdot \left(e^{1+\jmath} + e^{-(1+\jmath)}\right)$$

$$\therefore e^{z\jmath} - e^{-z\jmath} = e^{1+\jmath} \cdot e^{\jmath\frac{\pi}{2}} + e^{-(1+\jmath)} \cdot e^{\jmath\frac{\pi}{2}} = e^{1+\jmath} \cdot e^{\jmath\frac{\pi}{2}} - \left(-e^{-(1+\jmath)} \cdot e^{\jmath\frac{\pi}{2}}\right)$$

$$\therefore e^{z\jmath} - e^{-z\jmath} = e^{1+\jmath} \cdot e^{\jmath\frac{\pi}{2}} - e^{-\jmath\pi}e^{-(1+\jmath)} \cdot e^{\jmath\frac{\pi}{2}} = e^{1+\jmath+\jmath\frac{\pi}{2}} - e^{-\jmath\pi-(1+\jmath)+\jmath\frac{\pi}{2}}$$

$$\therefore e^{z\jmath} - e^{-z\jmath} = e^{1+\jmath+\jmath\frac{\pi}{2}} - e^{-(1+\jmath)-\jmath\frac{\pi}{2}} = e^{1+\jmath+\jmath\frac{\pi}{2}} - e^{-(1+\jmath+\jmath\frac{\pi}{2})} \quad ; \quad \therefore z\jmath = 1+\jmath+\jmath\frac{\pi}{2}$$

$$\therefore z = \frac{1+\jmath+\jmath\frac{\pi}{2}}{\jmath} = -\jmath\left(1+\jmath+\jmath\frac{\pi}{2}\right) = -\jmath(1+\jmath) - \jmath^2\frac{\pi}{2} = -\jmath(1+\jmath) + \frac{\pi}{2} = -\jmath+\jmath^2+\frac{\pi}{2} = -1+\frac{\pi}{2}-\jmath$$

99) Find the complex number z in $\cosh z = \sin(1+\jmath)$

$$\frac{e^z + e^{-z}}{2} = \frac{e^{\jmath(1+\jmath)} - e^{-\jmath(1+\jmath)}}{2\jmath} \quad ; \quad \therefore e^z + e^{-z} = -\jmath(e^{\jmath(1+\jmath)} - e^{-\jmath(1+\jmath)}) = e^{-\jmath\frac{\pi}{2}}(e^{\jmath(1+\jmath)} - e^{-\jmath(1+\jmath)})$$

$$\therefore e^z + e^{-z} = e^{\jmath(1+\jmath)} \cdot e^{-\jmath\frac{\pi}{2}} - e^{-\jmath(1+\jmath)} \cdot e^{-\jmath\frac{\pi}{2}} = e^{\jmath(1+\jmath)} \cdot e^{-\jmath\frac{\pi}{2}} + \left(-e^{-\jmath(1+\jmath)} \cdot e^{-\jmath\frac{\pi}{2}}\right)$$

$$\therefore e^z + e^{-z} = e^{\jmath(1+\jmath)} \cdot e^{-\jmath\frac{\pi}{2}} + e^{\jmath\pi}e^{-\jmath(1+\jmath)} \cdot e^{-\jmath\frac{\pi}{2}} = e^{\jmath(1+\jmath)-\jmath\frac{\pi}{2}} + e^{\jmath\pi-\jmath(1+\jmath)-\jmath\frac{\pi}{2}}$$

$$\therefore e^z + e^{-z} = e^{\jmath(1+\jmath)-\jmath\frac{\pi}{2}} + e^{-\jmath(1+\jmath)+\jmath\frac{\pi}{2}} = e^{\jmath(1+\jmath)-\jmath\frac{\pi}{2}} + e^{-(\jmath(1+\jmath)-\jmath\frac{\pi}{2})}$$

$$\therefore z = \jmath(1+\jmath) - \jmath\frac{\pi}{2} = \jmath+\jmath^2-\jmath\frac{\pi}{2} = -1+\jmath(1-\frac{\pi}{2})$$

100) Find the complex number z in $\sin z = \cosh(-1-\jmath)$

$$\frac{e^{z\jmath}-e^{-z\jmath}}{2\jmath} = \frac{e^{-1-\jmath}+e^{-(-1-\jmath)}}{2} \quad ; \quad \therefore e^{z\jmath}-e^{-z\jmath} = \jmath\left(e^{-1-\jmath}+e^{-(-1-\jmath)}\right) = e^{\jmath\frac{\pi}{2}}\cdot\left(e^{-1-\jmath}+e^{-(-1-\jmath)}\right)$$

$$\therefore e^{z\jmath}-e^{-z\jmath} = e^{-1-\jmath}\cdot e^{\jmath\frac{\pi}{2}} + e^{-(-1-\jmath)}\cdot e^{\jmath\frac{\pi}{2}} = e^{-1-\jmath}\cdot e^{\jmath\frac{\pi}{2}} - \left(-e^{-(-1-\jmath)}\cdot e^{\jmath\frac{\pi}{2}}\right)$$

$$\therefore e^{z\jmath}-e^{-z\jmath} = e^{-1-\jmath}\cdot e^{\jmath\frac{\pi}{2}} - e^{-\jmath\pi}e^{-(-1-\jmath)}\cdot e^{\jmath\frac{\pi}{2}} = e^{-1-\jmath+\jmath\frac{\pi}{2}} - e^{-\jmath\pi-(-1-\jmath)+\jmath\frac{\pi}{2}}$$

$$\therefore e^{z\jmath}-e^{-z\jmath} = e^{-1-\jmath+\jmath\frac{\pi}{2}} - e^{-(-1-\jmath)-\jmath\frac{\pi}{2}} = e^{-1-\jmath+\jmath\frac{\pi}{2}} - e^{-(-1-\jmath+\jmath\frac{\pi}{2})} \quad ; \quad \therefore z\jmath = -1-\jmath+\jmath\frac{\pi}{2}$$

$$\therefore z = \frac{-1-\jmath+\jmath\frac{\pi}{2}}{\jmath} = -\jmath\left(-1-\jmath+\jmath\frac{\pi}{2}\right) = -\jmath(-1-\jmath)-\jmath^2\frac{\pi}{2} = -\jmath(-1-\jmath)+\frac{\pi}{2} = \jmath+\jmath^2+\frac{\pi}{2} = -1+\frac{\pi}{2}+\jmath$$

101) Find the complex number z in $\cosh z = \sin(-1-\jmath)$

$$\frac{e^{z}+e^{-z}}{2} = \frac{e^{\jmath(-1-\jmath)}-e^{-\jmath(-1-\jmath)}}{2\jmath} \quad ; \quad \therefore e^{z}+e^{-z} = -\jmath(e^{\jmath(-1-\jmath)}-e^{-\jmath(-1-\jmath)}) = e^{-\jmath\frac{\pi}{2}}(e^{\jmath(-1-\jmath)}-e^{-\jmath(-1-\jmath)})$$

$$\therefore e^{z}+e^{-z} = e^{\jmath(-1-\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} - e^{-\jmath(-1-\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} = e^{\jmath(-1-\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} + \left(-e^{-\jmath(-1-\jmath)}\cdot e^{-\jmath\frac{\pi}{2}}\right)$$

$$\therefore e^{z}+e^{-z} = e^{\jmath(-1-\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} + e^{\jmath\pi}e^{-\jmath(-1-\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} = e^{\jmath(-1-\jmath)-\jmath\frac{\pi}{2}} + e^{\jmath\pi-\jmath(-1-\jmath)-\jmath\frac{\pi}{2}}$$

$$\therefore e^{z}+e^{-z} = e^{\jmath(-1-\jmath)-\jmath\frac{\pi}{2}} + e^{-\jmath(-1-\jmath)+\jmath\frac{\pi}{2}} = e^{\jmath(-1-\jmath)-\jmath\frac{\pi}{2}} + e^{-(\jmath(-1-\jmath)-\jmath\frac{\pi}{2})}$$

$$\therefore z = \jmath(-1-\jmath)-\jmath\frac{\pi}{2} = -\jmath-\jmath^2-\jmath\frac{\pi}{2} = 1+\jmath(-1-\frac{\pi}{2})$$

102) Find the complex number z in $\sin z = \cosh(-1+\jmath)$

$$\frac{e^{z\jmath}-e^{-z\jmath}}{2\jmath} = \frac{e^{-1+\jmath}+e^{-(-1+\jmath)}}{2} \quad ; \quad \therefore e^{z\jmath}-e^{-z\jmath} = \jmath\left(e^{-1+\jmath}+e^{-(-1+\jmath)}\right) = e^{\jmath\frac{\pi}{2}}\cdot\left(e^{-1+\jmath}+e^{-(-1+\jmath)}\right)$$

$$\therefore e^{z\jmath}-e^{-z\jmath} = e^{-1+\jmath}\cdot e^{\jmath\frac{\pi}{2}} + e^{-(-1+\jmath)}\cdot e^{\jmath\frac{\pi}{2}} = e^{-1+\jmath}\cdot e^{\jmath\frac{\pi}{2}} - \left(-e^{-(-1+\jmath)}\cdot e^{\jmath\frac{\pi}{2}}\right)$$

$$\therefore e^{z\jmath}-e^{-z\jmath} = e^{-1+\jmath}\cdot e^{\jmath\frac{\pi}{2}} - e^{-\jmath\pi}e^{-(-1+\jmath)}\cdot e^{\jmath\frac{\pi}{2}} = e^{-1+\jmath+\jmath\frac{\pi}{2}} - e^{-\jmath\pi-(-1+\jmath)+\jmath\frac{\pi}{2}}$$

$$\therefore e^{z\jmath}-e^{-z\jmath} = e^{-1+\jmath+\jmath\frac{\pi}{2}} - e^{-(-1+\jmath)-\jmath\frac{\pi}{2}} = e^{-1+\jmath+\jmath\frac{\pi}{2}} - e^{-(-1+\jmath+\jmath\frac{\pi}{2})} \quad ; \quad \therefore z\jmath = -1+\jmath+\jmath\frac{\pi}{2}$$

$$\therefore z = \frac{-1+\jmath+\jmath\frac{\pi}{2}}{\jmath} = -\jmath\left(-1+\jmath+\jmath\frac{\pi}{2}\right) = -\jmath(-1+\jmath)-\jmath^2\frac{\pi}{2} = -\jmath(-1+\jmath)+\frac{\pi}{2} = -\jmath-\jmath^2+\frac{\pi}{2} = 1+\frac{\pi}{2}-\jmath$$

103) Find the complex number z in $\cosh z = \sin(-1+\jmath)$

$$\frac{e^{z}+e^{-z}}{2} = \frac{e^{\jmath(-1+\jmath)}-e^{-\jmath(-1+\jmath)}}{2\jmath} \quad ; \quad \therefore e^{z}+e^{-z} = -\jmath(e^{\jmath(-1+\jmath)}-e^{-\jmath(-1+\jmath)}) = e^{-\jmath\frac{\pi}{2}}(e^{\jmath(-1+\jmath)}-e^{-\jmath(-1+\jmath)})$$

$$\therefore e^{z}+e^{-z} = e^{\jmath(-1+\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} - e^{-\jmath(-1+\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} = e^{\jmath(-1+\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} + \left(-e^{-\jmath(-1+\jmath)}\cdot e^{-\jmath\frac{\pi}{2}}\right)$$

$$\therefore e^{z}+e^{-z} = e^{\jmath(-1+\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} + e^{\jmath\pi}e^{-\jmath(-1+\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} = e^{\jmath(-1+\jmath)-\jmath\frac{\pi}{2}} + e^{\jmath\pi-\jmath(-1+\jmath)-\jmath\frac{\pi}{2}}$$

$$\therefore e^{z}+e^{-z} = e^{\jmath(-1+\jmath)-\jmath\frac{\pi}{2}} + e^{-\jmath(-1+\jmath)+\jmath\frac{\pi}{2}} = e^{\jmath(-1+\jmath)-\jmath\frac{\pi}{2}} + e^{-(\jmath(-1+\jmath)-\jmath\frac{\pi}{2})}$$

$$\therefore z = \jmath(-1+\jmath)-\jmath\frac{\pi}{2} = -\jmath+\jmath^2-\jmath\frac{\pi}{2} = -1-\jmath(1+\frac{\pi}{2})$$

104) Find the complex number z in $\sin z = \cosh(1-\jmath)$

$$\frac{e^{z\jmath}-e^{-z\jmath}}{2\jmath} = \frac{e^{1-\jmath}+e^{-(1-\jmath)}}{2} \quad ; \quad \therefore e^{z\jmath}-e^{-z\jmath} = \jmath\left(e^{1-\jmath}+e^{-(1-\jmath)}\right) = e^{\jmath\frac{\pi}{2}}\cdot\left(e^{1-\jmath}+e^{-(1-\jmath)}\right)$$

$$\therefore e^{z\jmath}-e^{-z\jmath} = e^{1-\jmath}\cdot e^{\jmath\frac{\pi}{2}} + e^{-(1-\jmath)}\cdot e^{\jmath\frac{\pi}{2}} = e^{1-\jmath}\cdot e^{\jmath\frac{\pi}{2}} - \left(-e^{-(1-\jmath)}\cdot e^{\jmath\frac{\pi}{2}}\right)$$

$$\therefore e^{z\jmath}-e^{-z\jmath} = e^{1-\jmath}\cdot e^{\jmath\frac{\pi}{2}} - e^{-\jmath\pi}e^{-(1-\jmath)}\cdot e^{\jmath\frac{\pi}{2}} = e^{1-\jmath+\jmath\frac{\pi}{2}} - e^{-\jmath\pi-(1-\jmath)+\jmath\frac{\pi}{2}}$$

$$\therefore e^{z\jmath}-e^{-z\jmath} = e^{1-\jmath+\jmath\frac{\pi}{2}} - e^{-(1-\jmath)-\jmath\frac{\pi}{2}} = e^{1-\jmath+\jmath\frac{\pi}{2}} - e^{-(1-\jmath+\jmath\frac{\pi}{2})} \quad ; \quad \therefore z\jmath = 1-\jmath+\jmath\frac{\pi}{2}$$

$$\therefore z = \frac{1-\jmath+\jmath\frac{\pi}{2}}{\jmath} = -\jmath\left(1-\jmath+\jmath\frac{\pi}{2}\right) = -\jmath(1-\jmath)-\jmath^2\frac{\pi}{2} = -\jmath(1-\jmath)+\frac{\pi}{2} = -\jmath+\jmath^2+\frac{\pi}{2} = -1+\frac{\pi}{2}-\jmath$$

105) Find the complex number z in $\cosh z = \sin(1-\jmath)$

$$\frac{e^{z}+e^{-z}}{2} = \frac{e^{\jmath(1-\jmath)}-e^{-\jmath(1-\jmath)}}{2\jmath} \quad ; \quad \therefore e^{z}+e^{-z} = -\jmath(e^{\jmath(1-\jmath)}-e^{-\jmath(1-\jmath)}) = e^{-\jmath\frac{\pi}{2}}(e^{\jmath(1-\jmath)}-e^{-\jmath(1-\jmath)})$$

$$\therefore e^{z}+e^{-z} = e^{\jmath(1-\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} - e^{-\jmath(1-\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} = e^{\jmath(1-\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} + \left(-e^{-\jmath(1-\jmath)}\cdot e^{-\jmath\frac{\pi}{2}}\right)$$

$$\therefore e^{z}+e^{-z} = e^{\jmath(1-\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} + e^{\jmath\pi}e^{-\jmath(1-\jmath)}\cdot e^{-\jmath\frac{\pi}{2}} = e^{\jmath(1-\jmath)-\jmath\frac{\pi}{2}} + e^{\jmath\pi-\jmath(1-\jmath)-\jmath\frac{\pi}{2}}$$

$$\therefore e^{z}+e^{-z} = e^{\jmath(1-\jmath)-\jmath\frac{\pi}{2}} + e^{-\jmath(1-\jmath)+\jmath\frac{\pi}{2}} = e^{\jmath(1-\jmath)-\jmath\frac{\pi}{2}} + e^{-(\jmath(1-\jmath)-\jmath\frac{\pi}{2})}$$

$$\therefore z = \jmath(1-\jmath)-\jmath\frac{\pi}{2} = \jmath-\jmath^2-\jmath\frac{\pi}{2} = 1+\jmath(1-\frac{\pi}{2})$$

1) **DAY1**

2) Simplify $x - \frac{\ln x}{x} - 2x + \frac{1}{x} - 2x\ln x + \frac{\ln x}{x}$.

$$x - \frac{\ln x}{x} - 2x + \frac{1}{x} - 2x\ln x + \frac{\ln x}{x} = -x + \frac{1}{x} - 2x\ln x$$

3) Simplify $\frac{10^{-4}}{10^{-6}}$.

$$\frac{10^{-4}}{10^{-6}} = 10^{-4+6} = 10^{2}$$

4) Write $\frac{1}{\sqrt{x}}$ without using fractions.

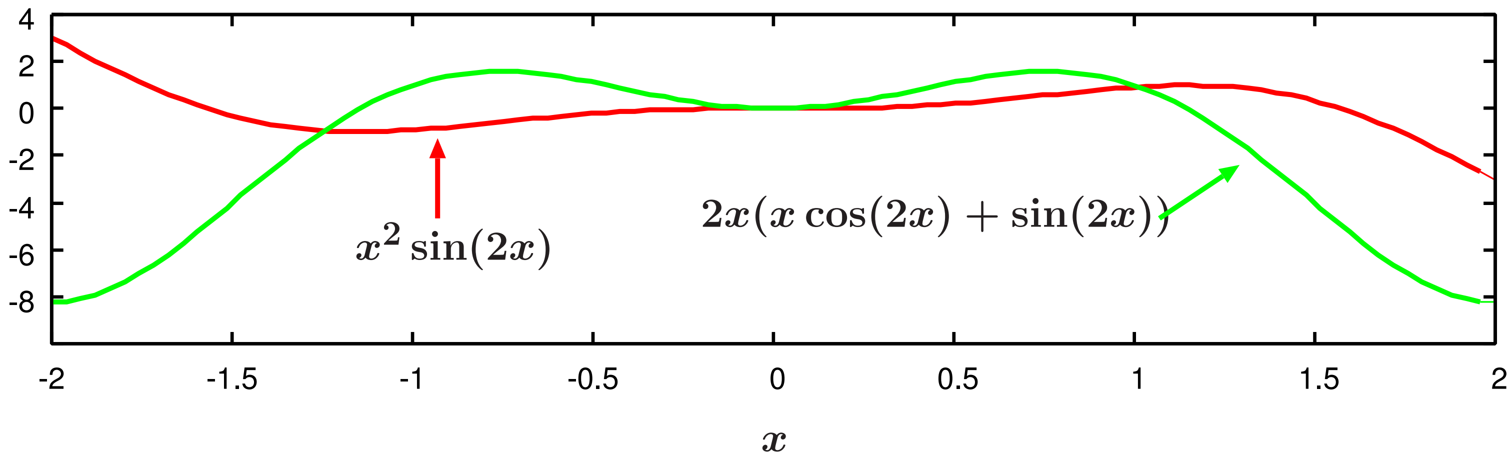
As we know $\sqrt{x} = x^{\frac{1}{2}}$ and $\frac{1}{x} = x^{-1}$

$$\frac{1}{\sqrt{x}} \equiv x^{-\frac{1}{2}}$$

5) What is the answer to the sum $0 - \frac{1}{x}$.

$$0 - a = -a \;\; ; \quad \therefore 0 - \frac{1}{x} = -\frac{1}{x}$$

6) Using the product rule, find $\frac{d\{y\}}{dx}$ of $y = x^2 \sin 2x$.



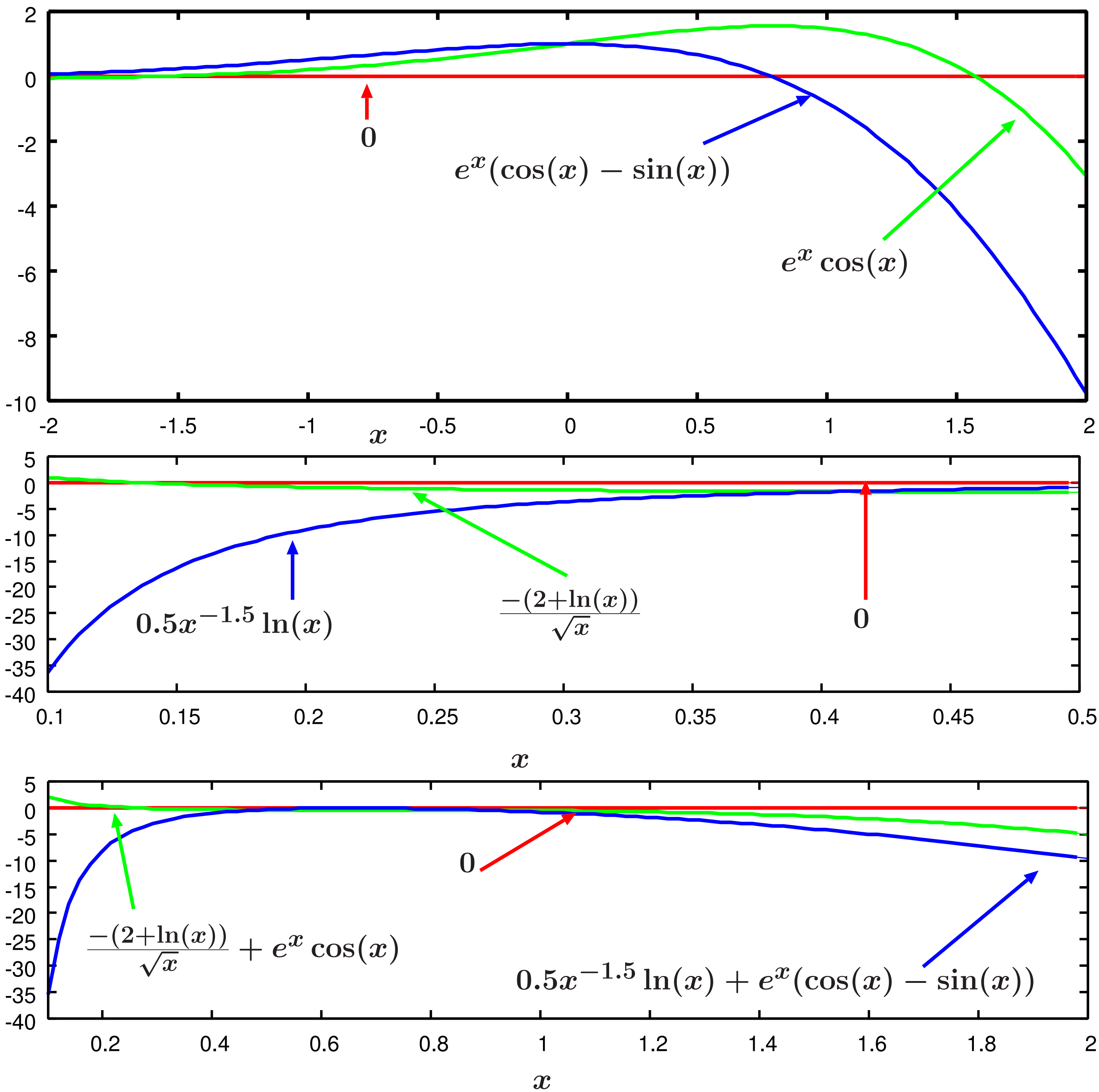
Because the function y is a product, we must split the function into two functions called $f(x)$ and $g(x)$. Letting $f(x) = x^2$ and $g(x) = \sin 2x$.

$$f(x) = x^2 \;\; ; \quad \therefore \frac{d\{f(x)\}}{dx} = 2x$$
$$g(x) = \sin 2x \;\; ; \quad \therefore \frac{d\{g(x)\}}{dx} = 2\cos 2x$$

Now applying the product rule as followed.

$$\frac{d\{y\}}{dx} = f(x) \cdot \frac{d\{g(x)\}}{dx} + g(x) \cdot \frac{d\{f(x)\}}{dx} = x^2 \cdot 2\cos 2x + \sin 2x \cdot 2x$$
$$= 2x^2 \cos 2x + 2x \sin 2x = 2x(x\cos 2x + \sin 2x)$$

7) Differentiate $f(x) = \frac{-2 - \ln x}{\sqrt{x}} + \mathfrak{e}^x \cos x$ with regard to x.



$$\frac{d\{f(x)\}}{dx} = \frac{d\left\{x^{-\frac{1}{2}}(-2-\ln x)+\mathfrak{e}^x\cos x\right\}}{dx}$$

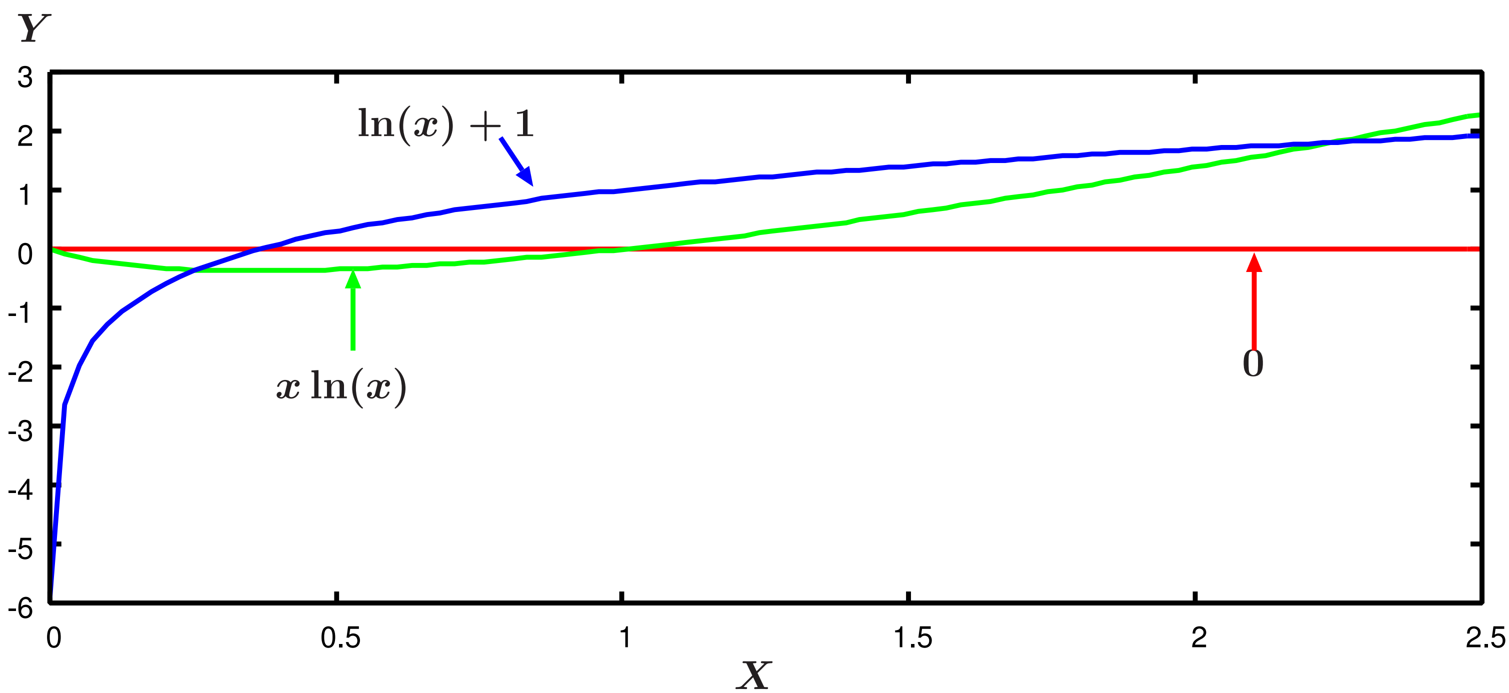
$$= \frac{d\left\{x^{-\frac{1}{2}}\right\}}{dx}(-2-\ln x) + x^{-\frac{1}{2}}\frac{d\{(-2-\ln x)\}}{dx} + \frac{d\{\mathfrak{e}^x\}}{dx}\cos x + \mathfrak{e}^x\frac{d\{\cos x\}}{dx}$$

$$= -\frac{1}{2}x^{-\frac{1}{2}-1}(-2-\ln x) + x^{-\frac{1}{2}}\left(0-\frac{1}{x}\right) + \mathfrak{e}^x\cos x - \mathfrak{e}^x\sin x$$

$$= -\frac{1}{2}x^{-\frac{3}{2}}(-2-\ln x) - x^{-\frac{1}{2}-1} + \mathfrak{e}^x\cos x - \mathfrak{e}^x\sin x$$

$$= x^{-\frac{3}{2}} + \frac{1}{2}x^{-\frac{3}{2}}\ln x - x^{-\frac{3}{2}} + \mathfrak{e}^x\cos x - \mathfrak{e}^x\sin x = \frac{1}{2}x^{-\frac{3}{2}}\ln x + \mathfrak{e}^x\cos x - \mathfrak{e}^x\sin x$$

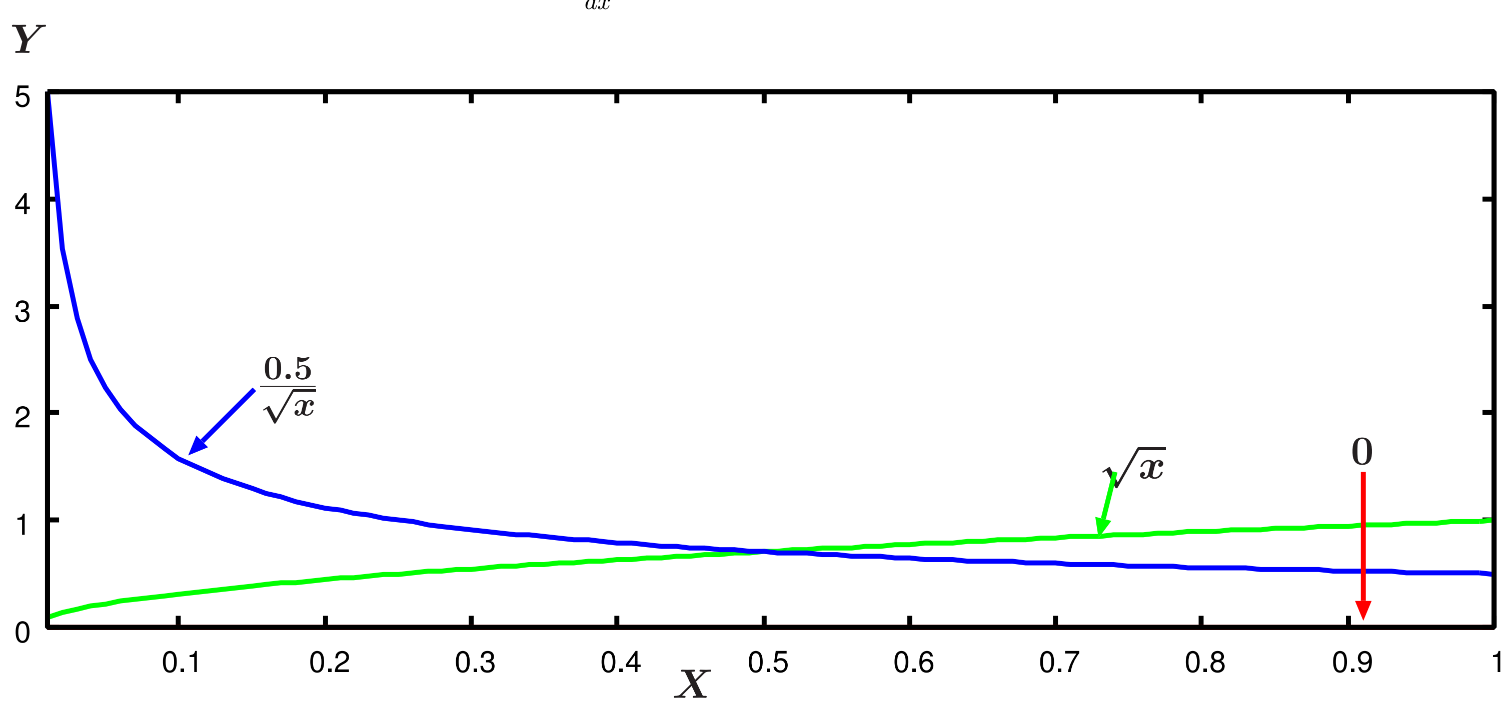
8) Differentiate $f(x) = x\ln x$ with regard to x and express $\frac{d\{f(x)\}}{dx}$ using $f(x)$ (*i.e.,* produce a differential equation)



$$\frac{d\{f(x)\}}{dx} = \frac{d\{x\}}{dx}\ln x + x\frac{d\{\ln x\}}{dx} = 1\cdot\ln x + x\frac{1}{x} = \ln x + 1$$

In order to produce a differential equation, we need to find $\ln x$ in terms of $f(x)$. Since $f(x) = x\ln x$ can be written as $\ln x = \frac{f(x)}{x}$, $\frac{d\{f(x)\}}{dx} = \ln x + 1 = \frac{f(x)}{x} + 1$. Therefore the differential equation is $\frac{d\{f(x)\}}{dx} - \frac{f(x)}{x} - 1 = 0$.

9) Differentiate $f(x) = \sqrt{x}$ with regard to x and express $\frac{d\{f(x)\}}{dx}$ using $f(x)$ (*i.e.,* produce a differential equation)

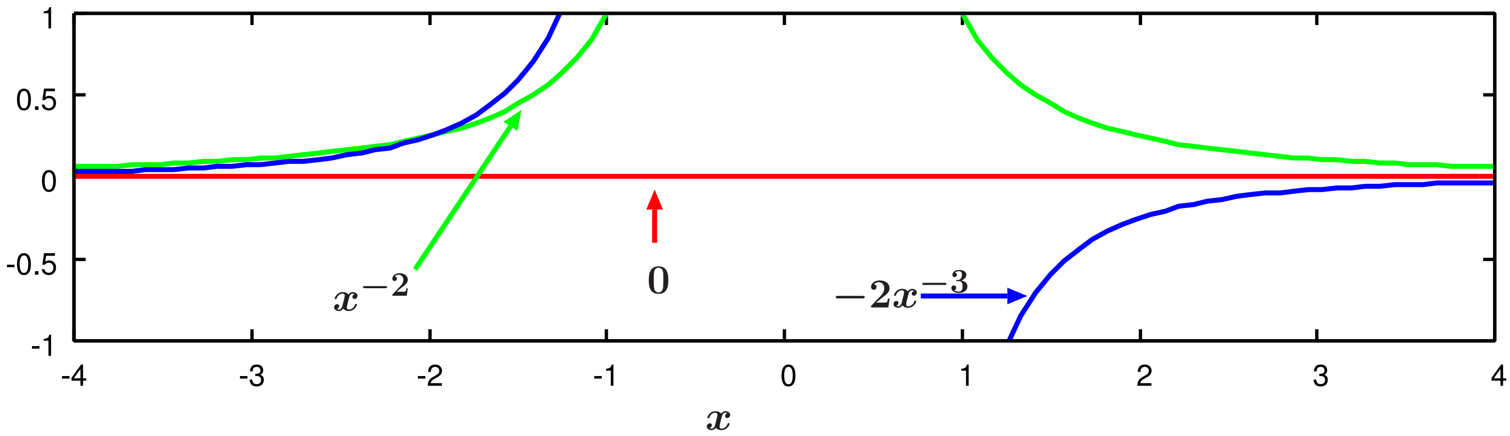


$$\frac{d\{f(x)\}}{dx} = \frac{d\left\{x^{\frac{1}{2}}\right\}}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} = \frac{1}{2f(x)}$$

Thus the differential equation is

$$\frac{d\left\{f(x)\right\}}{dx}=\frac{1}{2f(x)}\quad ;\quad \therefore 2f(x)\frac{d\left\{f(x)\right\}}{dx}=1$$

10) Differentiate $f(x)=x^{-2}$ with regard to x.

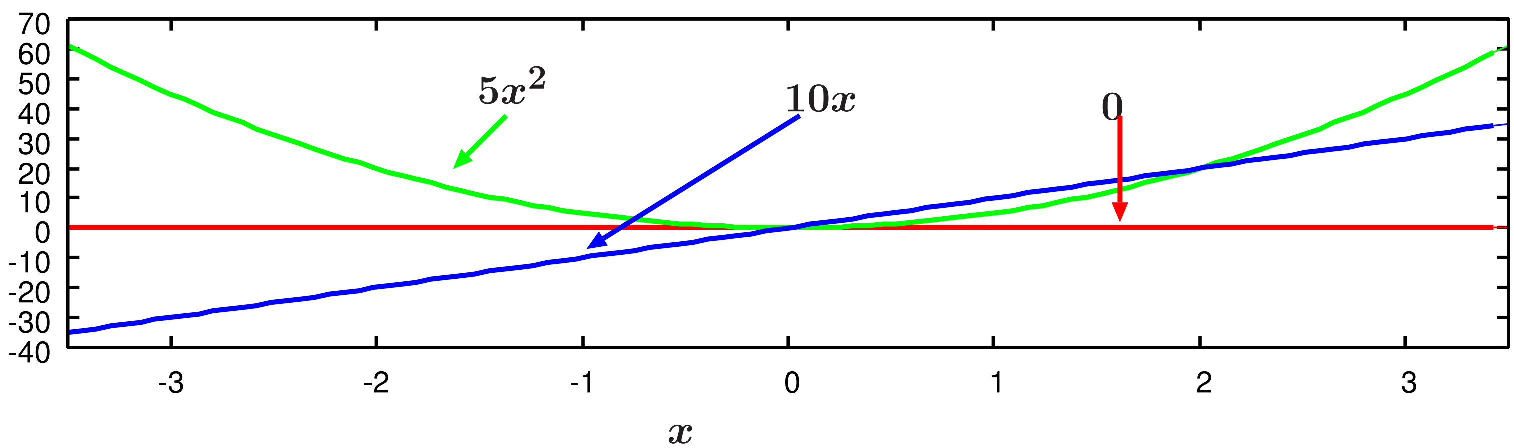


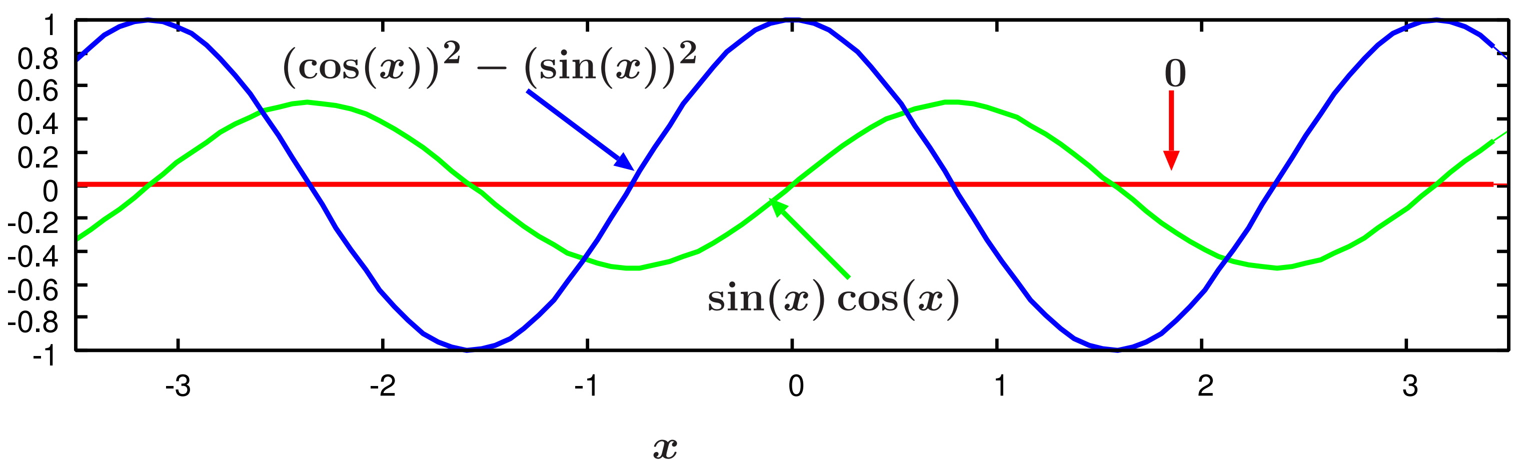
$$\frac{d\left\{f(x)\right\}}{dx}=\frac{d\left\{x^{-2}\right\}}{dx}=-2x^{-3}$$

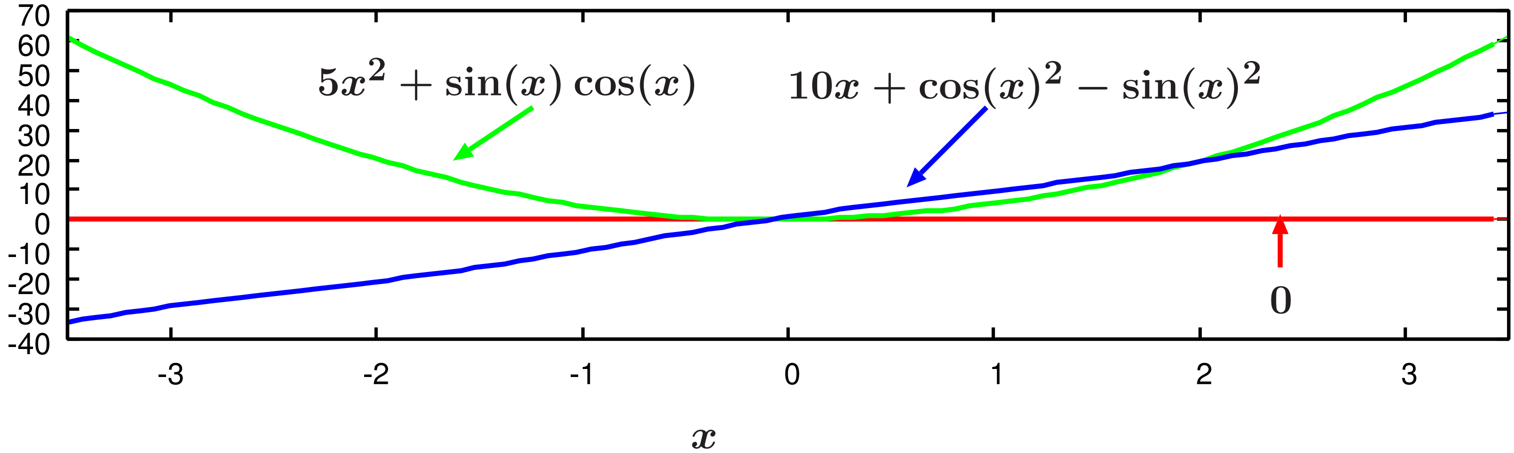
11) Differentiate $f(x)=x^{-2}\left(4+3x^{-3}\right)$ with regard to x.

$$\frac{d\left\{f(x)\right\}}{dx}=x^{-2}\frac{d\left\{4+3x^{-3}\right\}}{dx}+\frac{d\left\{x^{-2}\right\}}{dx}\left(4+3x^{-3}\right)=x^{-2}\left(-9x^{-4}\right)+\left(-2\,x^{-3}\right)\left(4+3x^{-3}\right)$$
$$=-9x^{-6}-8x^{-3}-6x^{-6}=-15x^{-6}-8x^{-3}$$

12) Differentiate $f(x)=5x^2+\sin x\cos x$ with regard to x.

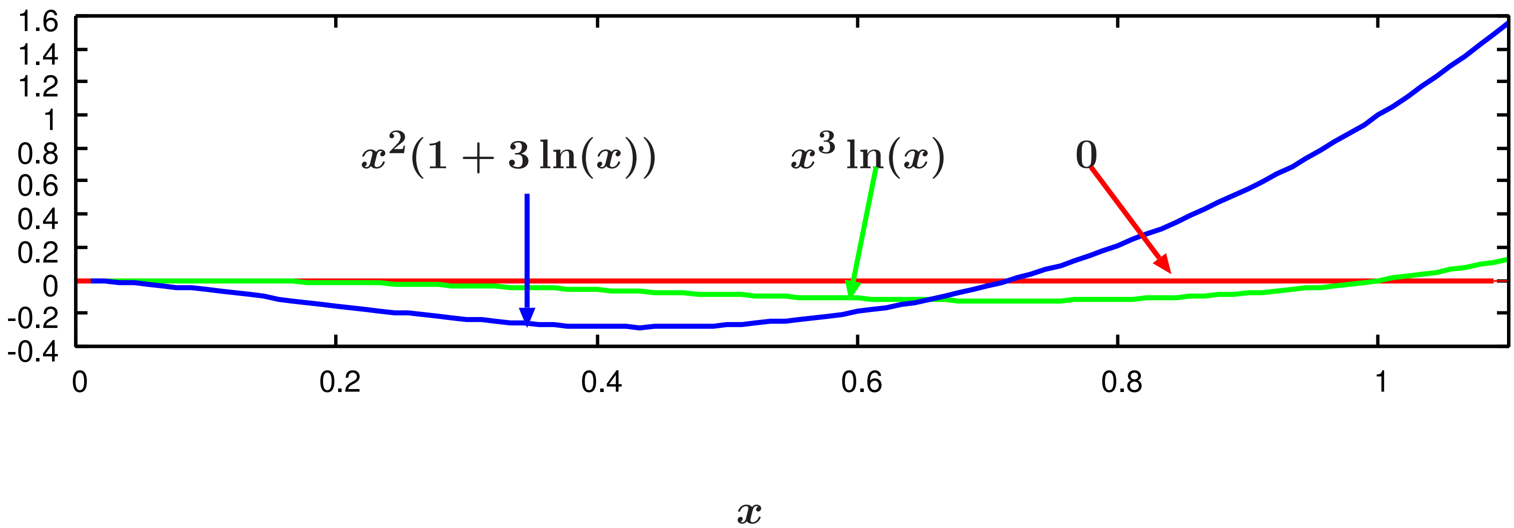






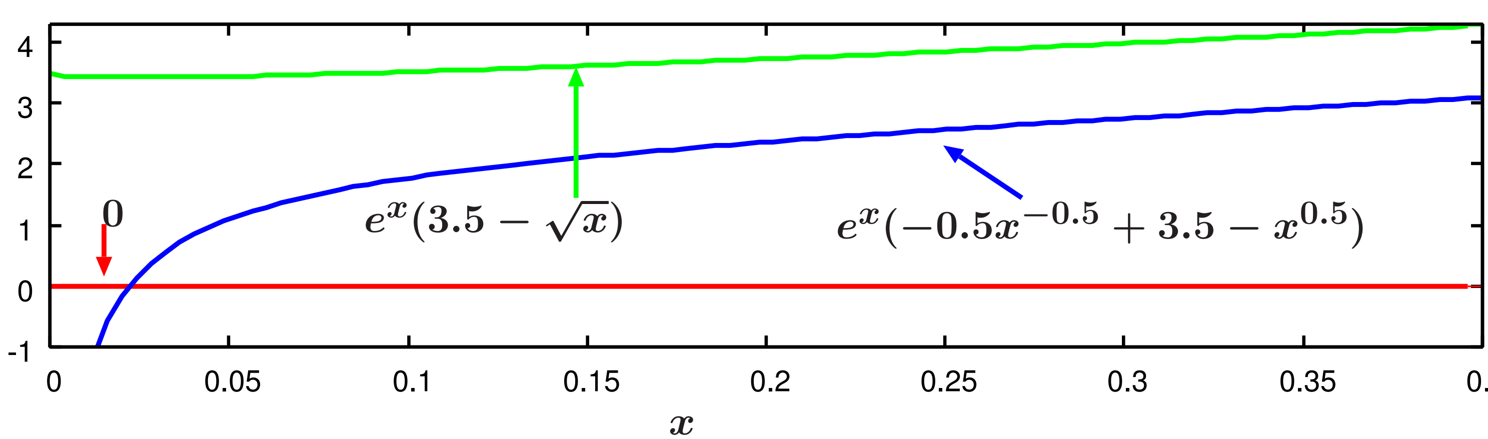
$$\frac{d\{f(x)\}}{dx} = \frac{d\{5x^2 + \sin x \cos x\}}{dx} = \frac{d\{5x^2\}}{dx} + \frac{d\{\sin x\}}{dx}\cos x + \sin x \frac{d\{\cos x\}}{dx}$$
$$= 10x + \cos x \cos x + \sin x\,(-\sin x) = 10x + \cos^2 x - \sin^2 x$$

13) Differentiate $f(x) = x^3 \ln x$ with regard to x.



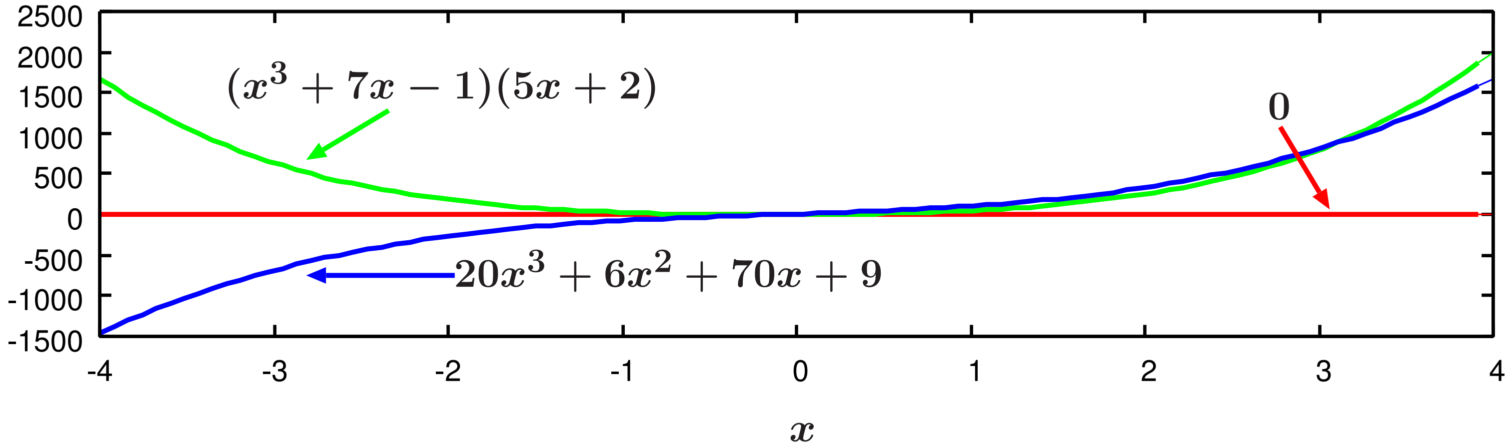
$$\frac{d\{f(x)\}}{dx} = x^3 \frac{d\{\ln x\}}{dx} + \frac{d\{x^3\}}{dx}\ln x = x^3 \frac{1}{x} + 3x^2 \ln x = x^2 + 3x^2 \ln x$$

14) Differentiate $f(x) = \mathfrak{e}^x\,(3.5 - \sqrt{x})$ with regard to x.



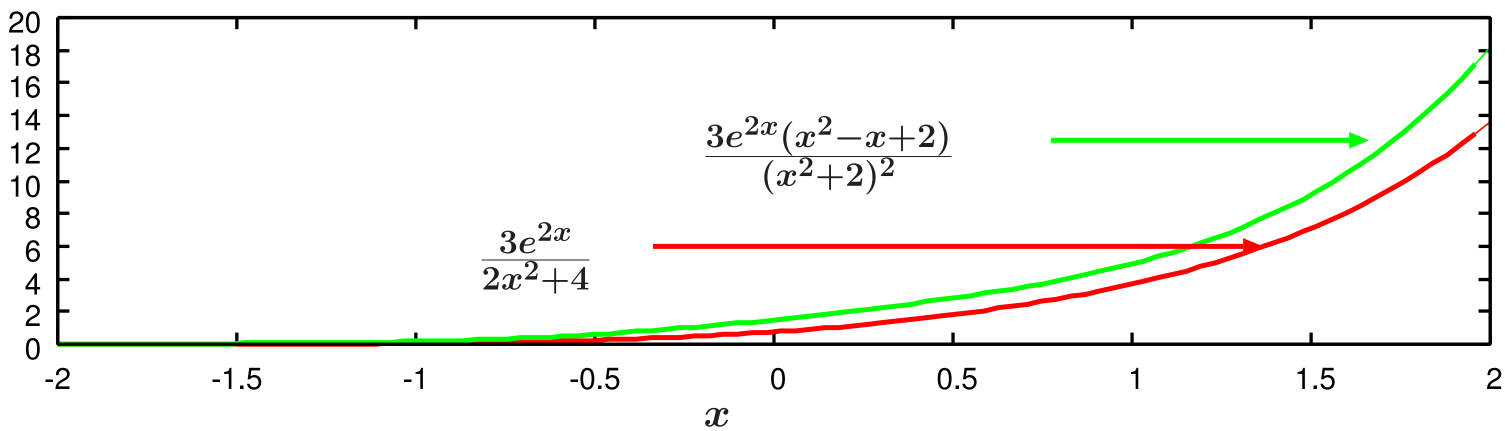
$$\frac{d\{f(x)\}}{dx} = \mathfrak{e}^x \frac{d\{3.5 - \sqrt{x}\}}{dx} + \frac{d\{\mathfrak{e}^x\}}{dx}(3.5 - \sqrt{x}) = \mathfrak{e}^x\left(-\frac{1}{2}x^{-\frac{1}{2}}\right) + \mathfrak{e}^x\,(3.5 - \sqrt{x}) = \mathfrak{e}^x\left(-\frac{1}{2}x^{-\frac{1}{2}} + 3.5 - \sqrt{x}\right)$$

15) Differentiate $f(x) = \left(x^3 + 7x - 1\right)(5x + 2)$ with respect to x.



$$\frac{d\{f(x)\}}{dx} = \left(x^3 + 7x - 1\right)\frac{d\{5x + 2\}}{dx} + \frac{d\left\{x^3 + 7x - 1\right\}}{dx}(5x + 2) = \left(x^3 + 7x - 1\right)(5) + \left(3x^2 + 7\right)(5x + 2)$$
$$= 5x^3 + 35x - 5 + 15x^3 + 35x + 6x^2 + 14 = 20x^3 + 6x^2 + 70x + 9$$

16) Find $\frac{d\{y\}}{dx}$ of $y = \frac{3e^{2x}}{(2x^2 + 4)}$. (Your answer may include y as well as x)



- Recommended approach

$$y = \frac{3e^{2x}}{(2x^2 + 4)}$$

When we take logarithm of the equation, we obtain

$$\ln y = \ln \frac{3e^{2x}}{(2x^2 + 4)} = \ln|3e^{2x}| - \ln|2x^2 + 4| = \ln 3 + \ln e^{2x} - \ln|2x^2 + 4| = \ln 3 + 2x - \ln|2x^2 + 4|$$

Now we differentiate the equation with respect to x as follows

$$\frac{d\{\ln y\}}{dx} = \frac{d\left\{\ln 3 + 2x - \ln|2x^2 + 4|\right\}}{dx} \quad ; \quad \therefore \frac{d\{y\}}{dx}\frac{d\{\ln y\}}{dy} = 2 - \frac{4x}{2x^2 + 4} \quad ; \quad \therefore \frac{d\{y\}}{dx}\frac{1}{y} = 2 - \frac{4x}{2x^2 + 4}$$
$$\therefore \frac{d\{y\}}{dx} = y\left(2 - \frac{4x}{2x^2 + 4}\right) \quad ; \quad \therefore \frac{d\{y\}}{dx} = y\left(\frac{2x^2 + 4 - 2x}{x^2 + 2}\right)$$

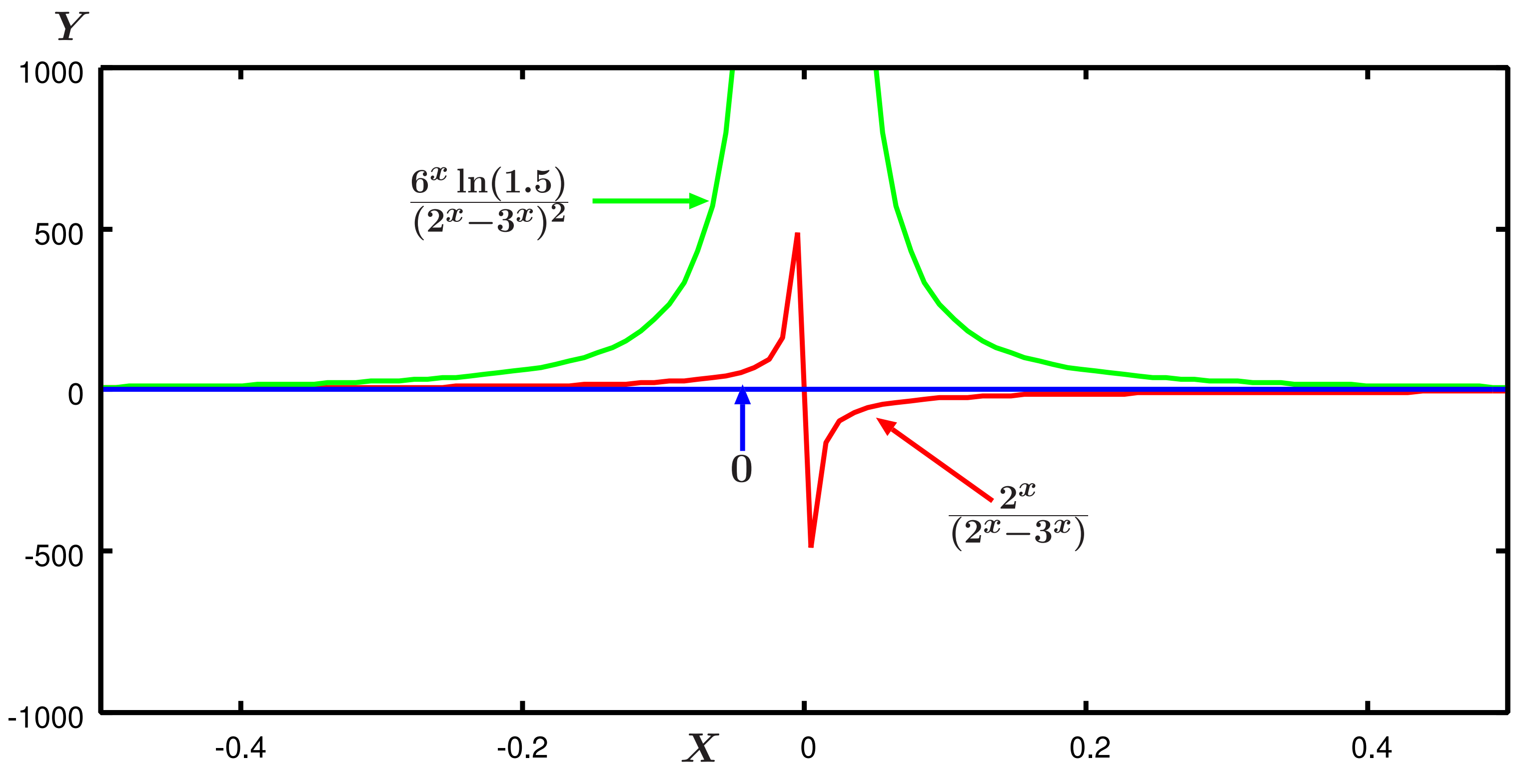
- Basic approach
The function y is fraction, therefore we must use the quotient rule. Splitting the function up in to $f(x)$ and $g(x)$. Letting $f(x) = 3e^{2x}$ and $g(x) = 2x^2 + 4$ because $y = \frac{f(x)}{g(x)}$.

$$f(x) = 3e^{2x} \quad ; \quad \therefore \frac{d\{f(x)\}}{dx} = 6e^{2x}$$
$$g(x) = 2x^2 + 4 \quad ; \quad \therefore \frac{d\{g(x)\}}{dx} = 4x$$

Now applying the quotient rule as followed (note the minus sign and the order of $f(x)$ and $g(x)$).

$$\frac{d\{y\}}{dx} = \frac{g(x)\cdot\frac{d\{f(x)\}}{dx} - f(x)\cdot\frac{d\{g(x)\}}{dx}}{g(x)^2} = \frac{(2x^2+4)\cdot(6e^{2x}) - (3e^{2x})\cdot(4x)}{(2x^2+4)^2}$$
$$= \frac{6(2x^2+4)e^{2x} - 4x(3e^{2x})}{(2x^2+4)^2} = \frac{12e^{2x}((x^2+2)-x)}{(2x^2+4)^2} = \frac{12e^{2x}(x^2-x+2)}{(2x^2+4)^2}$$
$$= \frac{12e^{2x}(x^2-x+2)}{4(x^2+2)^2} = \frac{3e^{2x}(x^2-x+2)}{(x^2+2)^2}$$

17) Differentiate $f(x) = y = \frac{2^x}{2^x - 3^x}$ with regard to x. (Your answer may include y as well as x)



- Recommended approach
  Let's take logarithm of the equation

$$\ln y = \ln|\frac{2^x}{2^x-3^x}| = \ln 2^x - \ln|2^x - 3^x| = x\ln 2 - \ln|2^x - 3^x|$$

Then let's differentiate the equation with respect to x as follows

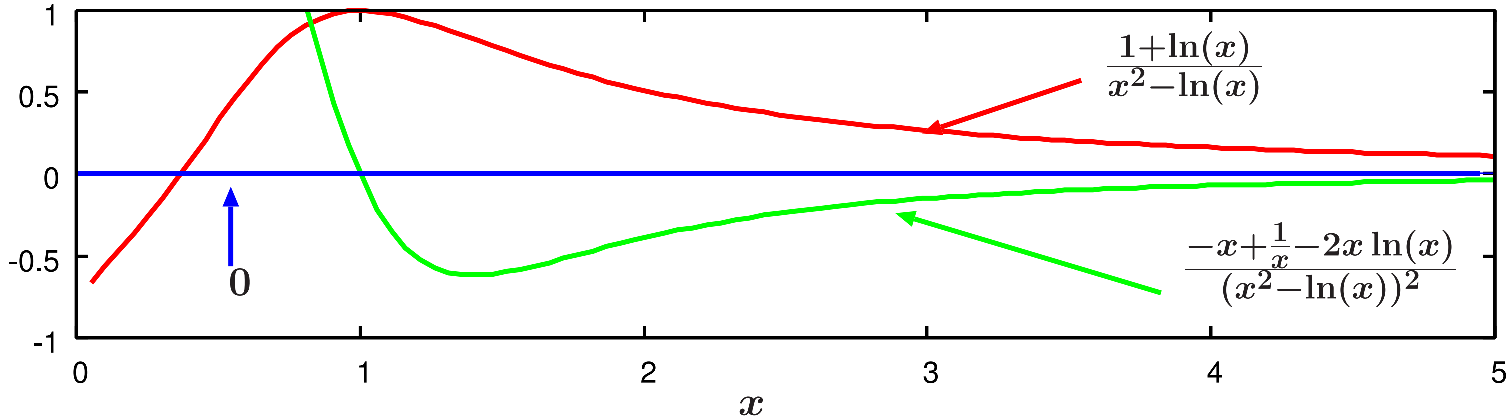
$$\frac{d\{\ln y\}}{dx} = \frac{d\{x\ln 2 - \ln|2^x-3^x|\}}{dx}$$
$$\therefore \frac{d\{y\}}{dx}\frac{d\{\ln y\}}{dy} = \frac{d\{y\}}{dx}\cdot\frac{1}{y} = \ln 2 - \frac{\frac{d\{2^x-3^x\}}{dx}}{2^x-3^x} = \ln 2 - \frac{2^x\ln 2 - 3^x \ln 3}{2^x - 3^x}$$
$$= \frac{2^x\ln 2 - 3^x\ln 2 - 2^x\ln 2 + 3^x\ln 3}{2^x-3^x} = \frac{-3^x\ln 2 + 3^x \ln 3}{2^x-3^x} = \frac{3^x\ln(3/2)}{2^x-3^x}$$
$$\therefore \frac{d\{y\}}{dx} = y\frac{3^x\ln(3/2)}{2^x-3^x} = \frac{2^x}{2^x-3^x}\cdot\frac{3^x\ln(3/2)}{2^x-3^x} = \frac{(2\cdot 3)^x\ln(3/2)}{(2^x-3^x)^2} = \frac{6^x\ln(3/2)}{(2^x-3^x)^2}$$

- Basic approach

$$\frac{d\{f(x)\}}{dx} = \frac{\frac{d\{2^x\}}{dx}(2^x-3^x) - 2^x\frac{d\{2^x-3^x\}}{dx}}{(2^x-3^x)^2} = \frac{(2^x\ln 2)(2^x-3^x) - 2^x(2^x\ln 2 - 3^x\ln 3)}{(2^x-3^x)^2}$$
$$= \frac{2^{2x}\ln 2 - 2^x3^x\ln 2 - 2^{2x}\ln 2 + 2^x 3^x \ln 3}{(2^x-3^x)^2} = \frac{2^x3^x(\ln 3 - \ln 2)}{(2^x-3^x)^2} = \frac{6^x(\ln\frac{3}{2})}{(2^x-3^x)^2}$$

$\because \frac{d\{a^x\}}{dx} = a^x\ln a$, $a^pa^q = a^{p+q}$, $a^xb^x = (ab)^x$, $\ln a - \ln b = \ln\frac{a}{b}$.

18) Differentiate $f(x) = y = \frac{1+\ln x}{x^2 - \ln x}$ with regard to x. (Your answer may include y as well as x)



- Recommended approach
  Let's take logarithm of the equation as follows

$$\ln y = \ln |\frac{1+\ln x}{x^2 - \ln x}| = \ln|1+\ln x| - \ln|x^2 - \ln x|$$

Now let's differentiate the equation with respect to x

$$\frac{d\{\ln y\}}{dx} = \frac{d\{\ln|1+\ln x| - \ln|x^2-\ln x|\}}{dx}$$

$$\therefore \frac{d\{y\}}{dx}\frac{d\{\ln y\}}{dy} = \frac{d\{y\}}{dx}\cdot\frac{1}{y} = \frac{\frac{1}{x}}{1+\ln x} - \frac{2x - \frac{1}{x}}{x^2-\ln x} = \frac{1}{x}\left(\frac{1}{1+\ln x} - \frac{2x^2-1}{x^2-\ln x}\right)$$

$$= \frac{1}{x}\frac{x^2 - \ln x - (1+\ln x)(2x^2-1)}{(1+\ln x)(x^2-\ln x)} = \frac{1}{x}\frac{x^2-\ln x - (2x^2 - 1 + 2x^2\ln x - \ln x)}{(1+\ln x)(x^2-\ln x)}$$

$$= \frac{1}{x}\frac{x^2 - \ln x - 2x^2 + 1 - 2x^2\ln x + \ln x}{(1+\ln x)(x^2-\ln x)} = \frac{1}{x}\frac{-x^2+1-2x^2\ln x}{(1+\ln x)(x^2-\ln x)}$$

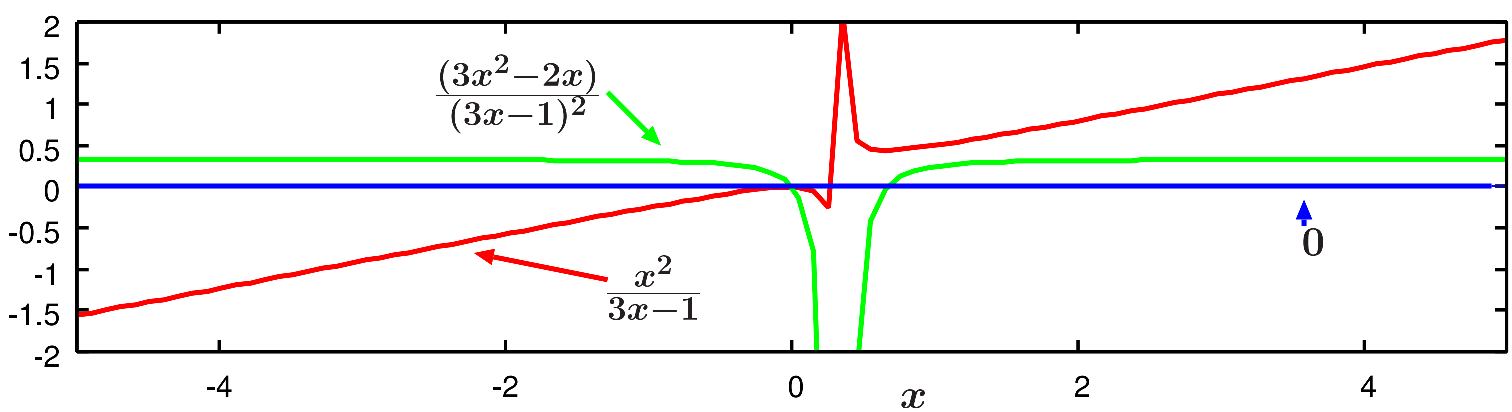
$$\therefore \frac{d\{y\}}{dx} = \frac{y}{x}\frac{-x^2+1-2x^2\ln x}{(1+\ln x)(x^2-\ln x)} = \frac{1+\ln x}{x^2-\ln x}\cdot\frac{1}{x}\frac{-x^2+1-2x^2\ln x}{(1+\ln x)(x^2-\ln x)} = \frac{-x^2+1-2x^2\ln x}{x(x^2-\ln x)^2}$$

- Basic approach

$$\frac{d\{f(x)\}}{dx} = \frac{\frac{d\{1+\ln x\}}{dx}(x^2-\ln x) - (1+\ln x)\frac{d\{x^2-\ln x\}}{dx}}{(x^2-\ln x)^2} = \frac{\frac{1}{x}(x^2-\ln x) - (1+\ln x)(2x - \frac{1}{x})}{(x^2-\ln x)^2}$$

$$= \frac{x - \frac{\ln x}{x} - 2x + \frac{1}{x} - 2x\ln x + \frac{\ln x}{x}}{(x^2-\ln x)^2} = \frac{-x + \frac{1}{x} - 2x\ln x}{(x^2-\ln x)^2}$$

19) Differentiate $y = \frac{x^2}{3x-1}$ (Your answer may include y as well as x)



- Recommended approach Let's take logarithm of the equation

$$\ln y = \ln\left|\frac{x^2}{3x-1}\right| = \ln|x^2| - \ln|3x-1| = 2\ln x - \ln|3x-1|$$

Now let's differentiate the equation with respect to x as follows

$$\frac{d\{\ln y\}}{dx} = \frac{d\{2\ln x - \ln|3x-1|\}}{dx} = \frac{2}{x} - \frac{3}{3x-1}$$

$$\therefore \frac{d\{y\}}{dx}\frac{d\{\ln y\}}{dy}\frac{d\{y\}}{dx}\cdot\frac{1}{y} = \frac{2(3x-1)-3x}{x(3x-1)} = \frac{6x-2-3x}{x(3x-1)} = \frac{3x-2}{x(3x-1)}$$

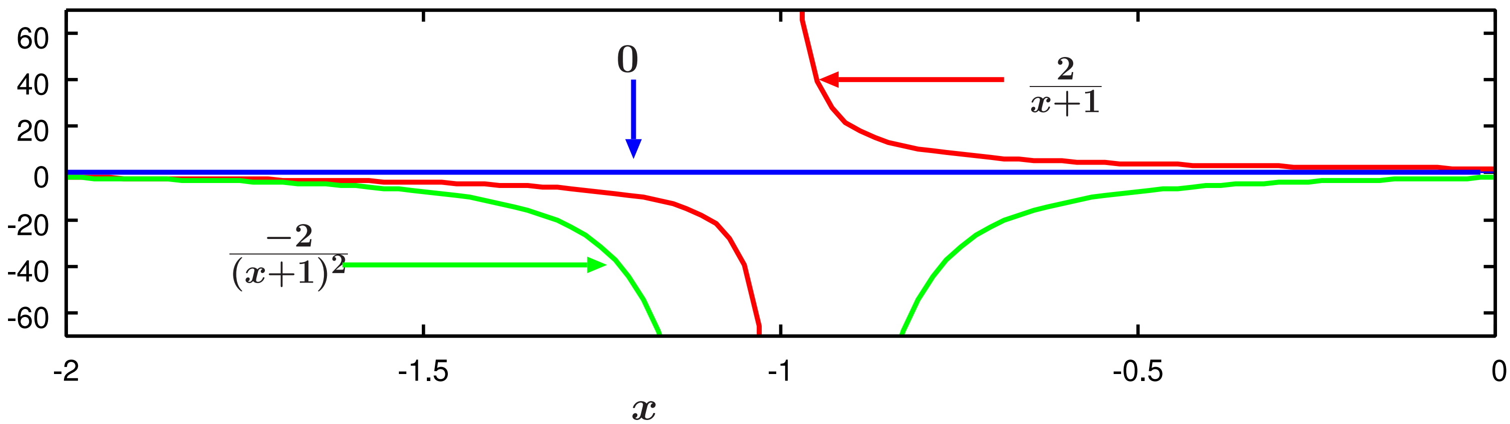
$$\therefore \frac{d\{y\}}{dx} = \frac{y(3x-2)}{x(3x-1)}$$

- Basic approach

$$\frac{d\{y\}}{dx} = \frac{d\left\{\frac{x^2}{3x-1}\right\}}{dx} = \frac{(3x-1)\frac{d\{x^2\}}{dx} - \frac{d\{(3x-1)\}}{dx}x^2}{(3x-1)^2} = \frac{(3x-1)\cdot(2x)-3x^2}{(3x-1)^2}$$

$$= \frac{6x^2-2x-3x^2}{(3x-1)^2} = \frac{3x^2-2x}{(3x-1)^2}$$

20) Differentiate $f(x) = \frac{2}{x+1}$ with regard to x and express $\frac{d\{f(x)\}}{dx}$ using $f(x)$ (*i.e.,* produce a differential equation).

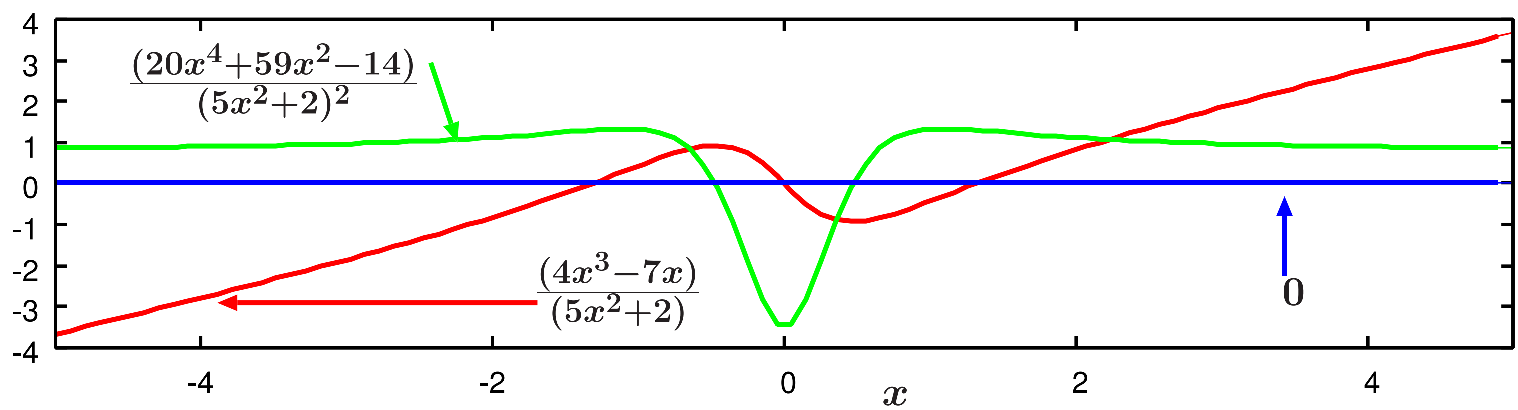


$$\frac{d\{f(x)\}}{dx} = \frac{d\left\{\frac{2}{x+1}\right\}}{dx} = \frac{\frac{d\{2\}}{dx}\cdot(x+1) - 2\cdot\frac{d\{x+1\}}{dx}}{(x+1)^2} = \frac{0\cdot(x+1)-2\cdot 1}{(x+1)^2} = \frac{-2}{(x+1)^2}$$

We now have to express $(x+1)^2$ using $f(x)$. Since $f(x) = \frac{2}{x+1}$ can be re-written as $(f(x))^2 = \frac{2^2}{(x+1)^2}$ or $\frac{1}{4}(f(x))^2 = \frac{1}{(x+1)^2}$, we get $\frac{d\{f(x)\}}{dx} = \frac{-2}{(x+1)^2} = -2\frac{1}{4}(f(x))^2$. Thus the differential equation is

$$\frac{d\{f(x)\}}{dx} = -\frac{1}{2}(f(x))^2 \quad ; \quad \therefore 2\frac{d\{f(x)\}}{dx} = -(f(x))^2 \quad ; \quad \therefore 2\frac{d\{f(x)\}}{dx} + (f(x))^2 = 0$$

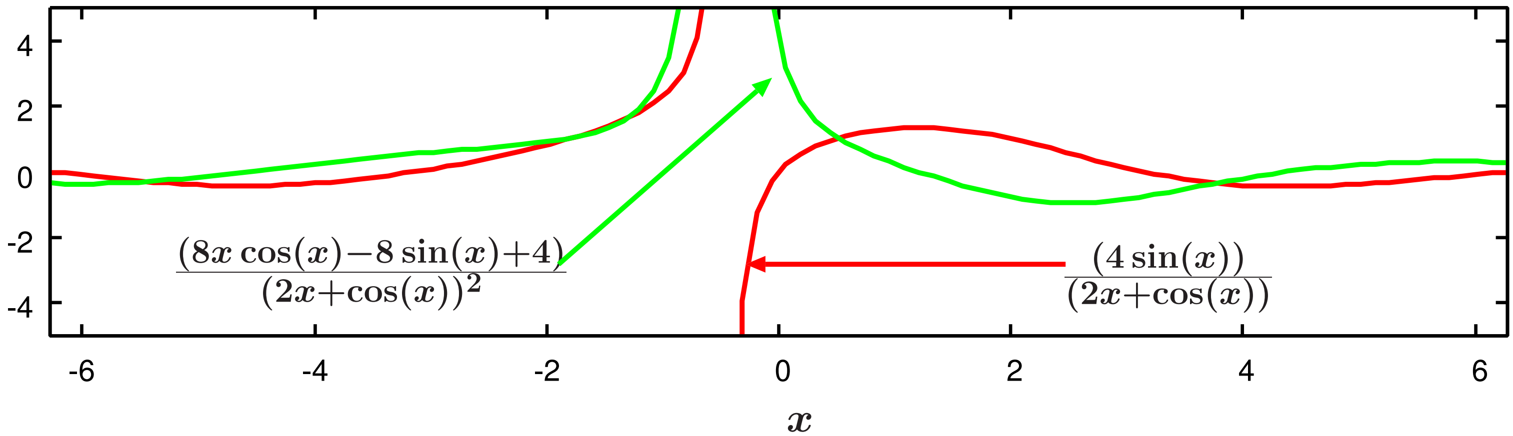
21) Differentiate $f(x) = \frac{4x^3-7x}{5x^2+2}$ with regard to x.



$$\frac{d\{f(x)\}}{dx} = \frac{d\left\{\dfrac{4x^3-7x}{5x^2+2}\right\}}{dx} = \frac{\dfrac{d\{4x^3-7x\}}{dx}(5x^2+2)-(4x^3-7x)\dfrac{d\{5x^2+2\}}{dx}}{(5x^2+2)^2}$$

$$= \frac{(12x^2-7)(5x^2+2)-(4x^3-7x)\cdot(10x)}{(5x^2+2)^2} = \frac{60x^4-35x^2+24x^2-14-40x^4+70x^2}{(5x^2+2)^2} = \frac{20x^4+59x^2-14}{(5x^2+2)^2}$$

22) Differentiate $f(x) = y = \frac{4\sin x}{2x+\cos x}$ with regard to x. (Your answer may include y as well as x)



- Recommended approach Let's take logarithm of the equation

$$\ln y = \ln\left|\frac{4\sin x}{2x+\cos x}\right| = \ln|4\sin x| - \ln|2x+\cos x| = \ln 4 + \ln|\sin x| - \ln|2x+\cos x|$$

Now let's differentiate the equation w.r.t. x

$$\frac{d\{\ln y\}}{dx} = \frac{d\{\ln 4 + \ln|\sin x| - \ln|2x+\cos x|\}}{dx} = \frac{\cos x}{\sin x} - \frac{2-\sin x}{2x+\cos x}$$

$$\therefore \frac{d\{y\}}{dx}\frac{d\{\ln y\}}{dy} = \frac{d\{y\}}{dx}\frac{1}{y} = \frac{(2x+\cos x)\cos x - \sin x(2-\sin x)}{\sin x(2x+\cos x)} = \frac{2x\cos x+\cos^2 x-2\sin x+\sin^2 x}{\sin x(2x+\cos x)}$$

$$= \frac{2x\cos x-2\sin x+1}{\sin x(2x+\cos x)} \;;\quad \therefore \frac{d\{y\}}{dx} = \frac{y(2x\cos x-2\sin x+1)}{\sin x(2x+\cos x)}$$

- Basic approach

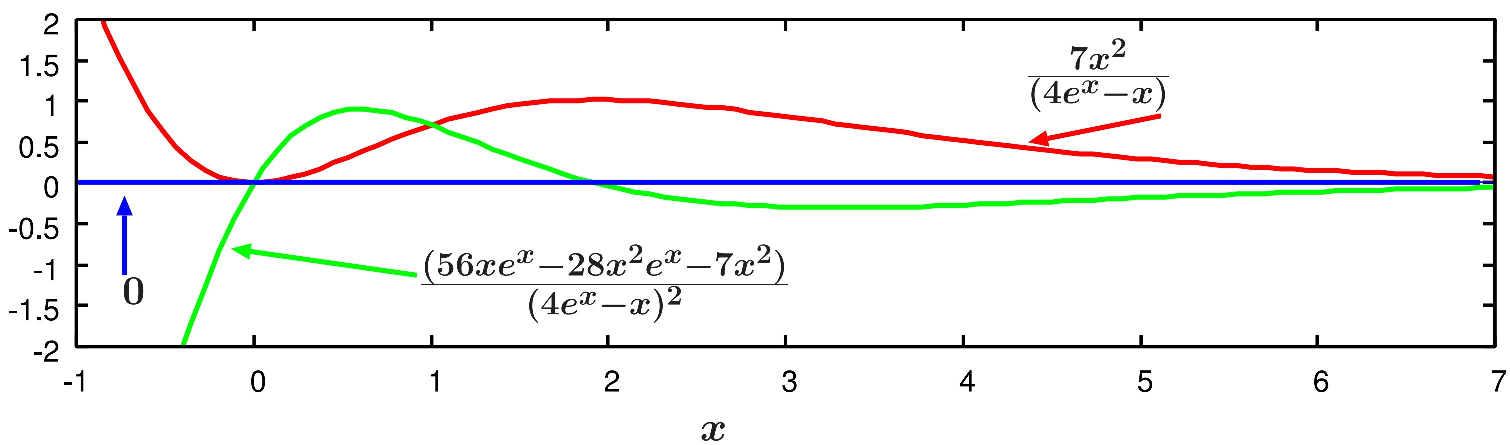
$$\frac{d\{f(x)\}}{dx} = \frac{d\left\{\dfrac{4\sin x}{2x+\cos x}\right\}}{dx} = \frac{\dfrac{d\{4\sin x\}}{dx}(2x+\cos x)-(4\sin x)\dfrac{d\{2x+\cos x\}}{dx}}{(2x+\cos x)^2}$$

$$= \frac{(4\cos x)(2x+\cos x)-(4\sin x)(2-\sin x)}{(2x+\cos x)^2} = \frac{8x\cos x+4\cos^2 x-8\sin x+4\sin^2 x}{(2x+\cos x)^2}$$

$$= \frac{8x\cos x-8\sin x+4}{(2x+\cos x)^2}$$

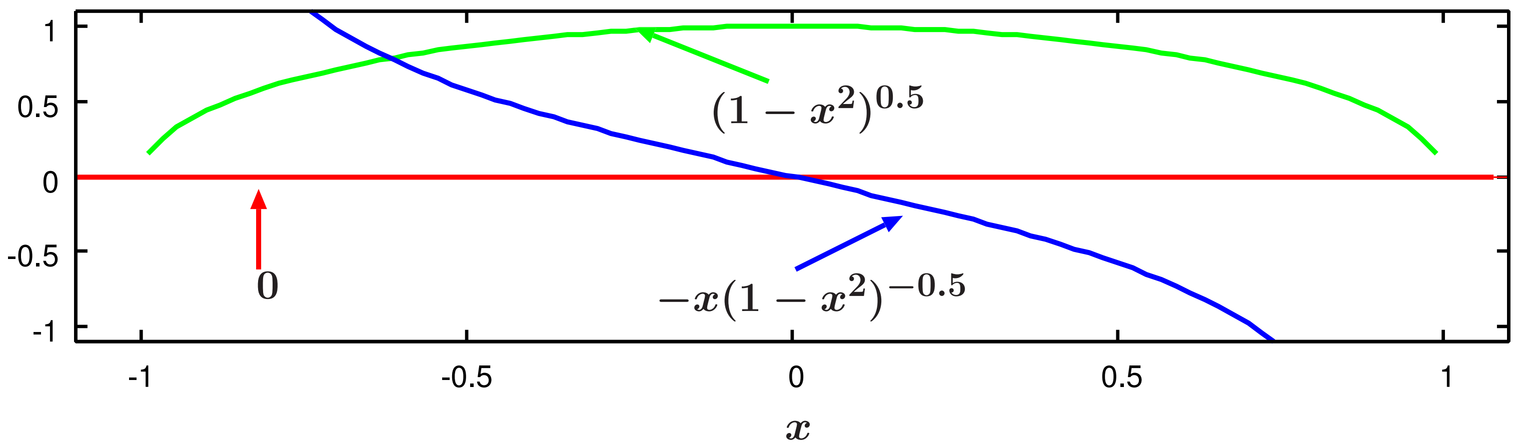
Note: the well-known trigonometry identity $\cos^2 x + \sin^2 x = 1$

23) Differentiate $f(x) = \frac{7x^2}{4e^x - x}$ with regard to x.



$$\frac{d\left\{\frac{7x^2}{4e^x - x}\right\}}{dx} = \frac{\frac{d\{7x^2\}}{dx}(4e^x - x) - 7x^2\frac{d\{4e^x - x\}}{dx}}{(4e^x - x)^2} = \frac{(14x)(4e^x - x) - 7x^2(4e^x - 1)}{(4e^x - x)^2}$$
$$= \frac{56xe^x - 14x^2 - 28x^2e^x + 7x^2}{(4e^x - x)^2} = \frac{56xe^x - 28x^2e^x - 7x^2}{(4e^x - x)^2}$$

24) Differentiate $y = \sqrt{c^2 - x^2}$ and express $\frac{d\{y\}}{dx}$ using y and x (*i.e.,* produce a differential equation).



When $u \triangleq c^2 - x^2$,

$$\frac{d\{y\}}{dx} = \frac{d\left\{\sqrt{c^2 - x^2}\right\}}{dx} = \frac{d\{\sqrt{u}\}}{dx} = \frac{d\{u\}}{dx}\frac{\partial u^{\frac{1}{2}}}{\partial u} = \frac{d\{c^2 - x^2\}}{dx}\frac{\partial\left\{u^{\frac{1}{2}}\right\}}{\partial u} = -2x \cdot \frac{1}{2}u^{-\frac{1}{2}}$$
$$= -xu^{-\frac{1}{2}} = -x(c^2 - x^2)^{-\frac{1}{2}}$$

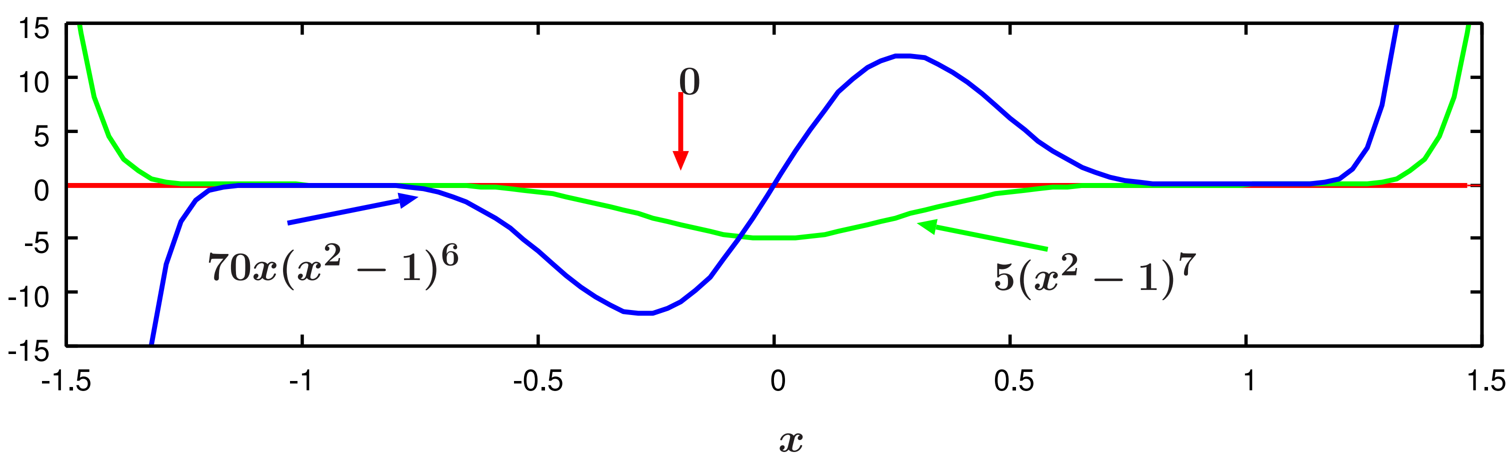
Since $y = u^{\frac{1}{2}}$, $y^{-1} = u^{-\frac{1}{2}}$. Therefore,

$$\frac{d\{y\}}{dx} = -xu^{-\frac{1}{2}} = -xy^{-1} = -\frac{x}{y}$$

This can be re-written as

$$\frac{d\{y\}}{dx} + \frac{x}{y} = 0 \ ; \quad \therefore y\frac{d\{y\}}{dx} + x = 0$$

25) Using the chain rule, find $\frac{d\{y\}}{dx}$ of $y = 5(x^2 - 1)^7$.



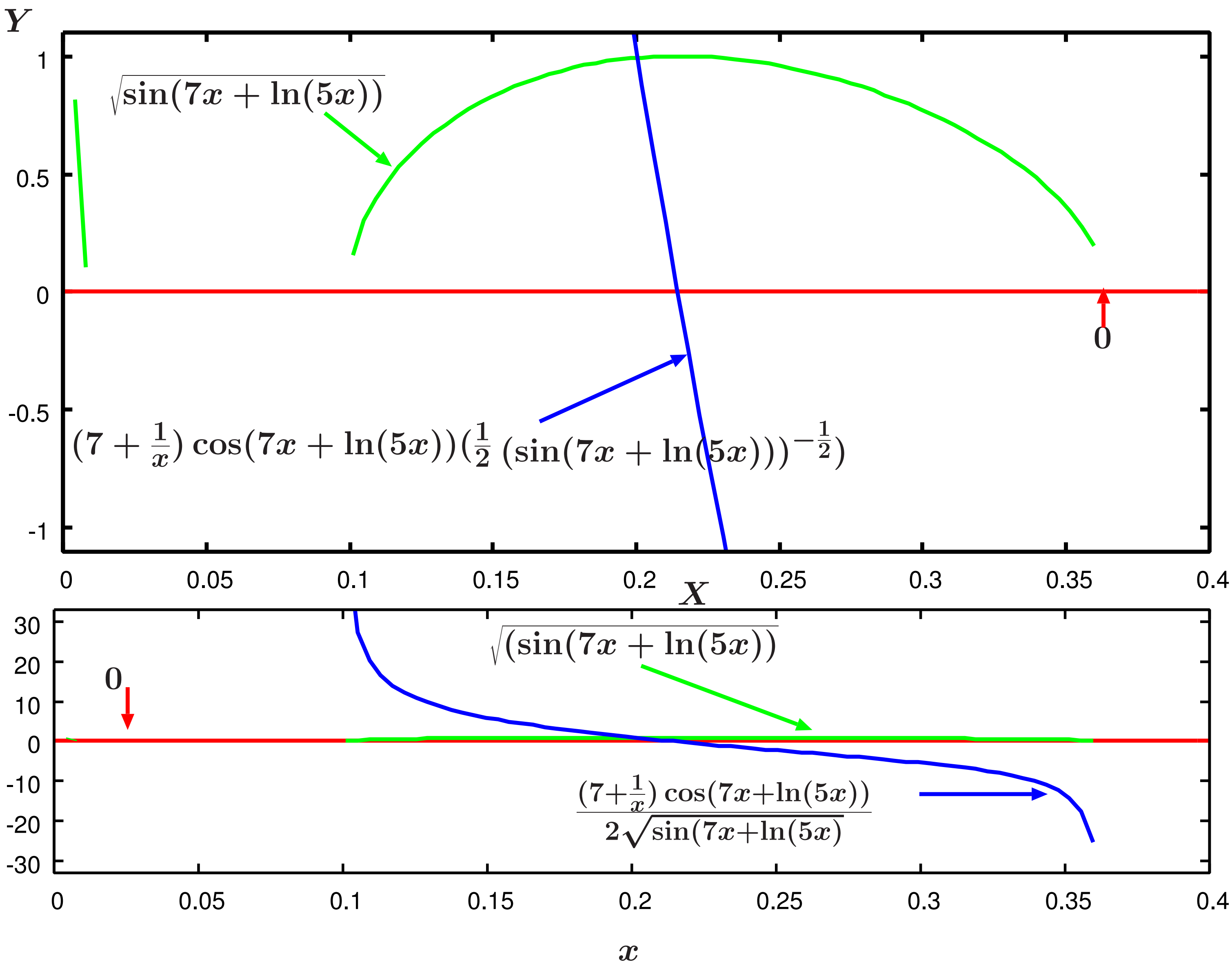
First let $u = x^2 - 1$. Therefore the function becomes $y = 5(u)^7$. Differentiate both of these equations.

$$u = x^2 - 1 \ ; \quad \therefore \frac{d\{u\}}{dx} = 2x$$
$$y = 5(u)^7 \ ; \quad \therefore \frac{\partial\{y\}}{\partial u} = 35(u)^6$$

Now using the chain rule formula below we can find $\frac{d\{y\}}{dx}$

$$\frac{d\{y\}}{dx} = \frac{\partial\{y\}}{\partial u} \cdot \frac{d\{u\}}{dx} = 2x \cdot 35(u)^6 = 70x(x^2-1)^6 \because u = x^2 - 1$$

26) Differentiate $f(x) = \sqrt{\sin(7x + \ln(5x))}$ with regard to x.



When we assume $u \triangleq 5x$, $v \triangleq 7x + \ln u$, $z \triangleq \sin v$, we can rewrite $f(x)$ as follows:

$$f(x) = \sqrt{\sin(7x + \ln(5x))} = \sqrt{\sin(7x + \ln(u))} = \sqrt{\sin(v)} = \sqrt{z}$$

Then we can start the differentiation,

$$\frac{d\left\{\sqrt{\sin(7x + \ln(5x))}\right\}}{dx} = \frac{d\{\sqrt{z}\}}{dx} = \frac{d\{z\}}{dx}\frac{d\{\sqrt{z}\}}{dz} = \frac{d\{\sin v\}}{dx}\frac{d\{\sqrt{z}\}}{dz} = \frac{d\{v\}}{dx}\frac{\partial\{\sin v\}}{\partial v}\frac{d\{\sqrt{z}\}}{dz}$$

$$= \frac{d\{7x + \ln u\}}{dx}\frac{\partial\{\sin v\}}{\partial v}\frac{d\{\sqrt{z}\}}{dz} = (\frac{d\{7x\}}{dx} + \frac{d\{\ln u\}}{dx})\frac{\partial\{\sin v\}}{\partial v}\frac{d\{\sqrt{z}\}}{dz}$$

$$= (\frac{d\{7x\}}{dx} + \frac{d\{u\}}{dx}\frac{\partial\{\ln u\}}{\partial u})\frac{\partial\{\sin v\}}{\partial v}\frac{d\{\sqrt{z}\}}{dz} = (7 + \frac{d\{5x\}}{dx}\frac{1}{u})\cos v(\frac{1}{2}z^{\frac{1}{2}-1})$$

$$= (7 + 5 \cdot \frac{1}{5x})\cos(7x + \ln(5x))(\frac{1}{2}(\sin v)^{-\frac{1}{2}}) = (7 + \frac{1}{x})\cos(7x + \ln(5x))(\frac{1}{2}(\sin(7x + \ln(5x)))^{-\frac{1}{2}})$$

Alternatively, $y = \sqrt{z}$,

$$\frac{d\{v\}}{dx} = 7 + \frac{\partial\{\ln(u)\}}{\partial u}\frac{d\{u\}}{dx} = 7 + \frac{1}{u}\cdot 5 = 7 + \frac{5}{u} \;\;;\;\; \frac{\partial\{z\}}{\partial v} = \cos(v) \;\;;\;\; \frac{d\{y\}}{dz} = \frac{1}{2}z^{-\frac{1}{2}}$$

$$\frac{d\{y\}}{dx} = \frac{d\{y\}}{dz}\frac{\partial\{z\}}{\partial v}\frac{d\{v\}}{dx} = \frac{1}{2}z^{-\frac{1}{2}}\cdot\cos(v)\cdot 7 + \frac{5}{u} = \frac{1}{2}(\sin(7x+\ln(5x)))^{-\frac{1}{2}}(\cos(7x+\ln(5x)))(7+\frac{1}{x})$$

27) Differentiate $f(x) = (1+\log_2(4x+1))^3$ with regard to x.

$$f(x) = (1+\log_2(4x+1))^3 \equiv v^3(\because v \triangleq 1+\log_2(4x+1)) \;\;;\;\; \therefore \frac{\partial\{f(x)\}}{\partial v} = \frac{\partial\{v^3\}}{\partial v} = 3v^2$$

$$\therefore \frac{d\{f(x)\}}{dx} = \frac{d\{v\}}{dx}\frac{\partial\{f(x)\}}{\partial v} = 3v^2\cdot\frac{d\{v\}}{dx}$$

Now we realize that we need $\frac{d\{v\}}{dx}$

$$v = 1+\log_2(4x+1) = 1+\frac{\log_{\mathfrak{e}}(4x+1)}{\log_{\mathfrak{e}} 2} \equiv 1+\frac{\log_{\mathfrak{e}} u}{\log_{\mathfrak{e}} 2}(\because u \triangleq 4x+1)$$

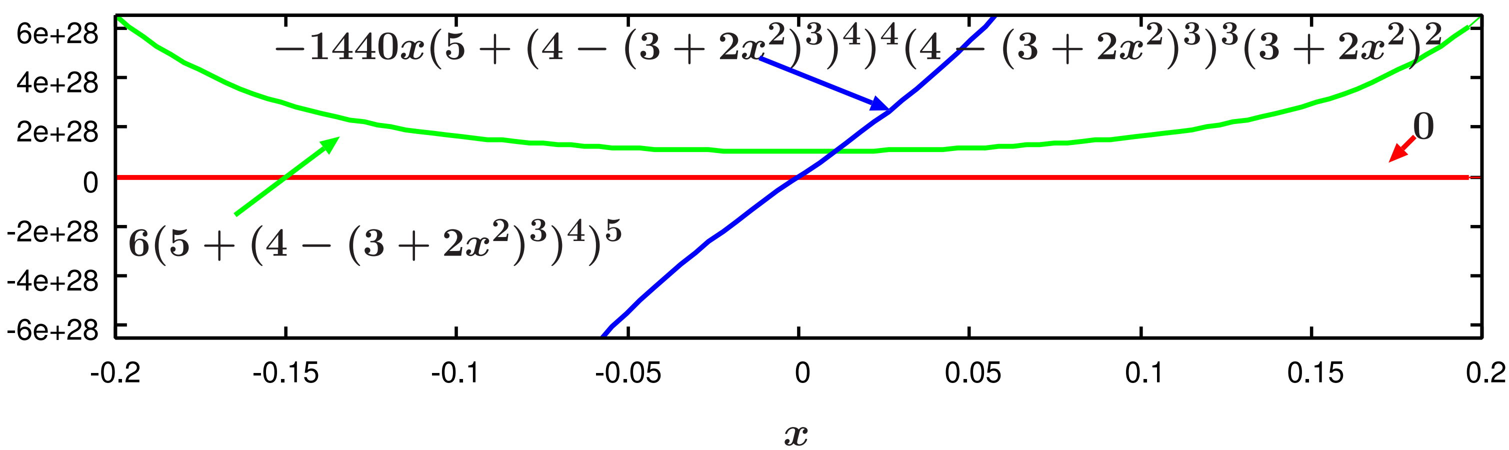
$$\therefore \frac{\partial\{v\}}{\partial u} = \frac{\partial\left\{1+\frac{\log_{\mathfrak{e}} u}{\log_{\mathfrak{e}} 2}\right\}}{\partial u} = \frac{1}{u\log_{\mathfrak{e}} 2}$$

$$\therefore \frac{d\{v\}}{dx} = \frac{d\{u\}}{dx}\frac{\partial\{v\}}{\partial u} = \frac{1}{u\log_{\mathfrak{e}} 2}\cdot\frac{d\{u\}}{dx} = \frac{1}{u\log_{\mathfrak{e}} 2}\frac{d\{4x+1\}}{dx} = \frac{4}{u\log_{\mathfrak{e}} 2}$$

Using $\frac{d\{v\}}{dx} = \frac{4}{u\log_{\mathfrak{e}} 2}$ and $\frac{d\{f(x)\}}{dx} = 3v^2\cdot\frac{d\{v\}}{dx}$, we obtain

$$\frac{d\{f(x)\}}{dx} = 3v^2\cdot\frac{d\{v\}}{dx} = 3v^2\cdot\frac{4}{u\log_{\mathfrak{e}} 2} = \frac{12v^2}{u\log_{\mathfrak{e}} 2} = \frac{12(1+\log_2(4x+1))^2}{(4x+1)\log_{\mathfrak{e}} 2}$$

28) Differentiate $f(x) = 6(5+(4-(3+2x^2)^3)^4)^5$ with regard to x.



We now define three variables u, v, z as follows: $u \triangleq 3+2x^2$, $v \triangleq 4-u^3$, $z \triangleq 5+v^4$. Then we can write

$$f(x) = 6(5+(4-(3+2x^2)^3)^4)^5 = 6(5+(4-(u)^3)^4)^5 = 6(5+(v)^4)^5 = 6(z)^5$$

Then now we differentiate $f(x)$ with respect to x.

$$\frac{d\{f(x)\}}{dx} = \frac{d\{6(5+(4-(3+2x^2)^3)^4)^5\}}{dx} = \frac{d\{6(z)^5\}}{dx} = \frac{d\{z\}}{dx}\frac{\partial\{6(z)^5\}}{\partial z} = \frac{d\{5+v^4\}}{dx}\frac{d\{6(z)^5\}}{dz}$$

$$= \frac{d\{v\}}{dx}\frac{\partial\{5+v^4\}}{\partial v}\frac{d\{6(z)^5\}}{dz} = \frac{d\{4-u^3\}}{dx}\frac{\partial\{5+v^4\}}{\partial v}\frac{d\{6(z)^5\}}{dz}$$

$$= \frac{d\{u\}}{dx}\frac{\partial\{4-u^3\}}{\partial u}\frac{\partial\{5+v^4\}}{\partial v}\frac{d\{6(z)^5\}}{dz} = \frac{d\{3+2x^2\}}{dx}\frac{\partial\{4-u^3\}}{\partial u}\frac{\partial\{5+v^4\}}{\partial v}\frac{d\{6(z)^5\}}{dz}$$

$$= (4x)(-3u^2)(4v^3)(30(z)^4) = (4x)(-3(3+2x^2)^2)(4(4-u^3)^3)(30(5+v^4)^4)$$

$$= (4x)(-3(3+2x^2)^2)\times(4(4-(3+2x^2)^3)^3)(30(5+(4-u^3)^4)^4)$$

$$= 4x(-3(3+2x^2)^2)(4(4-(3+2x^2)^3)^3)\times(30(5+(4-(3+2x^2)^3)^4)^4)$$

$$= -1440x(3+2x^2)^2\times(4-(3+2x^2)^3)^3(5+(4-(3+2x^2)^3)^4)^4$$

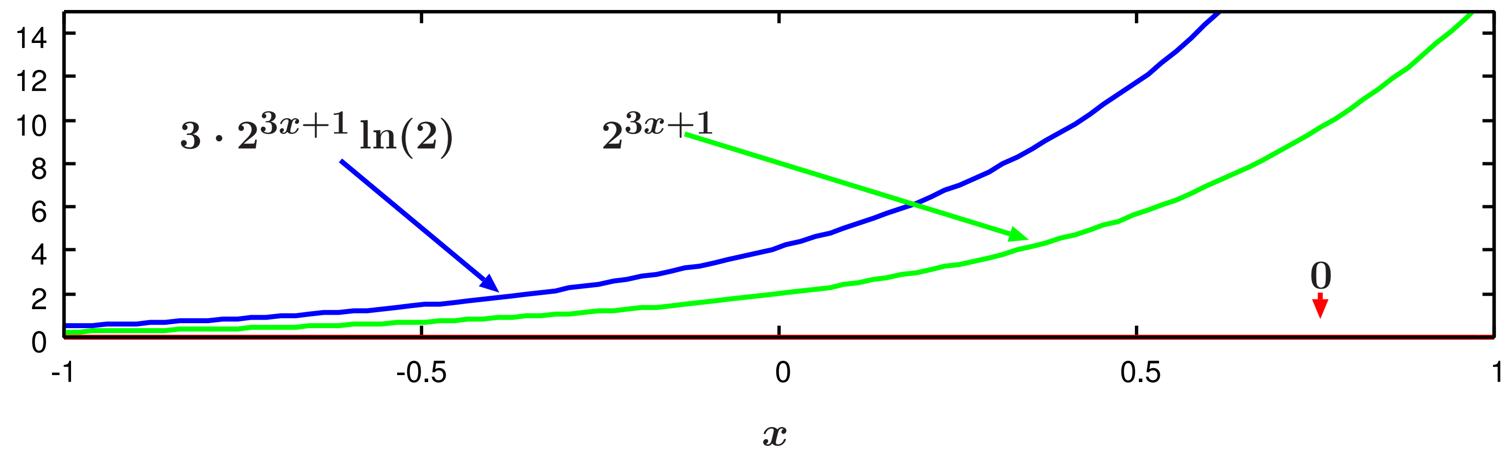
Alternatively, $y = 6z^5$.

$$\frac{d\{u\}}{dx} = 4x \;\;;\;\; \frac{\partial\{v\}}{\partial u} = -3u^2 \;\;;\;\; \frac{\partial\{z\}}{\partial v} = 4v^3 \;\;;\;\; \frac{d\{y\}}{dz} = 30z^4$$

Therefore,

$$\frac{d\{y\}}{dx} = \frac{d\{y\}}{dz}\frac{\partial\{z\}}{\partial v}\frac{\partial\{v\}}{\partial u}\frac{d\{u\}}{dx}$$
$$= 30(5+(4-(3+2x^2)^3)^4)^4(4(4-(3+3x^2)^3)^3)\times(-3(3+2x^2)^2)(4x)$$
$$= -1440x(5+(4-(3+2x^2)^3)^4)^4(4-(3+2x^2)^3)^3\times(3+2x^2)^2$$

29) Differentiate $y = 2^{3x+1}$ and express $\frac{d\{y\}}{dx}$ using y and x (*i.e.,* produce a differential equation).



We need $\frac{d\{2^{3x+1}\}}{dx}$. When $v \triangleq 3x+1$,

$$\frac{d\{2^{3x+1}\}}{dx} = \frac{d\{2^v\}}{dx} = \frac{d\{v\}}{dx}\frac{\partial\{2^v\}}{\partial v}.$$

We have to work out $\frac{\partial\{2^v\}}{\partial v}$. When $y \triangleq 2^v$ and apply the natural logarithm to both sides of the equation

$$y = 2^v$$

, we get

$$\ln(y) = \ln(2^v) = v\ln(2).$$

When we differentiate both sides of the equation

$$\ln(y) = v\ln(2),$$

with respect to v we get

$$\frac{\partial\{\ln(y)\}}{\partial v} = \frac{\partial\{v\ln(2)\}}{\partial v}.$$

$$\therefore \frac{\partial\{\ln(y)\}}{\partial v} = \frac{\partial\{y\}}{\partial v}\frac{\partial\{\ln(y)\}}{\partial y} = \frac{\partial\{v\ln(2)\}}{\partial v}.$$

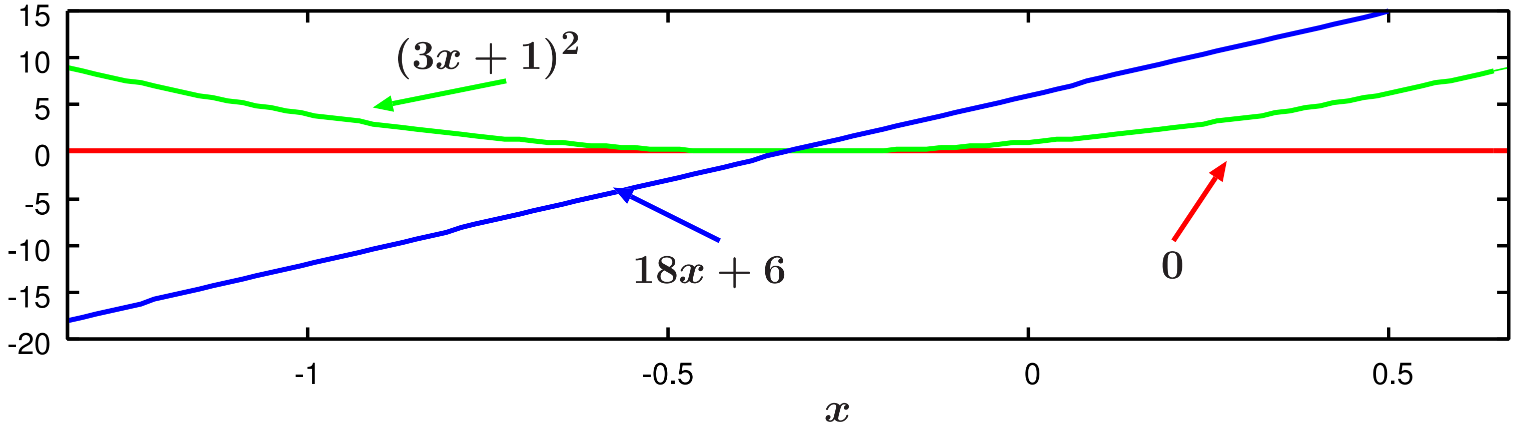
$$\therefore \frac{\partial\{y\}}{\partial v}\frac{1}{y} = \ln(2).$$

Thus

$$\frac{\partial\{y\}}{\partial v} = \frac{\partial\{2^v\}}{\partial v} = y\ln(2) = 2^v\ln(2).$$

Thus $\frac{d\{2^{3x+1}\}}{dx} = \frac{d\{v\}}{dx}\frac{\partial\{2^v\}}{\partial v} = \frac{d\{v\}}{dx}2^v\ln(2) = \frac{d\{3x+1\}}{dx}2^v\ln(2) = 3\cdot 2^v\ln(2) = 3\cdot 2^{3x+1}\ln(2)$. Therefore the differential equation is $\frac{d\{y\}}{dx} = 3\ln(2)y$.

30) Differentiate $f(x) = (3x+1)^2$ with regard to x and express express $\frac{d\{f(x)\}}{dx}$ using $f(x)$ (*i.e.,* produce a differential equation).



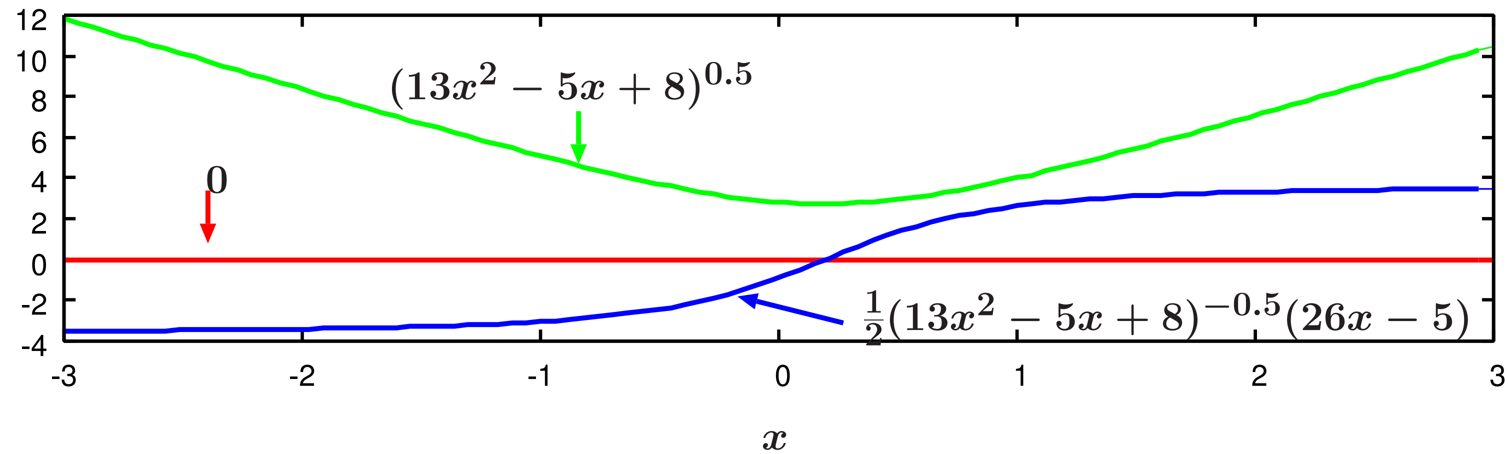
When $u \triangleq 3x+1$,

$$\frac{d\{f(x)\}}{dx} = \frac{d\{(3x+1)^2\}}{dx} = \frac{d\{(u)^2\}}{dx} = \frac{d\{u\}}{dx}\frac{\partial\{(u)^2\}}{\partial u} = \frac{d\{3x+1\}}{dx}\cdot 2u = 3\cdot 2(3x+1) = 6(3x+1)$$

Since $(\frac{d\{f(x)\}}{dx})^2 = (6(3x+1))^2 = 36(3x+1)^2 = 36f(x)$,

the differential equation is $\left(\frac{d\{f(x)\}}{dx}\right)^2 - 36f(x) = 0$.

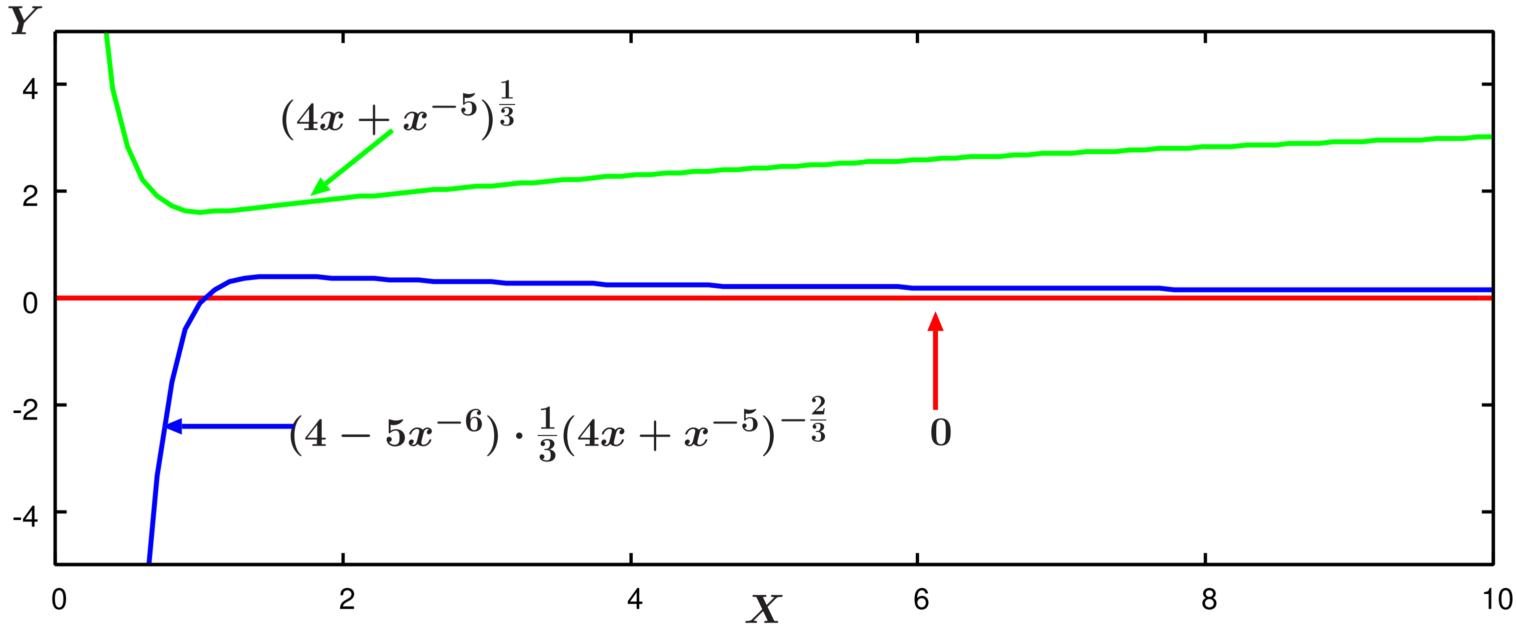
31) Differentiate $f(x) = \sqrt{13x^2-5x+8}$ with regard to x.



$u \triangleq 13x^2-5x+8$,

$$\frac{d\{f(x)\}}{dx} = \frac{d\left\{\sqrt{13x^2-5x+8}\right\}}{dx} = \frac{d\{\sqrt{u}\}}{dx} = \frac{d\{u\}}{dx}\frac{\partial\{\sqrt{u}\}}{\partial u}$$
$$= \frac{d\left\{13x^2-5x+8\right\}}{dx}\frac{1}{2}u^{-\frac{1}{2}} = (26x-5)\cdot\frac{1}{2}(13x^2-5x+8)^{-\frac{1}{2}}$$

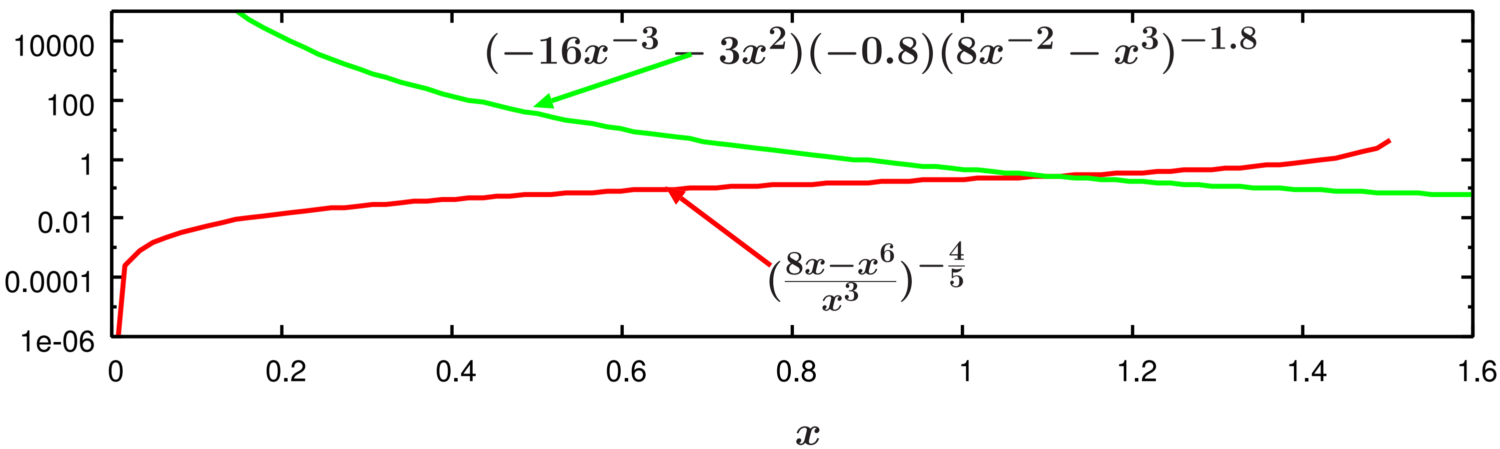
32) Differentiate $f(x) = (4x+x^{-5})^{\frac{1}{3}}$ with regard to x.



$u \triangleq 4x + x^{-5}$

$$\frac{d\{f(x)\}}{dx} = \frac{d\left\{(4x+x^{-5})^{\frac{1}{3}}\right\}}{dx} = \frac{d\left\{u^{\frac{1}{3}}\right\}}{dx} = \frac{d\{u\}}{dx}\frac{\partial\{u^{\frac{1}{3}}\}}{\partial u}$$
$$= \frac{d\{4x+x^{-5}\}}{dx}\frac{1}{3}u^{\frac{1}{3}-1} = (4-5x^{-6})\cdot\frac{1}{3}(4x+x^{-5})^{-\frac{2}{3}}$$

33) Differentiate $f(x) = (\frac{8x-x^6}{x^3})^{-\frac{4}{5}}$ with regard to x.

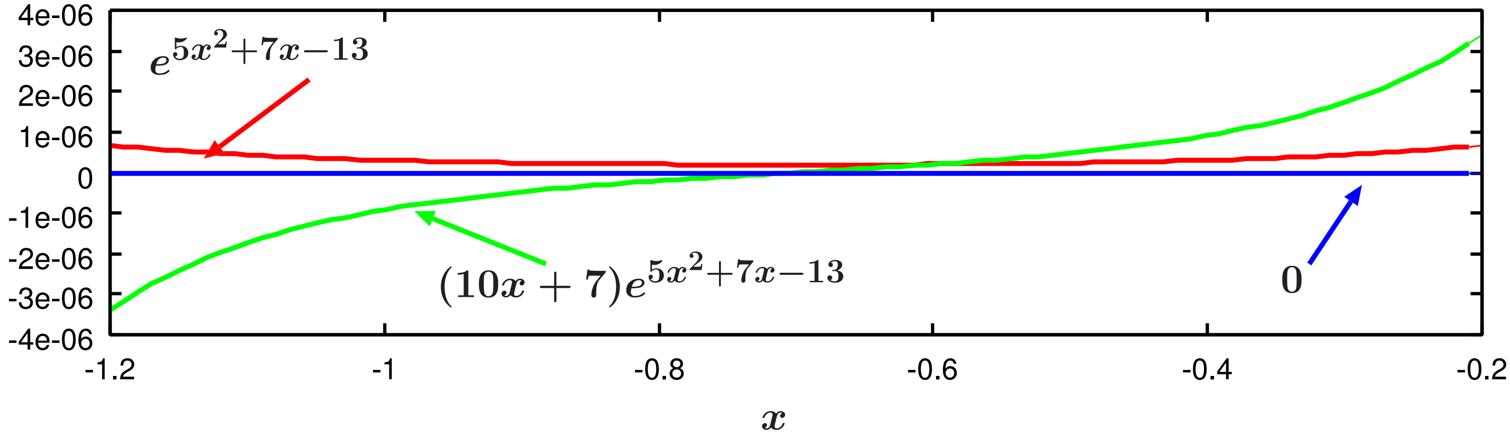


$$\frac{8x-x^6}{x^3} = \frac{8}{x^2} - x^3 = 8x^{-2} - x^3 \triangleq u$$

Then

$$\frac{d\{f(x)\}}{dx} = \frac{d\left\{(\frac{8x-x^6}{x^3})^{-\frac{4}{5}}\right\}}{dx} = \frac{d\left\{u^{-\frac{4}{5}}\right\}}{dx} = \frac{d\{u\}}{dx}\frac{\partial\left\{u^{-\frac{4}{5}}\right\}}{\partial u} = \frac{d\{8x^{-2}-x^3\}}{dx}\cdot(-\frac{4}{5})\cdot u^{-\frac{4}{5}-1}$$
$$= (-16x^{-3}-3x^2)\cdot(-\frac{4}{5})\cdot(8x^{-2}-x^3)^{-\frac{9}{5}}$$

34) Differentiate $f(x) = \mathfrak{e}^{5x^2+7x-13}$ with regard to x and express $\frac{d\{f(x)\}}{dx}$ using $f(x)$ and x (*i.e.,* produce a differential equation)

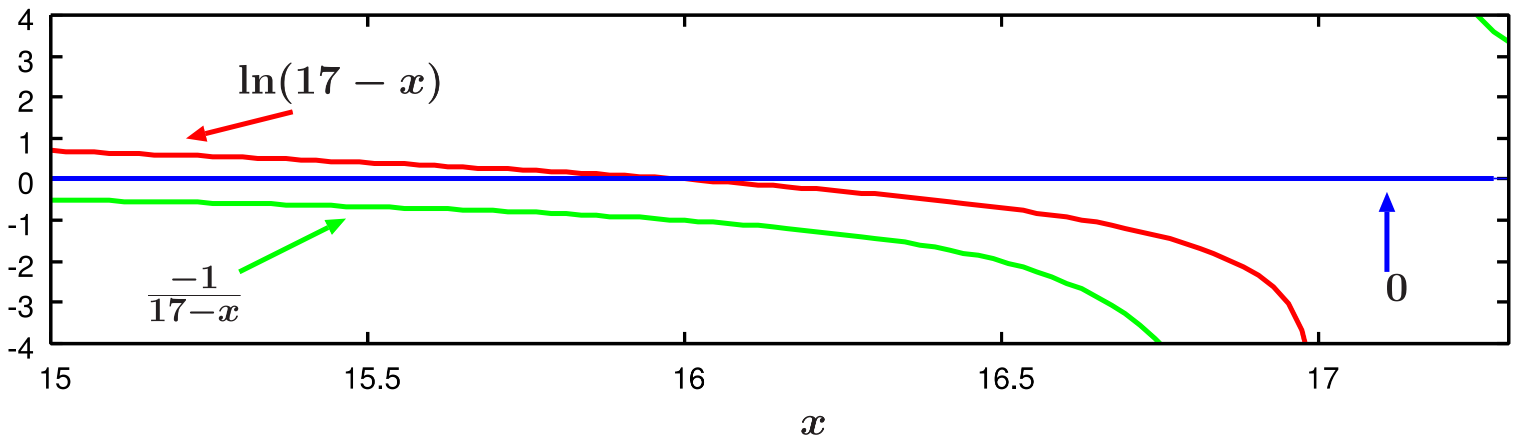


$u \triangleq 5x^2 + 7x - 13$

$$\frac{d\{f(x)\}}{dx} = \frac{d\left\{\mathfrak{e}^{5x^2+7x-13}\right\}}{dx} = \frac{d\{\mathfrak{e}^u\}}{dx} = \frac{d\{u\}}{dx}\frac{\partial\{\mathfrak{e}^u\}}{\partial u} = \frac{d\{5x^2+7x-13\}}{dx}\mathfrak{e}^u$$
$$= (10x+7)\mathfrak{e}^{5x^2+7x-13} = (10x+7)f(x)$$

Thus the differential equation is $\frac{d\{f(x)\}}{dx} - (10x+7)f(x) = 0$

35) Differentiate $f(x) = \ln(17-x)$ with regard to x and express $\frac{d\{f(x)\}}{dx}$ using $f(x)$ (*i.e.,* produce a differential equation).



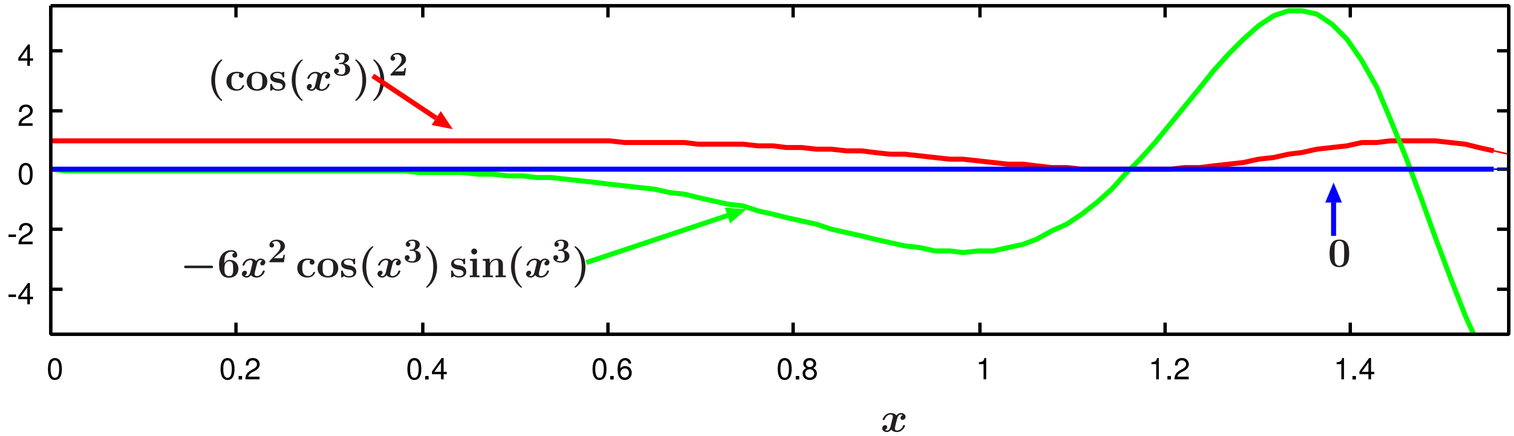
$u \triangleq 17 - x$

$$\frac{d\{f(x)\}}{dx} = \frac{d\{\ln(17-x)\}}{dx} = \frac{d\{\ln(u)\}}{dx} = \frac{d\{u\}}{dx}\frac{\partial\{\ln(u)\}}{\partial u} = \frac{d\{17-x\}}{dx}\frac{1}{u} = (-1)\frac{1}{17-x} = -\frac{1}{17-x}$$

We now have to express $17-x$ using $f(x)$. Since $f(x) = \ln \mathfrak{e}^{f(x)} = \ln(17-x)$, $\mathfrak{e}^{f(x)} = 17 - x$.
$\frac{d\{f(x)\}}{dx} = -\frac{1}{17-x}$ can be re-written as $(17-x)\frac{d\{f(x)\}}{dx} = -1$.

Therefore the differential equation is $\mathfrak{e}^{f(x)}\frac{d\{f(x)\}}{dx} + 1 = 0$.

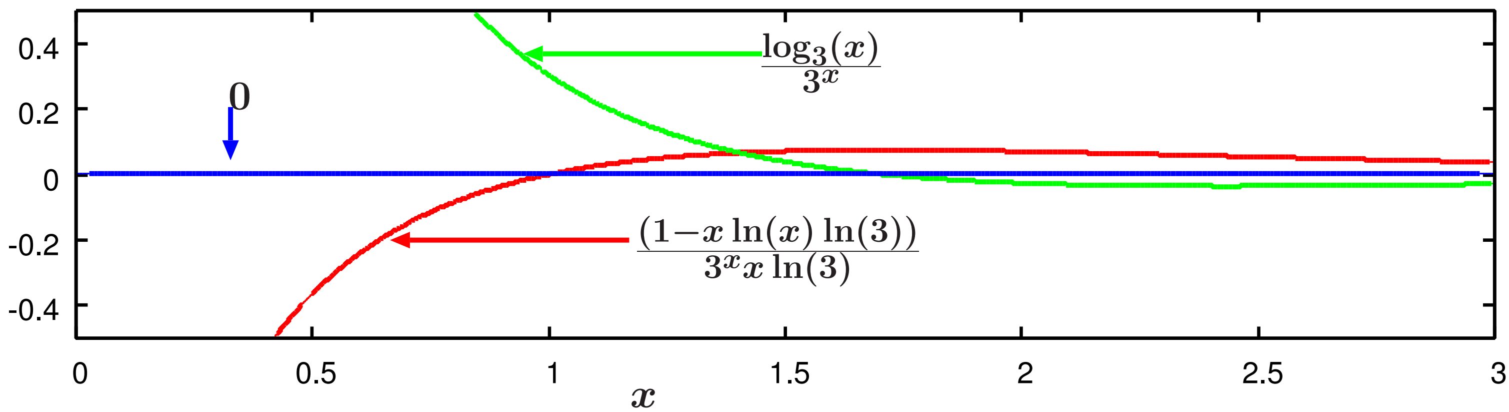
36) Differentiate $f(x) = \cos^2(x^3)$ with regard to x.



When we assume $u \triangleq x^3$, $v \triangleq \cos(u)$,

$$\begin{aligned}\frac{d\{f(x)\}}{dx} = \frac{d\{\cos^2(x^3)\}}{dx} = \frac{d\{\cos^2(u)\}}{dx} &= \frac{d\{u\}}{dx}\frac{\partial\{\cos^2(u)\}}{\partial u} = \frac{d\{x^3\}}{dx}\frac{\partial\{v^2\}}{\partial u} \\ &= \frac{d\{x^3\}}{dx}\frac{\partial\{v\}}{\partial u}\frac{\partial\{v^2\}}{\partial v} = \frac{d\{x^3\}}{dx}\frac{\partial\{\cos(u)\}}{\partial u}\frac{\partial\{v^2\}}{\partial v} \\ &= 3x^2(-\sin(u))(2v) = -6x^2\sin(x^3)\cos(u) = -6x^2\sin(x^3)\cos(x^3)\end{aligned}$$

37) Find $\frac{d\{y\}}{dx}$ of $y = \frac{\log_3 x}{3^x}$



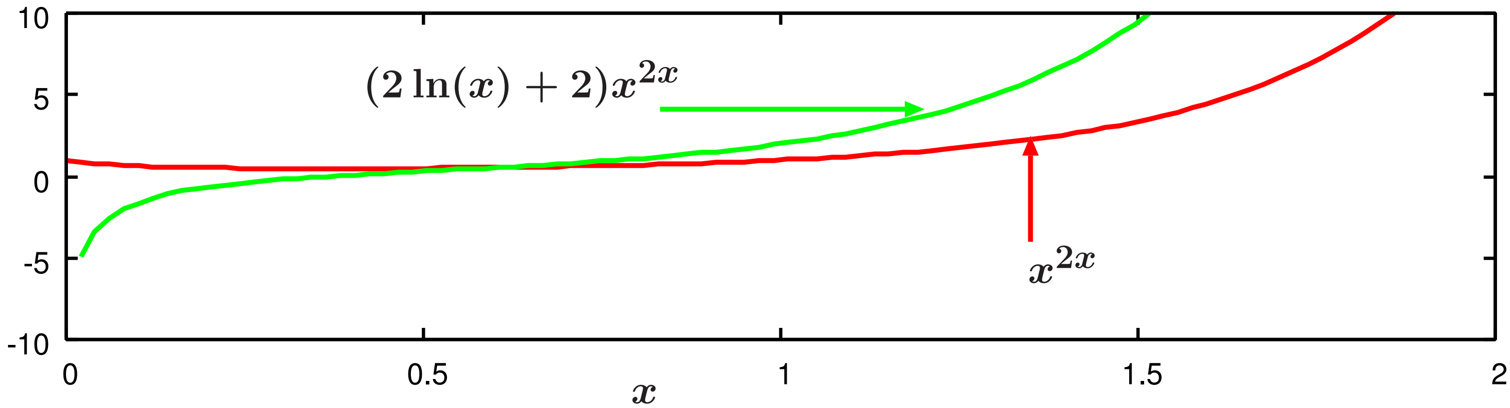
First we change the base of $\log$ so that we can use Equation (49)

$$y = \frac{\log_3 x}{3^x} = \frac{\frac{\log_e x}{\log_e 3}}{3^x} = \frac{1}{\ln 3}\frac{\ln x}{3^x}$$

Then we find $\frac{d\{y\}}{dx}$

$$\begin{aligned}\frac{d\{y\}}{dx} = \frac{1}{\ln 3}\frac{d\left\{\frac{\ln x}{3^x}\right\}}{dx} = \frac{1}{\ln 3}\frac{\frac{d\{\ln x\}}{dx}3^x - \ln x\frac{d\{3^x\}}{dx}}{3^{2x}} &= \frac{1}{\ln 3}\frac{\frac{1}{x}3^x - \ln x 3^x \ln 3}{3^{2x}} \\ &= \frac{1}{\ln 3}\frac{\frac{1}{x} - \ln x \ln 3}{3^x} = \frac{1 - x\ln x\ln 3}{3^x x\ln 3}\end{aligned}$$

38) Find $\frac{d\{y\}}{dx}$ of $y = x^{2x} \quad (x > 0)$



We apply the natural logarithm to both sides of the equation:

$$\ln y = \ln x^{2x} = 2x\ln x$$

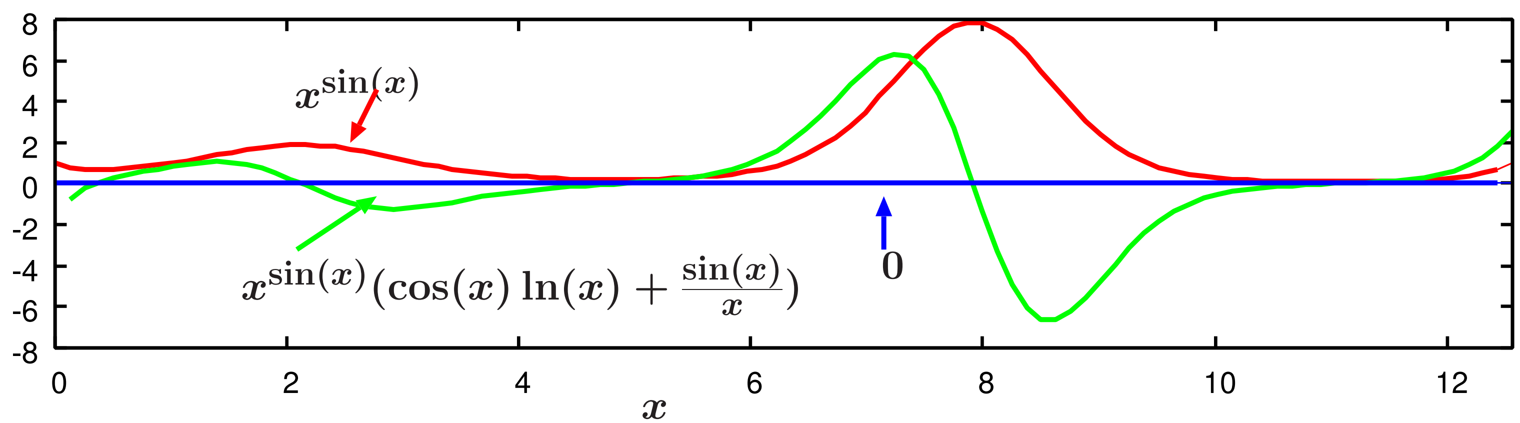
Here we differentiate both sides of $\ln y = 2x\ln x$ with respect to x:

$$\frac{d\{\ln y\}}{dx} = \frac{d\{2x\ln x\}}{dx}$$

$$\therefore \frac{d\{y\}}{dx}\frac{d\{\ln y\}}{dy} = \frac{d\{y\}}{dx}\frac{1}{y} = \frac{d\{2x\}}{dx}\ln x + 2x\frac{d\{\ln x\}}{dx} = 2\ln x + 2x\frac{1}{x} = 2\ln x + 2$$

$$\therefore \frac{d\{y\}}{dx} = (2\ln x + 2)y = (2\ln x + 2)x^{2x}$$

39) Find $\frac{d\{y\}}{dx}$ of $y = x^{\sin x}$ $\quad (x > 0)$



We apply the natural logarithm to both sides of the equation:

$$\ln y = \ln x^{\sin x} = \sin x \ln x$$

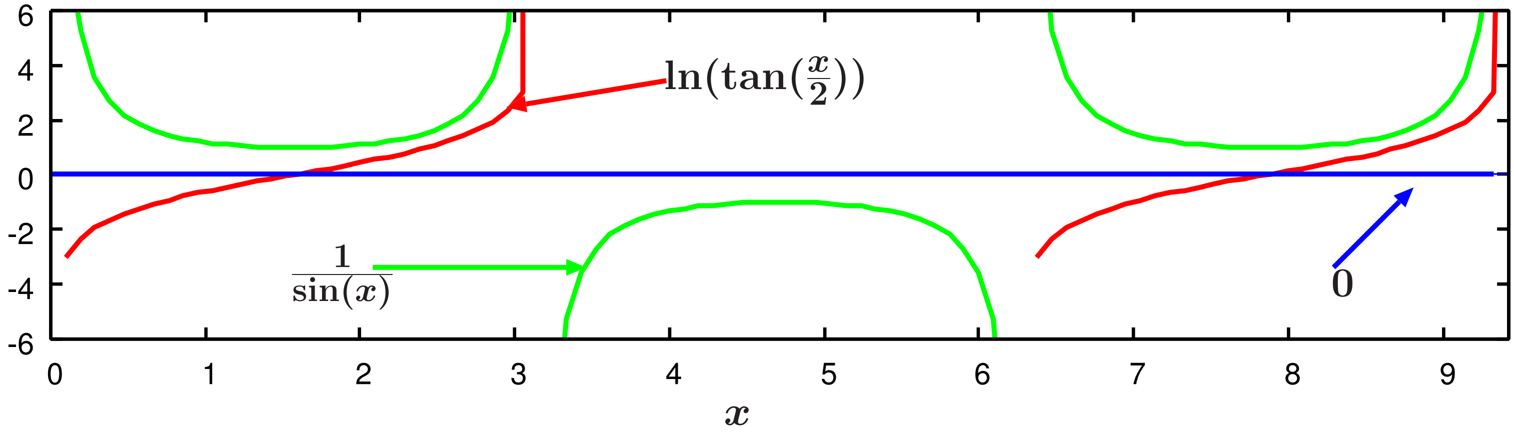
Here we differentiate both sides of $\ln y = \sin x\ln x$ with respect to x:

$$\frac{d\{\ln y\}}{dx} = \frac{d\{\sin x\ln x\}}{dx}$$

$$\therefore \frac{d\{y\}}{dx}\frac{d\{\ln y\}}{dy} = \frac{d\{y\}}{dx}\frac{1}{y} = \frac{d\{\sin x\}}{dx}\ln x + \sin x\frac{d\{\ln x\}}{dx} = \cos x\ln x + \sin x\frac{1}{x}$$

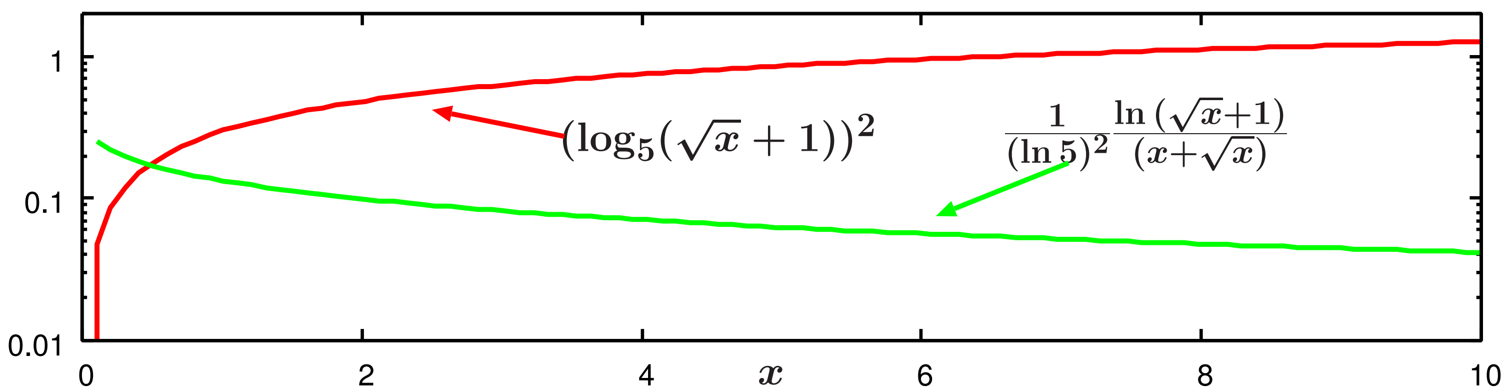
$$\therefore \frac{d\{y\}}{dx} = y(\cos x\ln x + \frac{\sin x}{x}) = x^{\sin x}(\cos x\ln x + \frac{\sin x}{x})$$

40) Differentiate $y = \ln(\tan\frac{x}{2})$ with respect to x



$$\frac{d\{y\}}{dx} = \frac{d\left\{\ln(\tan\frac{x}{2})\right\}}{dx} = \frac{d\{\ln u\}}{dx}(\because u \triangleq \tan\frac{x}{2}) = \frac{d\{u\}}{dx}\frac{\partial\{\ln u\}}{\partial u} = \frac{d\left\{\tan\frac{x}{2}\right\}}{dx}\frac{1}{u} = \frac{d\{\tan v\}}{dx}\frac{1}{u}(\because v \triangleq \frac{x}{2})$$
$$= \frac{d\{v\}}{dx}\frac{\partial\{\tan v\}}{\partial v}\frac{1}{u} = \frac{d\{v\}}{dx}\frac{1}{\cos^2 v}\frac{1}{u} = \frac{d\left\{\frac{x}{2}\right\}}{dx}\frac{1}{\cos^2 v}\frac{1}{u} = \frac{1}{2}\frac{1}{\cos^2\frac{x}{2}}\frac{1}{\tan\frac{x}{2}} = \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{1}{\sin x}$$

41) Differentiate $y = \{\log_5(\sqrt{x}+1)\}^2$ with respect to x



$$y = \{\log_5(\sqrt{x}+1)\}^2 = \left\{\frac{\log_e(\sqrt{x}+1)}{\log_e 5}\right\}^2 = \frac{1}{(\ln 5)^2}\{\log_e(\sqrt{x}+1)\}^2 = \frac{1}{(\ln 5)^2}\{\ln(\sqrt{x}+1)\}^2$$

Now we differentiate both sides

$$\frac{d\{y\}}{dx} = \frac{1}{(\ln 5)^2}\frac{d\{\{\ln(\sqrt{x}+1)\}^2\}}{dx} = \frac{1}{(\ln 5)^2}\frac{d\{u^2\}}{dx}(\because u \triangleq \ln(\sqrt{x}+1)) = \frac{1}{(\ln 5)^2}\frac{d\{u\}}{dx}\frac{\partial\{u^2\}}{\partial u}$$
$$= \frac{1}{(\ln 5)^2}\frac{d\{u\}}{dx}\cdot(2u) = \frac{1}{(\ln 5)^2}\frac{d\{\ln(\sqrt{x}+1)\}}{dx}\cdot(2u) = \frac{1}{(\ln 5)^2}\frac{d\{\ln v\}}{dx}\cdot(2u)(\because v \triangleq \sqrt{x}+1)$$
$$= \frac{1}{(\ln 5)^2}\frac{d\{v\}}{dx}\frac{\partial\{\ln v\}}{\partial v}\cdot(2u) = \frac{1}{(\ln 5)^2}\frac{d\{\sqrt{x}+1\}}{dx}\frac{1}{v}\cdot(2u) = \frac{1}{(\ln 5)^2}(\frac{1}{2}x^{-\frac{1}{2}})\frac{1}{v}\cdot(2u) = \frac{1}{(\ln 5)^2}x^{-\frac{1}{2}}\frac{u}{v}$$
$$= \frac{1}{(\ln 5)^2}x^{-\frac{1}{2}}\frac{\ln(\sqrt{x}+1)}{\sqrt{x}+1} = \frac{1}{(\ln 5)^2}\frac{\ln(\sqrt{x}+1)}{x+\sqrt{x}}$$

DAY2

42) Write $-yx^{-1}$ as a fraction.

$$a^{-1}=\frac{1}{a}\ ;\quad \therefore -yx^{-1}=-\frac{y}{x}$$

43) Write $\frac{1}{3}(4x+x^5)^{-\frac{2}{3}}$ as a fraction.

$$\frac{1}{3}\times\frac{1}{(4x+x^5)^{\frac{2}{3}}}=\frac{1}{3\sqrt[3]{(4x+x^5)^2}}$$

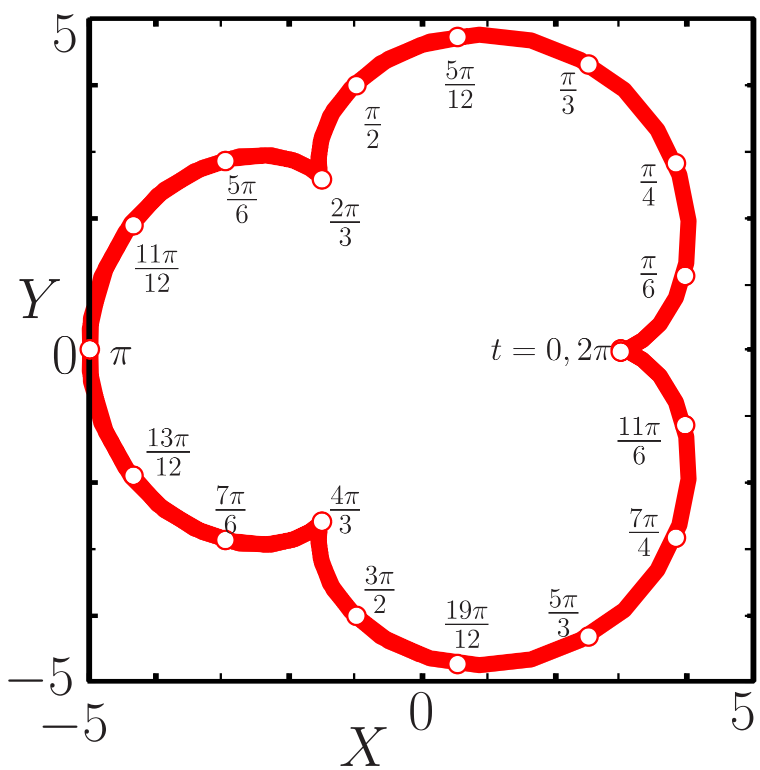
44) Write $-4x^{-5}$ as a fraction.

$$x^{-a}=\frac{1}{x^a}\ ;\quad \therefore -4x^{-5}=-\frac{4}{x^5}$$

45) y is the function of x. Using a parameter t, x and y are expressed as

$$x=4\cos(t)-\cos(4t)$$
$$y=4\sin(t)-\sin(4t)$$

Express $\frac{d\{y\}}{dx}$ and $\frac{d^2y}{dx^2}$ using t.



Since x and y are expressed using t, we can find $\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ with ease. So we are going to use $\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ to produce $\frac{d\{y\}}{dx}$ as follows:

$$\frac{d\{y\}}{dx}=\frac{d\{y\}}{dt}\cdot\frac{d\{t\}}{dx}=\frac{d\{y\}}{dt}\cdot(\frac{d\{x\}}{dt})^{-1}$$

$\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ are:

$$\frac{d\{x\}}{dt}=-4\sin(t)+4\sin(4t)$$
$$\frac{d\{y\}}{dt}=4\cos(t)-4\cos(4t)$$

$\frac{d\{y\}}{dx}$ is

$$\frac{d\{y\}}{dx}=\frac{d\{y\}}{dt}\cdot\frac{d\{t\}}{dx}=\frac{d\{y\}}{dt}\cdot(\frac{d\{x\}}{dt})^{-1}$$
$$=(4\cos(t)-4\cos(4t))\cdot(-4\sin(t)+4\sin(4t))^{-1}=(\cos(t)-\cos(4t))\cdot(-\sin(t)+\sin(4t))^{-1}$$

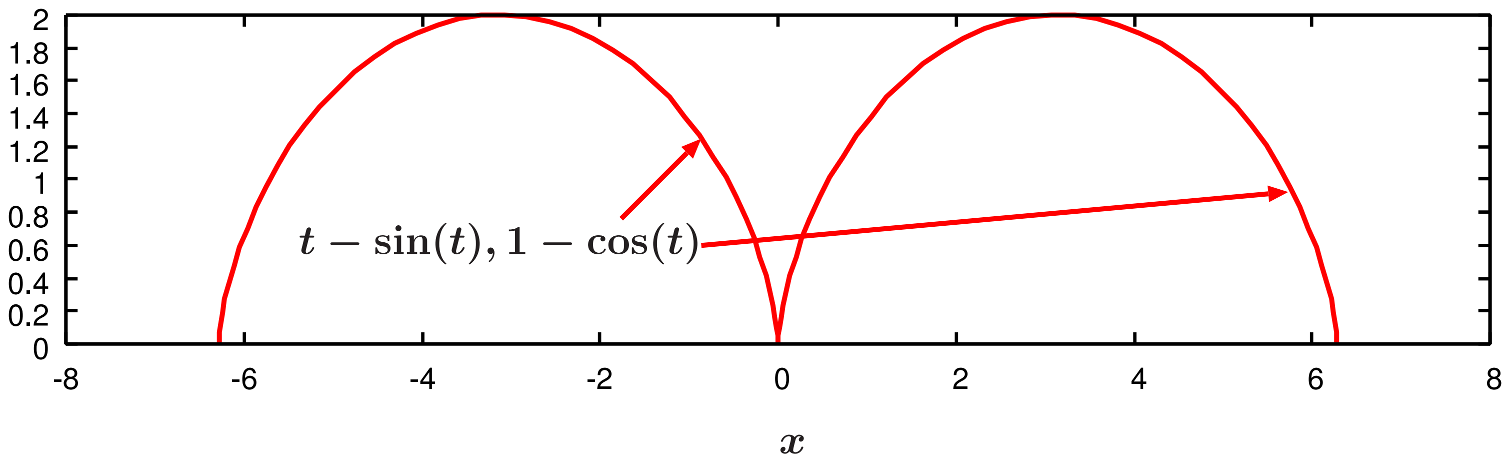
Similarly we can obtain $\frac{d^2y}{dx^2}$ as follows:

$$\frac{d^2y}{dx^2} = \frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dx} = \frac{d\{t\}}{dx}\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dt} = (\frac{d\{x\}}{dt})^{-1}\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dt}$$
$$= (\frac{d\{x\}}{dt})^{-1}\frac{d\left\{(\cos(t)-\cos(4t))\cdot(-\sin(t)+\sin(4t))^{-1}\right\}}{dt}$$
$$= \frac{1}{-4\sin(t)+4\sin(4t)}\left\{\frac{-\sin(t)+4\sin(4t)}{-\sin(t)+\sin(4t)} - \frac{(\cos(t)-\cos(4t))(-\cos(t)+4\cos(4t))}{(-\sin(t)+\sin(4t))^2}\right\}$$

46) y is the function of x. Using a parameter t where $0 < t < \pi$, x and y are expressed as

$$x = t - \sin t$$
$$y = 1 - \cos t$$

Express $\frac{d\{y\}}{dx}$ and $\frac{d^2y}{dx^2}$ using t.



Since x and y are expressed using t, we can find $\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ with ease. So we are going to use $\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ to produce $\frac{d\{y\}}{dx}$ as follows:

$$\frac{d\{y\}}{dx} = \frac{d\{y\}}{dt}\cdot\frac{d\{t\}}{dx} = \frac{d\{y\}}{dt}\cdot(\frac{d\{x\}}{dt})^{-1}$$

$\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ are:

$$\frac{d\{x\}}{dt} = 1 - \cos t$$
$$\frac{d\{y\}}{dt} = \sin t$$

$\frac{d\{y\}}{dx}$ is

$$\frac{d\{y\}}{dx} = \frac{d\{y\}}{dt}\cdot\frac{d\{t\}}{dx} = \frac{d\{y\}}{dt}\cdot(\frac{d\{x\}}{dt})^{-1} = \sin t\cdot(1-\cos t)^{-1} = \frac{\sin t}{1-\cos t}$$

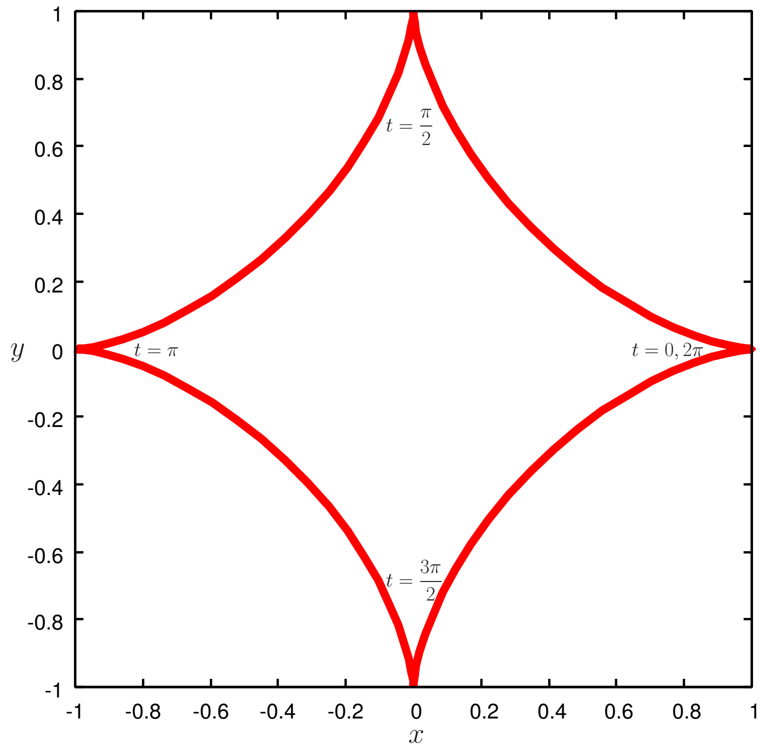
Similarly we can obtain $\frac{d^2y}{dx^2}$ as follows:

$$\frac{d^2y}{dx^2} = \frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dx} = \frac{d\{t\}}{dx}\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dt} = (\frac{d\{x\}}{dt})^{-1}\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dt} = (\frac{d\{x\}}{dt})^{-1}\frac{d\left\{\frac{\sin t}{1-\cos t}\right\}}{dt}$$
$$= (\frac{d\{x\}}{dt})^{-1}\frac{\frac{d\{\sin t\}}{dt}(1-\cos t) - \sin t\frac{d\{1-\cos t\}}{dt}}{(1-\cos t)^2} = (1-\cos t)^{-1}\frac{(\cos t)(1-\cos t) - \sin t\cdot\sin t}{(1-\cos t)^2}$$
$$= (1-\cos t)^{-1}\frac{\cos t - \cos^2 t - \sin^2 t}{(1-\cos t)^2} = (1-\cos t)^{-1}\frac{\cos t - 1}{(1-\cos t)^2} = -(1-\cos t)^{-1}\frac{1-\cos t}{(1-\cos t)^2}$$
$$= -\frac{1}{(1-\cos t)^2}$$

47) y is the function of x. Using a parameter t where $0 < t < \pi$, x and y are expressed as

$$x = \cos^3 t$$
$$y = \sin^3 t$$

Express $\frac{d\{y\}}{dx}$ and $\frac{d^2y}{dx^2}$ using t.



Since x and y are expressed using t, we can find $\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ with ease. So we are going to use $\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ to produce $\frac{d\{y\}}{dx}$ as follows:

$$\frac{d\{y\}}{dx} = \frac{d\{y\}}{dt} \cdot \frac{d\{t\}}{dx} = \frac{d\{y\}}{dt} \cdot (\frac{d\{x\}}{dt})^{-1}$$

$\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ are:

$$\frac{d\{x\}}{dt} = \frac{d\{\cos^3 t\}}{dt} = \frac{d\{a^3\}}{dt}(\because a \triangleq \cos t) = \frac{d\{a\}}{dt}\frac{\partial\{a^3\}}{\partial a} = \frac{d\{\cos t\}}{dt} \cdot (3a^2) = -\sin t \cdot (3a^2)$$
$$= -\sin t \cdot (3\cos^2 t) = -3\cos^2 t \sin t$$
$$\frac{d\{y\}}{dt} = \frac{d\{\sin^3 t\}}{dt} = \frac{d\{v^3\}}{dt}(\because v \triangleq \sin t) = \frac{d\{v\}}{dt}\frac{\partial\{v^3\}}{\partial v} = \frac{d\{\sin t\}}{dt}(3v^2) = \cos t(3\sin^2 t) = 3\sin^2 t \cos t$$

$\frac{d\{y\}}{dx}$ is

$$\frac{d\{y\}}{dx} = \frac{d\{y\}}{dt} \cdot (\frac{d\{x\}}{dt})^{-1} = 3\sin^2 t \cos t \cdot (-3\cos^2 t \sin t)^{-1} = -\sin t \cos^{-1} t = -\tan t$$

Similarly we can obtain $\frac{d^2y}{dx^2}$ as follows:

$$\frac{d^2y}{dx^2} = \frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dx} = \frac{d\{t\}}{dx}\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dt} = (\frac{d\{x\}}{dt})^{-1}\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dt} = (\frac{d\{x\}}{dt})^{-1}\frac{d\{-\tan t\}}{dt}$$
$$= (-3\cos^2 t \sin t)^{-1} \cdot \frac{-1}{\cos t^2} = \frac{1}{3\cos t^4 \sin t}$$

48) y is the function of x. Using a parameter t where $0 \leq t \leq \frac{\pi}{4}$, x and y are expressed as

$$x = \ln(2\cos t)$$
$$y = -t + \tan t$$

Express $\frac{d\{y\}}{dx}$ and $\frac{d^2y}{dx^2}$ using t.

Since x and y are expressed using t, we can find $\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ with ease. So we are going to use $\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ to produce $\frac{d\{y\}}{dx}$ as follows:

$$\frac{d\{y\}}{dx} = \frac{d\{y\}}{dt} \cdot \frac{d\{t\}}{dx} = \frac{d\{y\}}{dt} \cdot (\frac{d\{x\}}{dt})^{-1}$$

$\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ are:

$$\frac{d\{x\}}{dt} = \frac{d\{\ln(2\cos t)\}}{dt} = \frac{d\{\ln(A)\}}{dt} (\because A \triangleq 2\cos t) = \frac{d\{A\}}{dt}\frac{d\{\ln(A)\}}{dA} = \frac{d\{2\cos t\}}{dt}\frac{d\{\ln(A)\}}{dA}$$
$$= -2\sin t\frac{1}{A} = -2\sin t\frac{1}{2\cos t} = -\tan t$$
$$\frac{d\{y\}}{dt} = \frac{d\{-t+\tan t\}}{dt} = -1 + \frac{1}{\cos t^2} = -\frac{\cos t^2}{\cos t^2} + \frac{1}{\cos t^2} = \frac{1-\cos t^2}{\cos t^2} = \frac{\sin t^2}{\cos t^2} = \tan t^2$$

$\frac{d\{y\}}{dx}$ is

$$\frac{d\{y\}}{dx} = \frac{d\{y\}}{dt} \cdot \frac{d\{t\}}{dx} = \frac{d\{y\}}{dt} \cdot (\frac{d\{x\}}{dt})^{-1} = \tan t^2 \cdot (-\tan t)^{-1} = -\tan t$$

Similarly we can obtain $\frac{d^2y}{dx^2}$ as follows:

$$\frac{d^2y}{dx^2} = \frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dx} = \frac{d\{t\}}{dx}\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dt} = (\frac{d\{x\}}{dt})^{-1}\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dt} = (-\tan t)^{-1}\frac{d\{-\tan t\}}{dt} = \frac{1}{\tan t\cos t^2}$$

49) y is the function of x. Using a parameter t where $1 \leq t \leq 2$, x and y are expressed as

$$x = 1 + t$$
$$y = 1 - \frac{1}{t}$$

Express $\frac{d\{y\}}{dx}$ and $\frac{d^2y}{dx^2}$ using t.

Since x and y are expressed using t, we can find $\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ with ease. So we are going to use $\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ to produce $\frac{d\{y\}}{dx}$ as follows:

$$\frac{d\{y\}}{dx} = \frac{d\{y\}}{dt} \cdot \frac{d\{t\}}{dx} = \frac{d\{y\}}{dt} \cdot (\frac{d\{x\}}{dt})^{-1}$$

$\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$ are:

$$\frac{d\{x\}}{dt} = \frac{d\{1+t\}}{dt} = 1$$
$$\frac{d\{y\}}{dt} = \frac{d\left\{1-\frac{1}{t}\right\}}{dt} = \frac{d\left\{1-t^{-1}\right\}}{dt} = -(-1)t^{-2} = t^{-2}$$

$\frac{d\{y\}}{dx}$ is

$$\frac{d\{y\}}{dx} = \frac{d\{y\}}{dt} \cdot \frac{d\{t\}}{dx} = \frac{d\{y\}}{dt} \cdot (\frac{d\{x\}}{dt})^{-1} = t^{-2} \cdot (1)^{-1} = t^{-2}$$

Similarly we can obtain $\frac{d^2y}{dx^2}$ as follows:

$$\frac{d^2y}{dx^2} = \frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dx} = \frac{d\{t\}}{dx}\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dt} = (\frac{d\{x\}}{dt})^{-1}\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dt} = (1)^{-1}\frac{d\left\{t^{-2}\right\}}{dt} = -2t^{-3}$$

50) y is the function of x. Using a parameter t where $1 \leq t \leq 4.5$, x and y are expressed as

$$x = -\frac{1}{t}$$
$$y = \sin(t) + \ln(t)$$

Express $\frac{d\{y\}}{dx}$ and $\frac{d^2y}{dx^2}$ using t.

Since x and y are expressed using t, we can find $\frac{d\left\{x\right\}}{dt}$ and $\frac{d\left\{y\right\}}{dt}$ with ease. So we are going to use $\frac{d\left\{x\right\}}{dt}$ and $\frac{d\left\{y\right\}}{dt}$ to produce $\frac{d\left\{y\right\}}{dx}$ as follows:

$$\frac{d\left\{y\right\}}{dx} = \frac{d\left\{y\right\}}{dt} \cdot \frac{d\left\{t\right\}}{dx} = \frac{d\left\{y\right\}}{dt} \cdot (\frac{d\left\{x\right\}}{dt})^{-1}$$

$\frac{d\left\{x\right\}}{dt}$ and $\frac{d\left\{y\right\}}{dt}$ are:

$$\frac{d\left\{x\right\}}{dt} = \frac{d\left\{-\frac{1}{t}\right\}}{dt} = \frac{d\left\{-t^{-1}\right\}}{dt} = t^{-2}$$

$$\frac{d\left\{y\right\}}{dt} = \frac{d\left\{\sin(t) + \ln(t)\right\}}{dt} = \cos(t) + \frac{1}{t}$$

$\frac{d\left\{y\right\}}{dx}$ is

$$\frac{d\left\{y\right\}}{dx} = \frac{d\left\{y\right\}}{dt} \cdot \frac{d\left\{t\right\}}{dx} = \frac{d\left\{y\right\}}{dt} \cdot (\frac{d\left\{x\right\}}{dt})^{-1} = (\cos(t) + \frac{1}{t}) \cdot (t^{-2})^{-1} = (\cos(t) + \frac{1}{t}) \cdot t^2 = t^2\cos(t) + t$$

Similarly we can obtain $\frac{d^2y}{dx^2}$ as follows:

$$\frac{d^2y}{dx^2} = \frac{d\left\{\frac{d\left\{y\right\}}{dx}\right\}}{dx} = \frac{d\left\{t\right\}}{dx}\frac{d\left\{\frac{d\left\{y\right\}}{dx}\right\}}{dt} = (\frac{d\left\{x\right\}}{dt})^{-1}\frac{d\left\{\frac{d\left\{y\right\}}{dx}\right\}}{dt} = (t^{-2})^{-1}\frac{d\left\{t^2\cos(t) + t\right\}}{dt}$$

$$= t^2(2t\cos(t) - t^2\sin(t) + 1) = t^3\cos(t) - t^4\sin(t) + t^2$$

DAY3

In your answer to the following questions the expresson for $\frac{d\{y\}}{dx}$ and $\frac{d\{x\}}{dy}$ can be in terms of both x and y.

51) Simplify $\left(\frac{1}{3}\right)^{-2}$.

$$\left(\frac{1}{3}\right)^{-2} = \left(\frac{1}{3}\right)^{2\cdot(-1)} = \left(\frac{1}{9}\right)^{-1} = 9$$

52) What is $-1 \times \frac{1}{4+2x}$.

$$-1 \times \frac{a}{b} = \frac{-a}{b} \;\;;\;\; \therefore -1 \times \frac{1}{4+2x} = \frac{-1}{4+2x}$$

53) Simplify $\ln 2 - \ln 8$

$$\ln|2| - \ln|8| = \ln|\frac{2}{8}| = \ln|\frac{1}{4}| = \ln|1| - \ln|4| = -\ln|4|$$

54) Find $\frac{d\{f(x,y)\}}{dx}$ and $\frac{d\{f(x,y)\}}{dy}$ of $f(x,y) = y^2\ln x + \sin x + \mathfrak{e}^y$ and find $\frac{d\{y\}}{dx}$ and $\frac{d\{x\}}{dy}$ of $y^2\ln x + \sin x + \mathfrak{e}^y = 1$ when $x > 0$, $\frac{y^2}{x} + \cos x \neq 0$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\left\{y^2\ln x + \sin x + \mathfrak{e}^y\right\}}{dx} = \frac{d\left\{y^2\ln x\right\}}{dx} + \frac{d\{\sin x\}}{dx} + \frac{d\{\mathfrak{e}^y\}}{dx}$$
$$= \frac{d\left\{y^2\right\}}{dx}\ln x + y^2\frac{d\{\ln x\}}{dx} + \frac{d\{\sin x\}}{dx} + \frac{d\{\mathfrak{e}^y\}}{dx} = 0\cdot\ln x + y^2\cdot\frac{1}{x} + \cos x + 0 = \frac{y^2}{x} + \cos x$$
$$\frac{d\{f(x,y)\}}{dy} = \frac{d\left\{y^2\ln x + \sin x + \mathfrak{e}^y\right\}}{dy} = \frac{d\left\{y^2\ln x\right\}}{dy} + \frac{d\{\sin x\}}{dy} + \frac{d\{\mathfrak{e}^y\}}{dy}$$
$$= \frac{d\left\{y^2\right\}}{dy}\ln x + y^2\frac{d\{\ln x\}}{dy} + \frac{d\{\sin x\}}{dy} + \frac{d\{\mathfrak{e}^y\}}{dy} = 2y\ln x + y^2\cdot 0 + 0 + \mathfrak{e}^y = 2y\ln x + \mathfrak{e}^y$$
$$\frac{d\left\{y^2\ln x + \sin x + \mathfrak{e}^y\right\}}{dx} = \frac{d\{1\}}{dx} \;\;;\;\; \therefore \frac{d\left\{y^2\ln x + \sin x + \mathfrak{e}^y\right\}}{dx} = 0$$
$$\therefore \frac{d\left\{y^2\ln x\right\}}{dx} + \frac{d\{\sin x\}}{dx} + \frac{d\{\mathfrak{e}^y\}}{dx} = 0 \;\;;\;\; \therefore \frac{d\left\{y^2\right\}}{dx}\ln x + y^2\frac{d\{\ln x\}}{dx} + \frac{d\{\sin x\}}{dx} + \frac{d\{\mathfrak{e}^y\}}{dx} = 0$$
$$\therefore \frac{d\{y\}}{dx}\frac{d\left\{y^2\right\}}{dy}\ln x + y^2\cdot\frac{1}{x} + \cos x + \frac{d\{y\}}{dx}\frac{d\{\mathfrak{e}^y\}}{dy} = 0 \;\;;\;\; \therefore \frac{d\{y\}}{dx}\cdot(2y)\ln x + \frac{y^2}{x} + \cos x + \frac{d\{y\}}{dx}\mathfrak{e}^y = 0$$
$$\therefore \frac{d\{y\}}{dx}\cdot(2y)\ln x + \frac{d\{y\}}{dx}\mathfrak{e}^y = -\frac{y^2}{x} - \cos x \;\;;\;\; \therefore \frac{d\{y\}}{dx}\cdot(2y\ln x + \mathfrak{e}^y) = -\frac{y^2}{x} - \cos x$$
$$\therefore \frac{d\{y\}}{dx} = -\frac{\frac{y^2}{x} + \cos x}{2y\ln x + \mathfrak{e}^y}$$
$$\frac{d\left\{y^2\ln x + \sin x + \mathfrak{e}^y\right\}}{dy} = \frac{d\{1\}}{dy} \;\;;\;\; \therefore \frac{d\left\{y^2\ln x + \sin x + \mathfrak{e}^y\right\}}{dy} = 0$$
$$\therefore \frac{d\left\{y^2\ln x\right\}}{dy} + \frac{d\{\sin x\}}{dy} + \frac{d\{\mathfrak{e}^y\}}{dy} = 0 \;\;;\;\; \therefore \frac{d\left\{y^2\right\}}{dy}\ln x + y^2\frac{d\{\ln x\}}{dy} + \frac{d\{\sin x\}}{dy} + \frac{d\{\mathfrak{e}^y\}}{dy} = 0$$
$$\therefore 2y\ln x + y^2\frac{d\{x\}}{dy}\frac{d\{\ln x\}}{dx} + \frac{d\{x\}}{dy}\frac{d\{\sin x\}}{dx} + \mathfrak{e}^y = 0 \;\;;\;\; \therefore 2y\ln x + y^2\frac{d\{x\}}{dy}\frac{1}{x} + \frac{d\{x\}}{dy}\cos x + \mathfrak{e}^y = 0$$
$$\therefore \frac{d\{x\}}{dy}(\frac{y^2}{x} + \cos x) = -2y\ln x - \mathfrak{e}^y \;\;;\;\; \therefore \frac{d\{x\}}{dy} = -\frac{2y\ln x + \mathfrak{e}^y}{\frac{y^2}{x} + \cos x}$$

55) Find $\frac{d\{f(x,y)\}}{dx}$ and $\frac{d\{f(x,y)\}}{dy}$ of $f(x,y) = \mathfrak{e}^{xy^2}$ and find $\frac{d\{y\}}{dx}$ and $\frac{d\{x\}}{dy}$ of $\mathfrak{e}^{xy^2} = 1$ when $xy \neq 0$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\left\{\mathfrak{e}^{xy^2}\right\}}{dx} = \frac{d\{A\}}{dx}\frac{d\left\{\mathfrak{e}^A\right\}}{dA}(\because A \triangleq xy^2)$$
$$= \frac{d\left\{xy^2\right\}}{dx}\mathfrak{e}^A = (\frac{d\{x\}}{dx}y^2 + x\frac{d\left\{y^2\right\}}{dx})\mathfrak{e}^A = (y^2 + x\cdot 0)\mathfrak{e}^A = y^2\mathfrak{e}^{xy^2}$$
$$\frac{d\{f(x,y)\}}{dy} = \frac{d\left\{\mathfrak{e}^{xy^2}\right\}}{dy} = \frac{d\{A\}}{dy}\frac{d\left\{\mathfrak{e}^A\right\}}{dA}(\because A \triangleq xy^2)$$
$$= \frac{d\left\{xy^2\right\}}{dy}\mathfrak{e}^A = (\frac{d\{x\}}{dy}y^2 + x\frac{d\left\{y^2\right\}}{dy})\mathfrak{e}^A = (0\cdot y^2 + x\cdot(2y))\mathfrak{e}^A = 2xy\mathfrak{e}^{xy^2}$$
$$\frac{d\left\{\mathfrak{e}^{xy^2}\right\}}{dx} = \frac{d\{1\}}{dx} \;\;;\;\; \therefore \frac{d\{A\}}{dx}\frac{d\left\{\mathfrak{e}^A\right\}}{dA} = 0(\because A \triangleq xy^2) \;\;;\;\; \therefore \frac{d\left\{xy^2\right\}}{dx}\mathfrak{e}^A = 0 \;\;;\;\; \therefore (\frac{d\{x\}}{dx}y^2 + x\frac{d\left\{y^2\right\}}{dx})\mathfrak{e}^A = 0$$
$$\therefore (1\cdot y^2 + x\frac{d\{y\}}{dx}\frac{d\left\{y^2\right\}}{dy})\mathfrak{e}^A = 0 \;\;;\;\; \therefore y^2 + x\frac{d\{y\}}{dx}\cdot(2y) = 0(\because \mathfrak{e}^A \neq 0 \;\; \forall A)$$

$$\therefore 2yx\frac{d\{y\}}{dx} = -y^2 \ ; \quad \therefore \frac{d\{y\}}{dx} = -\frac{y^2}{2xy} = -\frac{y}{2x}$$

$$\frac{d\left\{\mathfrak{e}^{xy^2}\right\}}{dy} = \frac{d\{1\}}{dy} \ ; \quad \therefore \frac{d\{A\}}{dy}\frac{d\{\mathfrak{e}^A\}}{dA} = 0(\because A \triangleq xy^2) \ ; \quad \therefore \frac{d\{xy^2\}}{dy}\mathfrak{e}^A = 0 \ ; \quad \therefore (\frac{d\{x\}}{dy}y^2 + x\frac{d\{y^2\}}{dy})\mathfrak{e}^A = 0$$

$$\therefore (\frac{d\{x\}}{dy}y^2 + 2yx)\mathfrak{e}^A = 0 \ ; \quad \therefore \frac{d\{x\}}{dy}y^2 + 2yx = 0(\because \mathfrak{e}^A \neq 0 \ \ \forall A)$$

$$\therefore \frac{d\{x\}}{dy}y^2 = -2yx \ ; \quad \therefore \frac{d\{x\}}{dy} = -\frac{2yx}{y^2} = -\frac{2x}{y}$$

56) Find $\frac{d\{f(x,y)\}}{dx}$ and $\frac{d\{f(x,y)\}}{dy}$ of $f(x,y) = \mathfrak{e}^{x+y^2}$ and find $\frac{d\{y\}}{dx}$ and $\frac{d\{x\}}{dy}$ of $\mathfrak{e}^{x+y^2} = 1$ when $y \neq 0$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\left\{\mathfrak{e}^{x+y^2}\right\}}{dx} = \frac{d\{A\}}{dx}\frac{d\{\mathfrak{e}^A\}}{dA}(\because A \triangleq x + y^2)$$

$$= \frac{d\{x+y^2\}}{dx}\mathfrak{e}^A = (\frac{d\{x\}}{dx} + \frac{d\{y^2\}}{dx})\mathfrak{e}^A = (1+0)\mathfrak{e}^A = \mathfrak{e}^{x+y^2}$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\left\{\mathfrak{e}^{x+y^2}\right\}}{dy} = \frac{d\{A\}}{dy}\frac{d\{\mathfrak{e}^A\}}{dA}(\because A \triangleq x + y^2)$$

$$= \frac{d\{x+y^2\}}{dy}\mathfrak{e}^A = (\frac{d\{x\}}{dy} + \frac{d\{y^2\}}{dy})\mathfrak{e}^A = (0+2y)\mathfrak{e}^A = 2y\mathfrak{e}^{x+y^2}$$

$$\frac{d\left\{\mathfrak{e}^{x+y^2}\right\}}{dx} = \frac{d\{1\}}{dx} \ ; \quad \therefore \frac{d\{A\}}{dx}\frac{d\{\mathfrak{e}^A\}}{dA} = 0(\because A \triangleq x + y^2)$$

$$\therefore \frac{d\{x+y^2\}}{dx}\mathfrak{e}^A = 0 \ ; \quad \therefore (\frac{d\{x\}}{dx} + \frac{d\{y^2\}}{dx})\mathfrak{e}^A = 0$$

$$\therefore (1 + \frac{d\{y\}}{dx}\frac{d\{y^2\}}{dy})\mathfrak{e}^A = 0 \ ; \quad \therefore 1 + \frac{d\{y\}}{dx}\cdot(2y) = 0(\because \mathfrak{e}^A \neq 0 \ \ \forall A)$$

$$\therefore 2y\frac{d\{y\}}{dx} = -1 \ ; \quad \therefore \frac{d\{y\}}{dx} = -\frac{1}{2y}$$

$$\frac{d\left\{\mathfrak{e}^{x+y^2}\right\}}{dy} = \frac{d\{1\}}{dy} \ ; \quad \therefore \frac{d\{A\}}{dy}\frac{d\{\mathfrak{e}^A\}}{dA} = 0(\because A \triangleq x + y^2)$$

$$\therefore \frac{d\{x+y^2\}}{dy}\mathfrak{e}^A = 0 \ ; \quad \therefore (\frac{d\{x\}}{dy} + \frac{d\{y^2\}}{dy})\mathfrak{e}^A = 0$$

$$\therefore (\frac{d\{x\}}{dy} + 2y)\mathfrak{e}^A = 0 \ ; \quad \therefore \frac{d\{x\}}{dy} + 2y = 0(\because \mathfrak{e}^A \neq 0 \ \ \forall A) \ ; \quad \therefore \frac{d\{x\}}{dy} = -2y$$

57) Find $\frac{d\{f(x,y)\}}{dx}$ of $f(x,y) = y^2 + x^2 + xy$ and find $\frac{d\{y\}}{dx}$ of $y^2 + x^2 + xy = 1$ when $x + 2y \neq 0$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{y^2+x^2+xy\}}{dx} = \frac{d\{y^2\}}{dx} + \frac{d\{x^2\}}{dx} + \frac{d\{xy\}}{dx} = 0 + 2x + y\frac{d\{x\}}{dx} = 2x + y$$

$$y^2 + x^2 + xy = 1 \ ; \quad \therefore \frac{d\{y^2+x^2+xy\}}{dx} = \frac{d\{1\}}{dx} \ ; \quad \therefore \frac{d\{y^2\}}{dx} + \frac{d\{x^2\}}{dx} + \frac{d\{xy\}}{dx} = \frac{d\{1\}}{dx}$$

$$\therefore \frac{d\{y\}}{dx}\frac{d\{y^2\}}{dy} + 2x + x\frac{d\{y\}}{dx} + y\frac{d\{x\}}{dx} = \frac{d\{1\}}{dx} \ ; \quad \therefore 2y\frac{d\{y\}}{dx} + 2x + x\frac{d\{y\}}{dx} + y = 0$$

$$\therefore (2y + x)\frac{d\{y\}}{dx} = -y - 2x \ ; \quad \therefore \frac{d\{y\}}{dx} = \frac{-y-2x}{2y+x}$$

58) Find $\frac{d\{f(x,y)\}}{dx}$ of $f(x,y) = \ln y + x$ and find $\frac{d\{y\}}{dx}$ of $\ln y + x = 1$. when $y > 0$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{\ln y + x\}}{dx} = \frac{d\{\ln y\}}{dx} + \frac{d\{x\}}{dx} = 0 + 1 = 1$$

$$\frac{d\{\ln y + x\}}{dx} = \frac{d\{1\}}{dx} \ ; \quad \therefore \frac{d\{\ln y\}}{dx} + \frac{d\{x\}}{dx} = \frac{d\{1\}}{dx} \ ; \quad \therefore \frac{d\{y\}}{dx}\frac{d\{\ln y\}}{dy} + 1 = \frac{d\{1\}}{dx}$$

$$\therefore \frac{1}{y}\frac{d\{y\}}{dx} + 1 = 0 \ ; \quad \therefore \frac{1}{y}\frac{d\{y\}}{dx} = -1 \ ; \quad \therefore \frac{d\{y\}}{dx} = -y$$

59) Find $\frac{d\{f(x,y)\}}{dy}$ of $f(x,y) = 8y + x$ and find $\frac{d\{x\}}{dy}$ of $8y + x = 1$.

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{8y + x\}}{dy} = \frac{d\{8y\}}{dy} + \frac{d\{x\}}{dy} = 8 + 0 = 8$$

$$\frac{d\{8y + x\}}{dy} = \frac{d\{1\}}{dy} \;\; ; \;\; \therefore \frac{d\{8y\}}{dy} + \frac{d\{x\}}{dy} = 0 \;\; ; \;\; \therefore 8 + \frac{d\{x\}}{dy} = 0 \;\; ; \;\; \therefore \frac{d\{x\}}{dy} = -8$$

60) Find $\frac{d\{f(x,y)\}}{dy}$ of $f(x,y) = 4y^2 + xy^3 + 2$ and find $\frac{d\{x\}}{dy}$ of $4y^2 + xy^3 + 2 = 0$ when $y \neq 0$.

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{4y^2 + xy^3 + 2\}}{dy} = \frac{d\{4y^2\}}{dy} + \frac{d\{xy^3\}}{dy} + \frac{d\{2\}}{dy} = \frac{d\{4y^2\}}{dy} + x\frac{d\{y^3\}}{dy} + 0 = 8y + 3y^2x$$

$$\frac{d\{4y^2 + xy^3 + 2\}}{dy} = \frac{d\{0\}}{dy} \;\; ; \;\; \therefore \frac{d\{4y^2\}}{dy} + \frac{d\{xy^3\}}{dy} + \frac{d\{2\}}{dy} = 0 \;\; ; \;\; \therefore 8y + x\frac{d\{y^3\}}{dy} + y^3\frac{d\{x\}}{dy} + 0 = 0$$

$$\therefore 8y + 3y^2x + y^3\frac{d\{x\}}{dy} = 0 \;\; ; \;\; \therefore y^3\frac{d\{x\}}{dy} = -8y - 3y^2x \;\; ; \;\; \therefore \frac{d\{x\}}{dy} = \frac{-8y - 3y^2x}{y^3} = \frac{-8 - 3xy}{y^2}$$

61) Find $\frac{d\{f(x,y)\}}{dy}$ of $f(x,y) = \cos y + \sin x$ and find $\frac{d\{x\}}{dy}$ of $\cos y + \sin x = 1$ when $\cos x \neq 0$.

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{\cos y + \sin x\}}{dy} = \frac{d\{\cos y\}}{dy} + \frac{d\{\sin x\}}{dy} = -\sin y$$

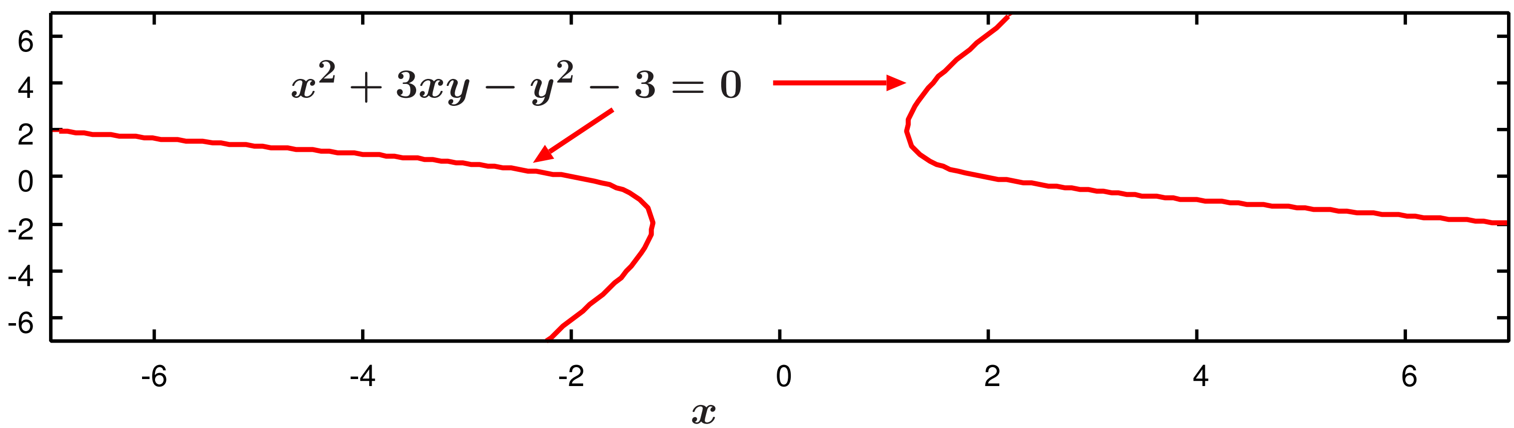
$$\frac{d\{\cos y + \sin x\}}{dy} = \frac{d\{1\}}{dy} \;\; ; \;\; \therefore \frac{d\{\cos y\}}{dy} + \frac{d\{\sin x\}}{dy} = 0 \;\; ; \;\; \therefore -\sin y + \frac{d\{x\}}{dy}\frac{d\{\sin x\}}{dx} = 0$$

$$\therefore -\sin y + \cos x\frac{d\{x\}}{dy} = 0 \;\; ; \;\; \therefore \cos x\frac{d\{x\}}{dy} = \sin y \;\; ; \;\; \therefore \frac{d\{x\}}{dy} = \frac{\sin y}{\cos x}$$

62) Find the equation of a tangent line of a curve

$$x^2 + 3xy - y^2 = 3$$

at the point $(1, 2)$



In general, a line which goes through $(1, 2)$ can be written as

$$y - 2 = g(x - 1)$$

where g is the gradient of the line. The gradient of a tangent line is $\frac{d\{y\}}{dx}$ Thus we need to find out $\frac{d\{y\}}{dx}$ as follows: We differentiate both sides of

$$x^2 + 3xy - y^2 = 3$$

with respect to x:

$$\frac{d\{x^2 + 3xy - y^2\}}{dx} = \frac{d\{3\}}{dx} \;\; ; \;\; \therefore \frac{d\{x^2\}}{dx} + \frac{d\{3xy\}}{dx} - \frac{d\{y^2\}}{dx} = \frac{d\{3\}}{dx}$$

$$\therefore 2x + \frac{d\{3x\}}{dx}y + 3x\frac{d\{y\}}{dx} - \frac{d\{y\}}{dx}\frac{d\{y^2\}}{dy} = 0 \;\; ; \;\; \therefore 2x + 3y + 3x\frac{d\{y\}}{dx} - \frac{d\{y\}}{dx} \cdot (2y) = 0$$

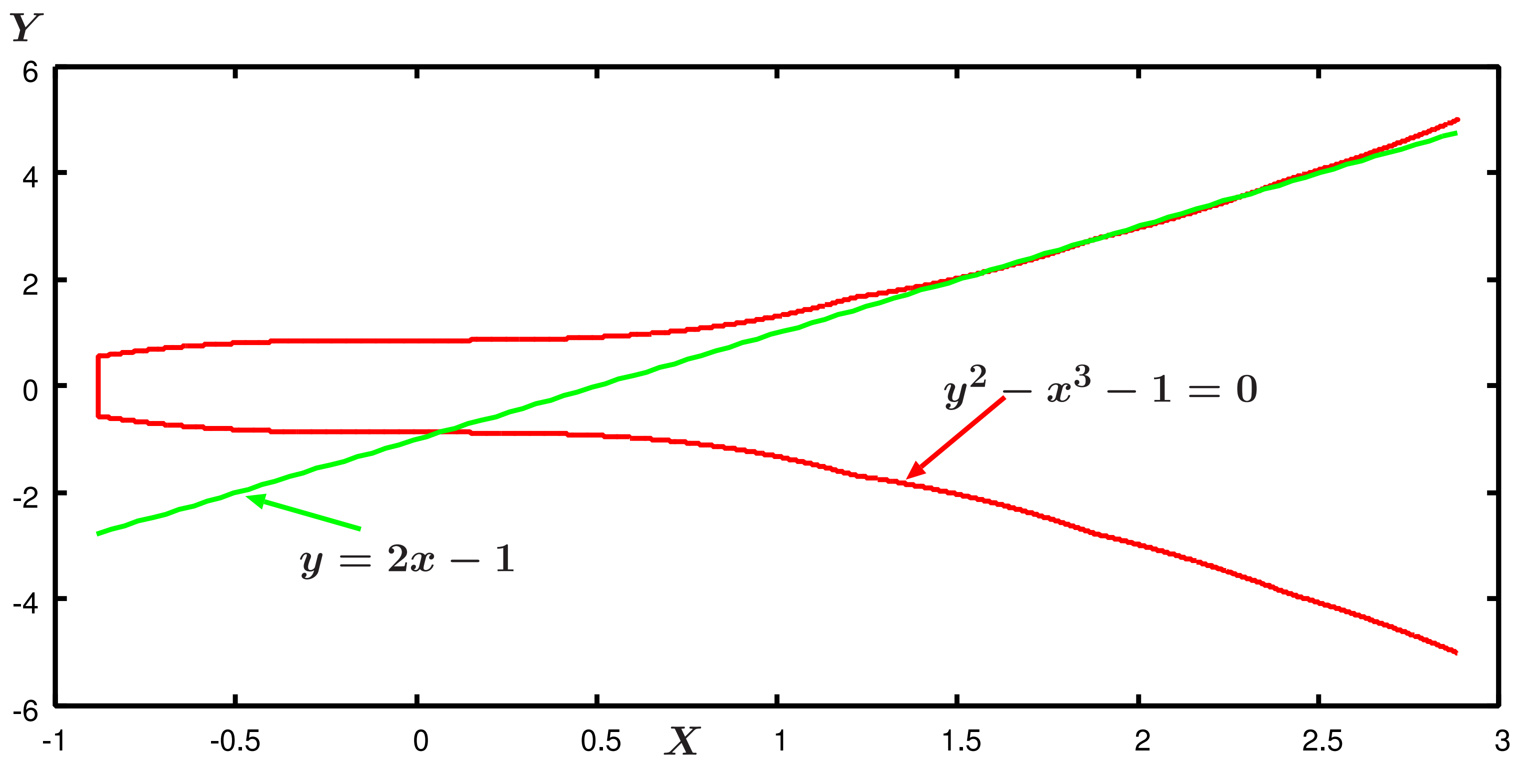
$$\therefore 2x + 3y + (3x - 2y)\frac{d\{y\}}{dx} = 0 \;\; ; \;\; \therefore \frac{d\{y\}}{dx} = \frac{2x + 3y}{2y - 3x}$$

Now that we have obtained $\frac{d\{y\}}{dx}$, we can find the tangent line as

$$y-2=\left.\frac{d\{y\}}{dx}\right|_{(x,y)=(1,2)}(x-1)=\frac{2x+3y}{2y-3x}|_{(x,y)=(1,2)}(x-1)=\frac{2\cdot 1+3\cdot 2}{2\cdot 2-3\cdot 1}(x-1)$$

$$\therefore y-2=\frac{2+6}{4-3}(x-1)=8(x-1)=8x-8\ ;\quad \therefore y=8x-6$$

63) A tangent line of a curve $y^2=x^3+1$ at a point (a,b) $((a,b)\neq(0,-1))$ goes through a point $(0,-1)$. Find a and b.



The tangent line of $y^2=x^3+1$ at (a,b) is

$$y-b=\left.\frac{d\{y\}}{dx}\right|_{(x,y)=(a,b)}(x-a)$$

In order to obtain $\frac{d\{y\}}{dx}$, we differentiate both sides of $y^2=x^3+1$ with respect to x:

$$\frac{d\{y^2\}}{dx}=\frac{d\{x^3+1\}}{dx}\ ;\quad \therefore \frac{d\{y\}}{dx}\frac{d\{y^2\}}{dy}=3x^2\ ;\quad \therefore \frac{d\{y\}}{dx}\cdot(2y)=3x^2\ ;\quad \therefore \frac{d\{y\}}{dx}=\frac{3x^2}{2y}$$

Thus The tangent line of $y^2=x^3+1$ at (a,b) is

$$y-b=\left.\frac{d\{y\}}{dx}\right|_{(x,y)=(a,b)}(x-a)=\left.\frac{3x^2}{2y}\right|_{(x,y)=(a,b)}(x-a)=\frac{3a^2}{2b}(x-a)$$

Since this line goes through $(0,-1)$, we put $(x,y)=(0,-1)$ into the tangent line:

$$-1-b=\frac{3a^2}{2b}(-a)\ ;\quad \therefore 2b(1+b)=3a^3\ ;\quad \therefore 2b+2b^2=3a^3 \qquad ①$$

At the same time, the point (a,b) is on the curve $y^2=x^3+1$. Thus a and b satisfy

$$b^2=a^3+1 \qquad ②$$

3×② - ① gives

$$b^2-2b=3\ ;\quad \therefore b^2-2b-3=0\ ;\quad \therefore (b-3)(b+1)=0\ ;\quad \therefore b=3(\because b\neq -1)$$

By putting $b=3$ into ②

$$3^2=a^3+1\ ;\quad \therefore a^3=8\ ;\quad \therefore a=2$$

64) Consider the equation $\log\sqrt{x^2+y^2}=\tan^{-1}\frac{y}{x}$. Find equations for $\dfrac{d\{y\}}{dx}$ in terms of x and y .

First of all we modify the original equation to get rid of $^{-1}$.

$$\tan^{-1}\frac{y}{x}=\log\sqrt{x^2+y^2}\ \ ;\ \ \therefore \frac{y}{x}=\tan\left(\log\sqrt{x^2+y^2}\right)\ \ ;\ \ \therefore y\cdot x^{-1}=\tan\left(\log\sqrt{x^2+y^2}\right)$$

We differentiate the equation with respect to x

$$\frac{d\{y\}}{dx}\cdot x^{-1}+y\frac{d\{x^{-1}\}}{dx}=\frac{d\left\{\tan\left(\log\sqrt{x^2+y^2}\right)\right\}}{dx}$$

$$\therefore \frac{d\{y\}}{dx}\cdot x^{-1}-y\cdot x^{-2}=\frac{\partial\left\{\tan\left(\log\sqrt{x^2+y^2}\right)\right\}}{\partial\left\{\log\sqrt{x^2+y^2}\right\}}\frac{\partial\left\{\log\sqrt{x^2+y^2}\right\}}{\partial\{x\}}$$

$$\therefore \frac{d\{y\}}{dx}\cdot x^{-1}-y\cdot x^{-2}=\frac{1}{\cos^2(\log\sqrt{x^2+y^2})}\frac{\partial\left\{\log\sqrt{x^2+y^2}\right\}}{\partial\{x\}}$$

$$\therefore \frac{d\{y\}}{dx}\cdot x^{-1}-y\cdot x^{-2}=\frac{1}{\cos^2(\log\sqrt{x^2+y^2})}\frac{\partial\left\{\log\sqrt{x^2+y^2}\right\}}{\partial\left\{\sqrt{x^2+y^2}\right\}}\frac{\partial\left\{\sqrt{x^2+y^2}\right\}}{\partial\{x\}}$$

$$\therefore \frac{d\{y\}}{dx}\cdot x^{-1}-y\cdot x^{-2}=\frac{1}{\cos^2(\log\sqrt{x^2+y^2})}\frac{1}{\sqrt{x^2+y^2}}\frac{1}{2}(x^2+y^2)^{-0.5}(2x+2y\frac{d\{y\}}{dx})$$

$$\therefore \frac{d\{y\}}{dx}\cdot x^{-1}-y\cdot x^{-2}=\frac{x+y\dfrac{d\{y\}}{dx}}{(x^2+y^2)\cos^2(\log\sqrt{x^2+y^2})}$$

$$\therefore (x^2+y^2)\cos^2(\log\sqrt{x^2+y^2})\frac{d\{y\}}{dx}\cdot x^{-1}-y\cdot x^{-2}\cdot(x^2+y^2)\cos^2(\log\sqrt{x^2+y^2})=x+y\frac{d\{y\}}{dx}$$

$$\therefore (x^{-1}\cdot(x^2+y^2)\cos^2(\log\sqrt{x^2+y^2})-y)\frac{d\{y\}}{dx}=x+y\cdot x^{-2}\cdot(x^2+y^2)\cos^2(\log\sqrt{x^2+y^2})$$

$$\therefore \frac{d\{y\}}{dx}=\frac{x+y\cdot x^{-2}\cdot(x^2+y^2)\cos^2(\log\sqrt{x^2+y^2})}{x^{-1}\cdot(x^2+y^2)\cos^2(\log\sqrt{x^2+y^2})-y} \qquad ①$$

Since

$$\cos^2(\log\sqrt{x^2+y^2})=\frac{1}{1+\tan^2(\log\sqrt{x^2+y^2})}=\frac{1}{1+y^2\cdot x^{-2}}=\frac{x^2}{x^2+y^2} \qquad ②$$

we put ② into ① as follows:

$$\frac{d\{y\}}{dx}=\frac{x+(x^2+y^2)\frac{x^2}{x^2+y^2}\cdot y\cdot x^{-2}}{(x^2+y^2)\frac{x^2}{x^2+y^2}\cdot x^{-1}-y}=\frac{x+y}{x-y}$$

65) Find all points (x,y) on the graph of $8x^2+4xy+5y^2=45$ where lines tangent to the graph at (x,y) have slope -2 .

First of all we find $\dfrac{d\{y\}}{dx}$ as follows:

$$\frac{d\left\{8x^2+4xy+5y^2\right\}}{dx}=\frac{d\{45\}}{dx}\ \ ;\ \ \therefore \frac{d\left\{8x^2\right\}}{dx}+4\frac{d\{xy\}}{dx}+\frac{d\left\{5y^2\right\}}{dx}=0$$

$$\therefore 16x+4(\frac{d\{x\}}{dx}y+x\frac{d\{y\}}{dx})+\frac{d\left\{5y^2\right\}}{dx}=0\ \ ;\ \ \therefore 16x+4(y+x\frac{d\{y\}}{dx})+10y\frac{d\{y\}}{dx}=0$$

$$\therefore (10y+4x)\frac{d\{y\}}{dx}=-16x-4y\ \ ;\ \ \therefore \frac{d\{y\}}{dx}=\frac{-8x-2y}{5y+2x}$$

As the slope is -2,

$$\frac{d\{y\}}{dx}=\frac{-8x-2y}{5y+2x}=-2\ \ ;\ \ \therefore 8x+2y=10y+4x\ \ ;\ \ \therefore 4x=8y\ \ ;\ \ \therefore x=2y$$

When we put $x=2y$ into $8x^2+4xy+5y^2=45$ we obtain

$$8(2y)^2+4(2y)y+5y^2=45\ \ ;\ \ \therefore 32y^2+8y^2+5y^2=45\ \ ;\ \ \therefore 45y^2=45\ \ ;\ \ \therefore y=\pm1$$

As $x=2y$, the answer is $(2,1)$ and $(-2,-1)$.

66) Consider the graph of $x^4+2x^2+y^3-y=0$. Find the locations of local minimum and the local maximum.

First of all we find $\dfrac{d\{y\}}{dx}$ by differentiating the equation with respect to x.

$$\frac{d\{x^4+2x^2+y^3-y\}}{dx}=0\ ;\quad \therefore\ \frac{d\{x^4\}}{dx}+\frac{d\{2x^2\}}{dx}+\frac{d\{y^3\}}{dx}-\frac{d\{y\}}{dx}=0$$
$$\therefore 4x^3+4x+3y^2\frac{d\{y\}}{dx}-\frac{d\{y\}}{dx}=0\ ;\quad \therefore 4x^3+4x+3y^2\frac{d\{y\}}{dx}-\frac{d\{y\}}{dx}=0$$
$$\therefore (3y^2-1)\frac{d\{y\}}{dx}=-4x^3-4x\ ;\quad \therefore\ \frac{d\{y\}}{dx}=\frac{-4x^3-4x}{3y^2-1}$$

When $\dfrac{d\{y\}}{dx}=0$, $-4x^3-4x=0$.

$$-4x^3-4x=0\ ;\quad \therefore x^3+x=0\ ;\quad \therefore x(x^2+1)=0\ ;\quad \therefore x=0$$

When $x=0$ is put into $x^4+2x^2+y^3-y=0$,

$$0^4+2\cdot 0^2+y^3-y=0\ ;\quad \therefore y^3-y=0\ ;\quad \therefore y(y^2-1)=0\ ;\quad \therefore y=0,\pm 1$$

Thus $(x,y)=(0,0),(0,1),(0,-1)$ are the local minimum or maximum. In order to find the nature of these three points, we will find $\dfrac{d^2y}{dx^2}$

$$\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dx}=\frac{\frac{d\{-4x^3-4x\}}{dx}(3y^2-1)-(-4x^3-4x)\frac{d\{3y^2-1\}}{dx}}{(3y^2-1)^2}$$
$$=\frac{(-12x^2-4)(3y^2-1)+(4x^3+4x)(6y\frac{d\{y\}}{dx})}{(3y^2-1)^2}=\frac{(-12x^2-4)(3y^2-1)+(4x^3+4x)\cdot 6y\frac{-4x^3-4x}{3y^2-1}}{(3y^2-1)^2}$$
$$=\frac{(-12x^2-4)(3y^2-1)+(4x^3+4x)\cdot 6y\frac{-4x^3-4x}{3y^2-1}}{(3y^2-1)^2}=\frac{(-12x^2-4)(3y^2-1)^2-6y(4x^3+4x)^2}{(3y^2-1)^3}$$

Therefore when we put $(x,y)=(0,0),(0,1),(0,-1)$ into $\dfrac{d\left\{\frac{d\{y\}}{dx}\right\}}{dx}$,

$$\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dx}\Big|_{(x,y)=(0,0)}=\frac{(-4)(-1)^2}{(-1)^3}=4>0$$

$$\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dx}\Big|_{(x,y)=(0,1)}=\frac{(-4)(3-1)^2}{(3-1)^3}=-2<0$$

$$\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dx}\Big|_{(x,y)=(0,-1)}=\frac{(-4)(3-1)^2}{(3-1)^3}=-2<0$$

Therefore $(x,y)=(0,0)$ is the local minimum and $(x,y)=(0,\pm 1)$ are the local maximum.

67) Find all points (x,y) on the graph of $x^2+xy+y^2=3$ where $\dfrac{d\{y\}}{dx}=0$.

First of all we find $\dfrac{d\{y\}}{dx}$ as follows:

$$\frac{d\{x^2+xy+y^2\}}{dx}=\frac{d\{3\}}{dx}\ ;\quad \therefore\ \frac{d\{x^2\}}{dx}+\frac{d\{xy\}}{dx}+\frac{d\{y^2\}}{dx}=0$$
$$\therefore 2x+\frac{d\{x\}}{dx}y+x\frac{d\{y\}}{dx}+2y\frac{d\{y\}}{dx}=0\ ;\quad \therefore 2x+y+x\frac{d\{y\}}{dx}+2y\frac{d\{y\}}{dx}=0$$
$$\therefore (x+2y)\frac{d\{y\}}{dx}=-(2x+y)\ ;\quad \therefore\ \frac{d\{y\}}{dx}=-\frac{2x+y}{x+2y}$$

When $\dfrac{d\{y\}}{dx}=0$, we obtain $2x+y=0$. When we substitute $y=-2x$ in $x^2+xy+y^2=3$

$$x^2+x(-2x)+(-2x)^2=3\ ;\quad \therefore x^2-2x^2+4x^2=3\ ;\quad \therefore 3x^2=3\ ;\quad \therefore x=\pm 1$$

When we put $x=-1$ into $x^2+xy+y^2=3$,

$$(-1)^2+(-1)\cdot y+y^2=3\ ;\quad \therefore 1-y+y^2=3\ ;\quad \therefore y^2-y-2=0\ ;\quad \therefore y=\frac{1\pm\sqrt{1-4(-2)}}{2}$$
$$\therefore y=\frac{1\pm\sqrt{1+8}}{2}\ ;\quad \therefore y=\frac{1\pm 3}{2}\ ;\quad \therefore y=2,-1$$

When we put $x = 1$ into $x^2 + xy + y^2 = 3$,

$$1^2 + y + y^2 = 3 \ ; \quad \therefore 1 + y + y^2 = 3 \ ; \quad \therefore y^2 + y - 2 = 0 \ ; \quad \therefore y = \frac{-1 \pm \sqrt{1 - 4(-2)}}{2}$$
$$\therefore y = \frac{-1 \pm \sqrt{1+8}}{2} \quad ; \quad \therefore y = \frac{-1 \pm 3}{2} \quad ; \quad \therefore y = -2, 1$$

In summary we find $(-1, 2)$, $(-1, -1)$, $(1, -2)$, $(1, 1)$. However, $(-1, -1)$ and $(1, 1)$ do not satisfy $2x + y = 0$. Therefore the answer is $(-1, 2)$ and $(1, -2)$.

68) Find $\frac{d\{f(x,y)\}}{dx}$ and $\frac{d\{f(x,y)\}}{dy}$ of $f(x,y) = x^3y^2$ and find $\frac{d\{y\}}{dx}$ and $\frac{d\{x\}}{dy}$ of $x^3y^2 = 1$ when $xy \neq 0$.

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{x^3y^2\}}{dx} = \frac{d\{x^3\}}{dx}y^2 + x^3\frac{d\{y^2\}}{dx} = y^2\frac{d\{x^3\}}{dx} + x^3 \cdot 0 = y^2 \cdot (3x^2)$$
$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{x^3y^2\}}{dy} = \frac{d\{x^3\}}{dy}y^2 + x^3\frac{d\{y^2\}}{dy} = 0 \cdot y^2 + x^3\frac{d\{y^2\}}{dy} = x^3 \cdot (2y)$$
$$\frac{d\{x^3y^2\}}{dx} = \frac{d\{1\}}{dx} \ ; \quad \therefore \frac{d\{x^3\}}{dx}y^2 + x^3\frac{d\{y^2\}}{dx} = 0 \ ; \quad \therefore 3x^2y^2 + x^3\frac{d\{y\}}{dx}\frac{d\{y^2\}}{dy} = 0$$
$$\therefore 3x^2y^2 + x^3\frac{d\{y\}}{dx} \cdot (2y) = 0 \ ; \quad \therefore x^3\frac{d\{y\}}{dx} \cdot (2y) = -3x^2y^2 \ ; \quad \therefore \frac{d\{y\}}{dx} = -\frac{3x^2y^2}{2yx^3} = -\frac{3y}{2x}$$
$$\frac{d\{x^3y^2\}}{dy} = \frac{d\{1\}}{dy} \ ; \quad \therefore \frac{d\{x^3\}}{dy}y^2 + x^3\frac{d\{y^2\}}{dy} = 0 \ ; \quad \therefore \frac{d\{x\}}{dy}\frac{d\{x^3\}}{dx}y^2 + x^3\frac{d\{y^2\}}{dy} = 0$$
$$\therefore \frac{d\{x\}}{dy} \cdot (3x^2) \cdot y^2 + x^3 \cdot (2y) = 0 \ ; \quad \therefore \frac{d\{x\}}{dy} \cdot (3x^2) \cdot y^2 = -x^3 \cdot (2y) \ ; \quad \therefore \frac{d\{x\}}{dy} = -\frac{x^3 \cdot (2y)}{3x^2y^2} = -\frac{2x}{3y}$$

69) Find $\frac{d\{f(x,y)\}}{dx}$ and $\frac{d\{f(x,y)\}}{dy}$ of $f(x,y) = x^3 + y^2$ and find $\frac{d\{y\}}{dx}$ and $\frac{d\{x\}}{dy}$ of $x^3 + y^2 = 1$ when $xy \neq 0$.

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{x^3 + y^2\}}{dx} = \frac{d\{x^3\}}{dx} + \frac{d\{y^2\}}{dx} = 3x^2$$
$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{x^3 + y^2\}}{dy} = \frac{d\{x^3\}}{dy} + \frac{d\{y^2\}}{dy} = 2y$$
$$\frac{d\{x^3 + y^2\}}{dx} = \frac{d\{1\}}{dx} \ ; \quad \therefore \frac{d\{x^3 + y^2\}}{dx} = \frac{d\{1\}}{dx} \ ; \quad \therefore \frac{d\{x^3\}}{dx} + \frac{d\{y^2\}}{dx} = 0$$
$$\therefore 3x^2 + \frac{d\{y\}}{dx}\frac{d\{y^2\}}{dy} = 0 \ ; \quad \therefore \frac{d\{y\}}{dx} \cdot (2y) = -3x^2 \ ; \quad \therefore \frac{d\{y\}}{dx} = -\frac{3x^2}{2y}$$
$$\frac{d\{x^3 + y^2\}}{dy} = \frac{d\{1\}}{dy} \ ; \quad \therefore \frac{d\{x^3\}}{dy} + \frac{d\{y^2\}}{dy} = 0 \ ; \quad \therefore \frac{d\{x\}}{dy}\frac{d\{x^3\}}{dx} + 2y = 0$$
$$\therefore \frac{d\{x\}}{dy} \cdot (3x^2) = -2y \ ; \quad \therefore \frac{d\{x\}}{dy} = -\frac{2y}{3x^2}$$

70) Find $\frac{d\{f(x,y)\}}{dy}$ and $\frac{d\{f(x,y)\}}{dx}$ of $f(x,y) = 2 + y + 2x$ and find $\frac{d\{x\}}{dy}$ and $\frac{d\{y\}}{dx}$ of $2 + y + 2x = 0$

As for $\frac{d\{f(x,y)\}}{dy}$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{2 + y + 2x\}}{dy} = \frac{d\{2\}}{dy} + \frac{d\{y\}}{dy} + \frac{d\{2x\}}{dy} = 0 + 1 + 0 = 1$$

As for $\frac{d\{f(x,y)\}}{dx}$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{2 + y + 2x\}}{dx} = \frac{d\{2\}}{dx} + \frac{d\{y\}}{dx} + \frac{d\{2x\}}{dx} = 0 + 0 + 2 = 2$$

As for $\frac{d\{x\}}{dy}$

$$\frac{d\{2 + y + 2x\}}{dy} = \frac{d\{0\}}{dy} \ ; \quad \therefore \frac{d\{2\}}{dy} + \frac{d\{y\}}{dy} + \frac{d\{2x\}}{dy} = \frac{d\{0\}}{dy} \ ; \quad \therefore 0 + 1 + 2\frac{d\{x\}}{dy} = 0$$
$$\therefore 2\frac{d\{x\}}{dy} = -1 \ ; \quad \therefore \frac{d\{x\}}{dy} = -\frac{1}{2}$$

As for $\frac{d\{y\}}{dx}$

$$\frac{d\{2 + y + 2x\}}{dx} = \frac{d\{0\}}{dx} \ ; \quad \therefore \frac{d\{2\}}{dx} + \frac{d\{y\}}{dx} + \frac{d\{2x\}}{dx} = \frac{d\{0\}}{dx} \ ; \quad \therefore 0 + \frac{d\{y\}}{dx} + 2 = 0 \ ; \quad \therefore \frac{d\{y\}}{dx} = -2$$

DAY4

71) Simplify $\frac{\ln x^9}{18}$

$$\frac{\ln x^9}{18} = \frac{9 \ln x}{18} = \frac{1}{2} \ln x$$

72) Simplify $4^5 + 4^9$

$$4^5 + 4^9 = 4^5(1 + 4^4) = 257 \cdot 4^5$$

73) Simplify $(x^2)^7$.

$$(x^2)^7 = x^{2\times7} = x^{14}$$

74) By using the Newton-Raphson method, find that root of the equation $f(x) = x^3 - 2x^2 + 3x - 4$ which is near to $x = 2$. Give your answer correct to 3 significant figures.

$$\frac{d\{f(x)\}}{dx} = 3x^2 - 4x + 3$$

Now we set $x_0 = 2$ and we find

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - 0.285714 = 1.71429$$

In the same way we find the next guess as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.71429 - 0.0611445 = 1.65315$$

In the same way we find the next guess as

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.65315 - 0.00251671 = 1.65063$$

Clearly x_3 agrees with x_2 to 3 significant figures. Thus the root is 1.65 to 3 significant figures. If you find the next guess, you will get

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.65063 - 8.0856 \cdot 10^{-7} = 1.65063$$

75) By using the Newton-Raphson method, find that root of the equation $f(x) = \sin(2x) - x + 1$ which is near to $x = 2$. Give your answer correct to 4 significant figures.

$$\frac{d\{f(x)\}}{dx} = 2\cos(2x) - 1$$

Now we set $x_0 = 2$ and we find

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - 0.761415 = 1.23859$$

In the same way we find the next guess as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.23859 + 0.146824 = 1.38541$$

In the same way we find the next guess as

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.38541 - 0.0080564 = 1.37735$$

In the same way we find the next guess as

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.37735 - 1.31228 \cdot 10^{-5} = 1.37734$$

Clearly x_4 agrees with x_3 to 4 significant figures. Thus the root is 1.377 to 4 significant figures.

76) By using the Newton-Raphson method, find that root of the equation $f(x) = \ln(x + 3) - 1$ which is near to $x = -1$. Give your answer correct to 5 significant figures.

$$\frac{d\{f(x)\}}{dx} = \frac{1}{x+3}$$

Now we set $x_0 = -1$ and we find

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1 + 0.613706 = -0.386294$$

In the same way we find the next guess as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -0.386294 + 0.102538 = -0.283756$$

In the same way we find the next guess as

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -0.283756 + 0.00203706 = -0.281719$$

In the same way we find the next guess as

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = -0.281719 - 8.28459 \cdot 10^{-7} = -0.281718$$

Clearly x_4 agrees with x_3 to 5 significant figures. Thus the root is -0.28171 to 5 significant figures.

77) By using the Newton-Raphson method, find that root of the equation $f(x) = \frac{\sin(x)}{x} + 0.1$ which is near to $x = 5$. Give your answer correct to 6 significant figures.

$$\frac{d\{f(x)\}}{dx} = \frac{x\cos(x) - \sin(x)}{x^2}$$

Now we set $x_0 = 5$ and we find

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 5 + 0.965248 = 5.96525$$

In the same way we find the next guess as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 5.96525 - 0.283271 = 5.68198$$

In the same way we find the next guess as

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 5.68198 - 0.0027712 = 5.67921$$

In the same way we find the next guess as

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 5.67921 - 2.20368 \cdot 10^{-6} = 5.67921$$

Clearly x_4 agrees with x_3 to 6 significant figures. Thus the root is 5.67921 to 6 significant figures.

78) Evaluate the limit $\lim_{x\to 2\pi} \frac{x\cos(\frac{x}{2} - \frac{2\pi}{3}) - \pi}{\sin\left(\frac{x}{2}\right)}$

When $x = 2\pi$, the numerator of the fraction becomes

$$x\cos(\frac{x}{2} - \frac{2\pi}{3}) - \pi\Big|_{x=2\pi} = 2\pi\cos(\pi - \frac{2\pi}{3}) - \pi = 2\pi\cos(\frac{\pi}{3}) - \pi = 2\pi\frac{1}{2} - \pi = 0$$

and the denominator becomes

$$\sin\left(\frac{x}{2}\right)\Big|_{x=2\pi} = \sin(\pi) = 0.$$

Therefore we use L'Hôpitals rule as follows.

$$\lim_{x\to 2\pi} \frac{x\cos(\frac{x}{2} - \frac{2\pi}{3}) - \pi}{\sin\left(\frac{x}{2}\right)} = \left.\frac{\frac{d\left\{x\cos(\frac{x}{2} - \frac{2\pi}{3}) - \pi\right\}}{dx}}{\frac{d\left\{\sin\left(\frac{x}{2}\right)\right\}}{dx}}\right|_{x=2\pi} = \left.\frac{\cos(\frac{x}{2} - \frac{2\pi}{3}) - \frac{x}{2}\sin(\frac{x}{2} - \frac{2\pi}{3})}{\frac{1}{2}\cos\left(\frac{x}{2}\right)}\right|_{x=2\pi}$$

$$= \frac{\cos(\pi - \frac{2\pi}{3}) - \pi\sin(\pi - \frac{2\pi}{3})}{\frac{1}{2}\cos(\pi)} = \frac{\frac{1}{2} - \frac{\sqrt{3}\pi}{2}}{-\frac{1}{2}} = \sqrt{3}\pi - 1$$

79) Evaluate the limit $\lim_{x\to 12} \frac{x^3 - 35x^2 + 366x - 1080}{-\frac{x^2}{2} + 10x - 48}$

When $x = 12$, we find

$$x^3 - 35x^2 + 366x - 1080\big|_{x=12} = 1728 - 5040 + 4392 - 1080 = 0$$

and

$$-\frac{x^2}{2} + 10x - 48\Big|_{x=12} = -72 + 120 - 48 = 0$$

Therefore we use L'Hôpitals rule as follows.

$$\lim_{x\to 12}\frac{x^3-35x^2+366x-1080}{-\frac{x^2}{2}+10x-48}=\left.\frac{\frac{d\left\{x^3-35x^2+366x-1080\right\}}{dx}}{\frac{d\left\{-\frac{x^2}{2}+10x-48\right\}}{dx}}\right|_{x=12}$$
$$=\left.\frac{3x^2-70x+366}{-x+10}\right|_{x=12}=\frac{3\cdot 12^2-70\cdot 12+366}{-12+10}=\frac{-42}{-2}=21$$

80) Evaluate the limit $\lim_{x\to 11}\frac{\log_{10}\left(\sqrt{x^2-21}\right)-1}{\mathfrak{e}^{x-10}-\mathfrak{e}}$

When $x=11$, we find

$$\left.\log_{10}\left(\sqrt{x^2-21}\right)-1\right|_{x=11}=\log_{10}\left(\sqrt{121-21}\right)-1=\log_{10}\left(\sqrt{100}\right)-1=\log_{10}10-1=1-1=0$$

and

$$\left.\mathfrak{e}^{x-10}-\mathfrak{e}\right|_{x=11}=\mathfrak{e}^{11-10}-\mathfrak{e}=\mathfrak{e}-\mathfrak{e}=0$$

Therefore we use L'Hôpitals rule as follows.

$$\lim_{x\to 11}\frac{\log_{10}\left(\sqrt{x^2-21}\right)-1}{\mathfrak{e}^{x-10}-\mathfrak{e}}=\left.\frac{\frac{d\left\{\log_{10}\left(\sqrt{x^2-21}\right)-1\right\}}{dx}}{\frac{d\left\{\mathfrak{e}^{x-10}-\mathfrak{e}\right\}}{dx}}\right|_{x=11}=\left.\frac{\frac{d\left\{\frac{\ln\sqrt{x^2-21}}{\ln 10}-1\right\}}{dx}}{\mathfrak{e}^{x-10}}\right|_{x=11}$$
$$=\left.\frac{\frac{1}{\ln 10}\frac{d\left\{\ln\sqrt{x^2-21}-1\right\}}{dx}}{\mathfrak{e}^{x-10}}\right|_{x=11}=\left.\frac{\frac{1}{\ln 10}\frac{d\left\{\ln\sqrt{x^2-21}\right\}}{dx}}{\mathfrak{e}^{x-10}}\right|_{x=11}=\left.\frac{\frac{1}{\ln 10}\frac{\ln\sqrt{x^2-21}}{d\sqrt{x^2-21}}\times\frac{d\sqrt{x^2-21}}{dx}}{\mathfrak{e}^{x-10}}\right|_{x=11}$$
$$=\left.\frac{\frac{1}{\ln 10}\frac{\ln\sqrt{x^2-21}}{d\sqrt{x^2-21}}\times\frac{d\sqrt{x^2-21}}{d(x^2-21)}\times\frac{d(x^2-21)}{dx}}{\mathfrak{e}^{x-10}}\right|_{x=11}=\left.\frac{\frac{1}{\ln 10}\frac{1}{\sqrt{x^2-21}}\times\frac{1}{2}\times\frac{1}{\sqrt{(x^2-21)}}\times(2x)}{\mathfrak{e}^{x-10}}\right|_{x=11}$$
$$=\left.\frac{\frac{x}{(x^2-21)\ln 10}}{\mathfrak{e}^{x-10}}\right|_{x=11}=\frac{\frac{11}{(121-21)\ln 10}}{\mathfrak{e}^{11-10}}=\frac{\frac{11}{(100)\ln 10}}{\mathfrak{e}}=\frac{11}{100\mathfrak{e}\ln 10}$$

81) Evaluate the limit $\lim_{x\to 7}\frac{-x^2+11x-28}{-x^3+18x^2-95x+126}$

When $x=7$, we find

$$\left.-x^2+11x-28\right|_{x=7}=-49+11\times 7-28=-49+77-28=0$$

and

$$\left.-x^3+18x^2-95x+126\right|_{x=7}=-7^3+18\times 7^2-95\times 7+126$$
$$=-343+882-665+126=1008-1008=0$$

Therefore we use the L'Hôpitals rule as follows.

$$\lim_{x\to 7}\frac{-x^2+11x-28}{-x^3+18x^2-95x+126}=\left.\frac{\frac{d\left\{-x^2+11x-28\right\}}{dx}}{\frac{d\left\{-x^3+18x^2-95x+126\right\}}{dx}}\right|_{x=7}=\left.\frac{-2x+11}{-3x^2+36x-95}\right|_{x=7}$$
$$=\frac{-2\times 7+11}{-3\times 7^2+36\times 7-95}=\frac{-14+11}{-3\times 49+36\times 7-95}=\frac{-3}{-147+252-95}=\frac{-3}{10}=-\frac{3}{10}$$

XIII. Exercises on Integrals

integralsem1all.tex

1) **DAY1**
2) What is $\frac{1}{4}+\frac{7}{12}+\frac{0}{5}$.

$$\frac{1}{4}+\frac{7}{12}+\frac{0}{5}=\frac{1\times 3}{4\times 3}+\frac{7}{12}=\frac{3}{12}+\frac{7}{12}=\frac{10}{12}=\frac{5}{6}$$

3) Solve $2\log_4(x)+3=0$
We work under the condition of $x>0$.

$$2\log_4(x)+3=0$$
$$\therefore \log_4(x)=-\frac{3}{2}=-\frac{3}{2}\log_4 4=\log_4(4^{-\frac{3}{2}})=\log_4\left((2^2)^{-\frac{3}{2}}\right)=\log_4\left(2^{2\times(-\frac{3}{2})}\right)=\log_4\left(2^{-3}\right)=\log_4\left(\frac{1}{8}\right)$$
$$\therefore x=\frac{1}{8}$$

4) Calculate $\frac{1}{2}a-0$ for $a=2$

$$\frac{1}{2}a-0=\frac{1}{2}a;\therefore \frac{1}{2}\cdot 2=\frac{2}{2}=1$$

5) Rearrange $t=\sqrt[4]{x}$ to make x the subject.

$$t=\sqrt[4]{x};\therefore t^4=x$$

6) Find the $\int x^0 dx$.

$$x^0=1;\therefore \int 1\cdot dx=x+c$$

7) Find the $\int \frac{1}{2x}dx$.

$$\int \frac{1}{2x}dx=\frac{1}{2}\int \frac{1}{x}dx=\frac{1}{2}\ln|x|+c$$

8) Find the $\int \frac{1}{-x}dx$.

$$\int \frac{1}{-x}dx=-\int \frac{1}{x}dx=-\ln|x|+c$$

9) Find the $\int \frac{1}{-x^2}dx$.

$$\int \frac{1}{-x^2}dx=-\int \frac{1}{x^2}dx=-\int x^{-2}dx=-\frac{1}{-2+1}x^{-2+1}+c=-\frac{1}{-1}x^{-1}+c=x^{-1}+c=\frac{1}{x}+c$$

10) Find the $\int -x^2dx$.

$$\int -x^2dx=-\int x^2dx=-\frac{1}{2+1}x^{2+1}+c=-\frac{1}{3}x^3+c$$

11)
12) Find the $\int \frac{1}{x+1}dx$.

$$\int \frac{1}{x+1}dx$$
$$=\ln|x+1|+c$$

13) Find the $\int \frac{1}{2x+1}dx$.

- First method

$$\int \frac{1}{2x+1}dx$$
$$=\int \frac{1}{2(x+\frac{1}{2})}dx=\frac{1}{2}\int \frac{1}{x+\frac{1}{2}}dx$$
$$=\frac{1}{2}\ln\left|x+\frac{1}{2}\right|+c \qquad ①$$

- Second method If we let $u = 2x+1$, then we have $du = 2dx$ and the integral becomes

$$\int \frac{1}{2x+1}dx$$
$$= \int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\int \frac{1}{u}du$$
$$= \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|2x+1| + c \qquad ②$$

Although you seem to be getting the different results from these two methods, they are identical from the view point of integration, taking into account "c". When you furthermore manipulate ②,

$$\frac{\ln|2x+1|}{2} + c = \frac{\ln\left|2(x+\frac{1}{2})\right|}{2} + c = \frac{\ln 2 + \ln\left|x+\frac{1}{2}\right|}{2} + c$$
$$= \frac{\ln 2}{2} + \frac{\ln\left|x+\frac{1}{2}\right|}{2} + c = \frac{\ln\left|x+\frac{1}{2}\right|}{2} + \frac{\ln 2}{2} + c = \frac{\ln\left|x+\frac{1}{2}\right|}{2} + C \qquad ③$$

where $C = c + \frac{\ln 2}{2}$ and ③ and ① are mathematically identical. Therefore ① and ② are both correct. When you have a definite integral, from the first method, you get

$$\int_a^b \frac{1}{2x+1}dx = \left[\frac{1}{2}\ln\left|x+\frac{1}{2}\right|\right]_a^b = \frac{\ln\left|b+\frac{1}{2}\right| - \ln\left|a+\frac{1}{2}\right|}{2} = \frac{\ln\left|\frac{b+\frac{1}{2}}{a+\frac{1}{2}}\right|}{2} = \frac{\ln\left|\frac{2b+1}{2a+1}\right|}{2} \qquad ④$$

From the second method

$$\int_a^b \frac{1}{2x+1}dx = \left[\frac{1}{2}\ln|2x+1|\right]_a^b = \frac{\ln|2b+1| - \ln|2a+1|}{2} = \frac{\ln\left|\frac{2b+1}{2a+1}\right|}{2} \qquad ⑤$$

④ and ⑤ are identical.

14) Find the $\int \frac{1}{1-2x}dx$.

$$\int \frac{1}{1-2x}dx = \int \frac{1}{-2(x-\frac{1}{2})}dx = \frac{1}{-2}\int \frac{1}{x-\frac{1}{2}}dx = -\frac{1}{2}\ln|x-\frac{1}{2}| + c$$

15) Find the $\int \sin(-t)dt$.

$$\int \sin(-t)dt = -\frac{1}{-1}\cos(-t) + c = \cos(-t) + c = \cos(t) + c$$

16) Find the $\int \cos(-t)dt$.

$$\int \cos(-t)dt = \frac{1}{-1}\sin(-t) + c = -\sin(-t) + c = \sin(t) + c$$

17) Find the $\int \tan(-t)dt$.

$$\int \tan(-t)dt = -\frac{1}{-1}\ln|\cos(-t)| + c = \ln|\cos(t)| + c$$

18) Find the $\int \mathfrak{e}^{-\theta}d\theta$.

$$\int \mathfrak{e}^{-\theta}\theta = \frac{1}{-1}\mathfrak{e}^{-\theta} + c = -\mathfrak{e}^{-\theta} + c$$

19) Find the $\int 3^{-\theta}d\theta$.

$$\int 3^{-\theta}d\theta = \frac{3^{-\theta}}{(-1)\cdot\ln(3)} + c = -\frac{3^{-\theta}}{\ln(3)} + c$$

20) Find the $\int \cos^2(-t)dt$.

$$\int \cos^2(-t)dt = \int (\cos(-t))^2\,dt = \int (\cos(t))^2\,dt = \int \cos^2(t)dt = \int \frac{1+\cos(2t)}{2}dt$$
$$= \frac{1}{2}\int 1+\cos(2t)dt = \frac{1}{2}(t + \frac{1}{2}\sin(2t)) + c = \frac{t+\sin(t)\cos(t)}{2} + c$$

21) Find the $\int \sin^2(-\theta)d\theta$.

$$\int \sin^2(-\theta)d\theta = \int (\sin(-\theta))^2\, d\theta = \int (-\sin(\theta))^2\, d\theta = \int (\sin(\theta))^2\, d\theta = \int \sin^2(\theta)d\theta$$
$$= \int \frac{1-\cos(2\theta)}{2}d\theta = \frac{1}{2}\int 1-\cos(2\theta)d\theta = \frac{1}{2}(\theta - \frac{1}{2}\sin(2\theta)) + c = \frac{\theta - \sin(\theta)\cos(\theta)}{2} + c$$

22) Find the $\int \ln(2\theta)d\theta$.

$$\int \ln(2\theta)d\theta = \theta\ln(2\theta) - \theta + c$$

23) Find the $\int \frac{1}{\cos^2(-\theta)}d\theta$.

$$\int \frac{1}{\cos^2(-\theta)}d\theta = \frac{\tan(-\theta)}{-1} + c = -\tan(-\theta) + c = -(-\tan(\theta)) + c = \tan(\theta) + c$$

24) Find the $\int \frac{1}{\sin^2(-\theta)}d\theta$.

$$\int \frac{1}{\sin^2(-\theta)}d\theta = -\frac{1}{(-1)\tan(-\theta)} + c = \frac{1}{\tan(-\theta)} + c = \frac{1}{-\tan(\theta)} + c = -\frac{1}{\tan(\theta)} + c$$

25) Find $\int 12dx$ and $\int 12dt$ and $\int 12xtdx$ and $\int 12xtdt$

$$\int 12dx = 12\int dx = 12\int x^0 dx = 12\cdot\frac{1}{1}x^1 + c = 12x + c$$

In the same way

$$\int 12dt = 12\int dt = 12\int t^0 dt = 12\cdot\frac{1}{1}t^1 + c = 12t + c$$

$$\int 12xtdx = 12t\int xdx = 12t\frac{1}{2}x^2 + c = 6tx^2 + c$$

$$\int 12xtdt = 12x\int tdt = 12x\frac{1}{2}t^2 + c = 6xt^2 + c$$

26) Find

$$\int 12t^2x^3dx \quad \text{and} \quad \int 12t^2x^3dt$$

In case $\int 12t^2x^3dx$, we see $12t^2x^3$ as a function of x. Therefore

$$\int 12t^2x^3dx = 12t^2\int x^3dx = 12t^2\cdot\frac{1}{4}x^4 = 3t^2x^4 + c.$$

In the same way,

$$\int 12t^2x^3dt = 12x^3\int t^2dt = 12x^3\cdot\frac{1}{3}t^3 = 4t^3x^3 + c.$$

27) Find

$$\int (\sqrt{t} + \sin x)dt \quad \text{and} \quad \int (\sqrt{t} + \sin x)dx$$

$$\int (\sqrt{t} + \sin x)dt = \int \sqrt{t}dt + \int \sin xdt = \int t^{\frac{1}{2}}dt + \sin x\int dt = \frac{1}{1+\frac{1}{2}}t^{1+\frac{1}{2}} + t\sin x + c$$
$$= \frac{1}{\frac{3}{2}}t^{1+\frac{1}{2}} + t\sin x + c = \frac{2}{3}t^{\frac{3}{2}} + t\sin x + c$$

$$\int (\sqrt{t} + \sin x)dx = \int \sqrt{t}dx + \int \sin xdx = \sqrt{t}\int dx + \int \sin xdx = \sqrt{t}\cdot x - \cos x + c$$

28)

a) Find $p(x)$ and $q(x)$ when

$$p(x) = \int_0^1 q(t)(3t - x/2)dt,$$

$$q(x) = -2x + \int_0^2 p(t)dt.$$

When you find a function which reduces its order by differentiation, then set the function to $f(x)$.
When you find a function which does not change its order by integration, then set the function to $g(x)$.
Let

$$a = \int_0^2 p(t)dt,$$

then

$$q(x) = -2x + a.$$

Thus

$$\begin{aligned} p(x) &= \int_0^1 q(t)(3t - x/2)dt \\ &= \int_0^1 (-2t + a)(3t - x/2)dt \\ &= \int_0^1 (-6t^2 + tx + 3at - ax/2)dt \\ &= \left[-2t^3 + (x + 3a)\frac{t^2}{2} - axt/2 \right]_0^1 \\ &= -2 + x/2 + 3a/2 - ax/2 \\ &= -2 + \frac{3a}{2} + \frac{(1-a)x}{2} \end{aligned}$$

We now put

$$p(x) = -2 + \frac{3a}{2} + \frac{(1-a)x}{2}$$

into

$$a = \int_0^2 p(t)dt.$$

$$\begin{aligned} a &= \int_0^2 p(t)dt \\ &= \int_0^2 \left(-2 + \frac{3a}{2} + \frac{(1-a)t}{2} \right) dt \\ &= \left[(-2 + \frac{3a}{2})t + \frac{(1-a)t^2}{4} \right]_0^2 \\ &= -4 + 3a + 1 - a \\ &= -3 + 2a \\ \therefore a &= -3 + 2a \\ \therefore 3 &= a \end{aligned}$$

Therefore,

$$\begin{aligned} p(x) &= -2 + \frac{3a}{2} + \frac{(1-a)x}{2} \\ &= -2 + \frac{9}{2} + \frac{(1-3)x}{2} \\ &= \frac{9-4}{2} + \frac{-2x}{2} \\ &= \frac{5}{2} - x \end{aligned}$$

and

$$q(x) = x + a = x + 3$$

b) Find $p(x)$ and $q(x)$ when

$$p(x) = \int_0^1 x \cdot q(t)dt,$$

$$q(x) = x + \int_0^1 p(t)dt.$$

Let

$$a = \int_0^1 p(t)dt,$$

then

$$q(x) = x + a.$$

Thus

$$\begin{aligned} p(x) &= \int_0^1 x \cdot q(t)dt \\ &= \int_0^1 x(t+a)dt \\ &= x\int_0^1 (t+a)dt \\ &= x\left[\frac{1}{2}t^2 + at\right]_0^1 \\ &= x\left[\frac{1}{2} + a\right] \end{aligned}$$

We now put

$$p(x) = x\left(a + \frac{1}{2}\right)$$

into

$$a = \int_0^1 p(t)dt.$$

$$\begin{aligned} a &= \int_0^1 p(t)dt \\ &= \int_0^1 t\left(a + \frac{1}{2}\right)dt \\ &= \left(a + \frac{1}{2}\right)\int_0^1 tdt \\ &= \left(a + \frac{1}{2}\right)[\frac{1}{2}t^2]_0^1 \\ &= \frac{1}{2}\left(a + \frac{1}{2}\right) \\ \therefore 2a &= a + \frac{1}{2} \\ \therefore a &= \frac{1}{2} \end{aligned}$$

Therefore,

$$p(x) = x\left[\frac{1}{2} + a\right] = x\left[\frac{1}{2} + \frac{1}{2}\right] = x$$

and

$$q(x) = x + a = x + \frac{1}{2}$$

DAY2

29) Evaluate $f(2) - f(0)$ when $f(x) = \ln|1+x|$

$$\begin{aligned} &\ln|1+2| - \ln|1+0| \\ &= \ln|3| - \ln|1| \\ &= \ln|3| \because \ln|1| = 0 \end{aligned}$$

30) Rearrange $2x + 7 = \frac{y+8}{y-4}$ to make y the subject.

$$\begin{aligned} 2x + 7x &= \frac{y+8}{y-4} \\ \therefore (2x+7)(y-4) &= y+8 \\ \therefore 2xy + 7y - 8x - 28 &= y+8 \\ \therefore 2xy + 7y - y &= 8 + 28 + 8x \\ \therefore y(2x+7-1) &= 36 + 8x \\ \therefore y &= \frac{36+8x}{2x+6} \\ \therefore y &= \frac{18+4x}{x+3} \end{aligned}$$

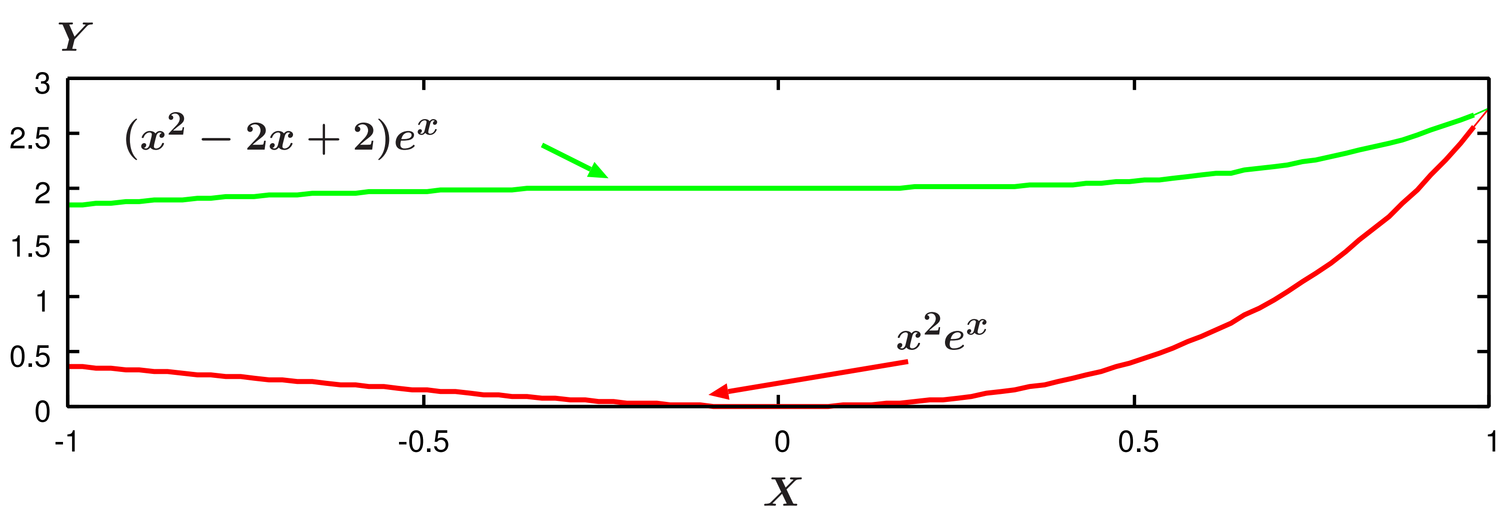
31) Solve $y = \frac{2x+3}{4x+5} + 1$ for x.

$$\begin{aligned} y &= \frac{2x+3}{4x+5} + 1 \\ \therefore y - 1 &= \frac{2x+3}{4x+5} \\ \therefore (y-1)(4x+5) &= 2x+3 \\ \therefore x(4y-4) + 5y - 5 &= 2x+3 \\ \therefore x(4y-4) - 2x &= 5 - 5y + 3 \\ \therefore x(4y-6) &= 8 - 5y \\ \therefore x &= \frac{8-5y}{4y-6} \end{aligned}$$

32) Solve for x of the following $|x+3| = 4$.

$$\begin{aligned} |x+3| &= 4 \\ \therefore x + 3 &= \pm 4 \\ \therefore x &= \pm 4 - 3 \\ \therefore x &= 4 - 3, -4 - 3 \\ \therefore x &= 1, -7 \end{aligned}$$

33) Evaluate $\int x^2 \mathfrak{e}^x \, dx$.



Usually we let $f(x)$ equal the polynomial function in this case $f(x) = x^2$ and let $g(x)$ equal the non linear function, in this case, $g(x) = \mathfrak{e}^x$.

$$\begin{aligned} &\int f(x) \cdot g(x) dx \\ &= f(x) \cdot \int g(x) dx - \int \left(\frac{d\{f(x)\}}{dx} \cdot \int g(x) dx \right) dx \\ &= x^2 \cdot \int \mathfrak{e}^x dx - \int \left(\frac{d\{x^2\}}{dx} \cdot \int \mathfrak{e}^x dx \right) dx \\ &= x^2 \cdot \mathfrak{e}^x - \int (2x \cdot \mathfrak{e}^x) dx \\ &= x^2 \cdot \mathfrak{e}^x - 2 \int (x\mathfrak{e}^x) dx \end{aligned}$$

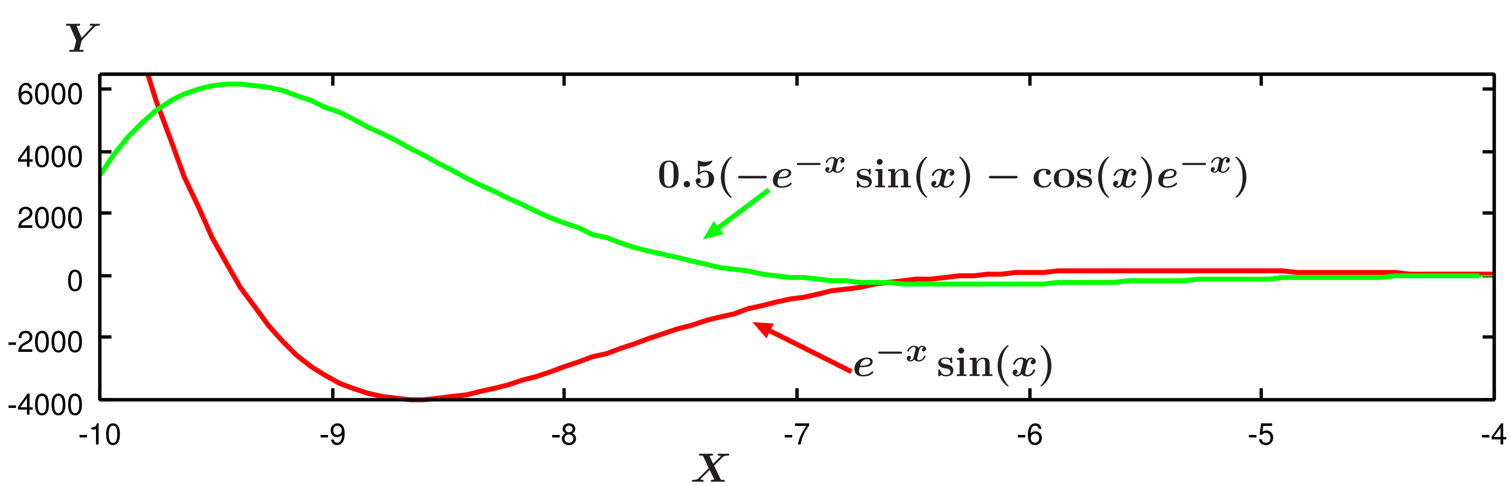
In order to find $\int (x\mathfrak{e}^x) dx$, we apply "by-parts" again as follows: This time, $f(x)$ is changed to $f(x) = x$ whilst $g(x)$ is the same as before.

$$\begin{aligned} &\int (x\mathfrak{e}^x) dx \\ &= \int f(x) \cdot g(x) dx \\ &= f(x) \cdot \int g(x) dx - \int \left(\frac{d\{f(x)\}}{dx} \cdot \int g(x) dx \right) dx \\ &= x \cdot \int \mathfrak{e}^x dx - \int \left(\frac{d\{x\}}{dx} \cdot \int \mathfrak{e}^x dx \right) dx \\ &= x \cdot \mathfrak{e}^x - \int (1 \cdot \mathfrak{e}^x) dx \\ &= x \cdot \mathfrak{e}^x - \mathfrak{e}^x + c \end{aligned}$$

By putting
$\int (x\mathfrak{e}^x) dx = x \cdot \mathfrak{e}^x - \mathfrak{e}^x$ into
$\int x^2 \mathfrak{e}^x dx = x^2 \cdot \mathfrak{e}^x - 2 \int (x\mathfrak{e}^x) dx$
we get:

$$\begin{aligned} &\int x^2 \mathfrak{e}^x dx \\ &= x^2 \cdot \mathfrak{e}^x - 2 \int (x\mathfrak{e}^x) dx \\ &= x^2 \cdot \mathfrak{e}^x - 2(x \cdot \mathfrak{e}^x - \mathfrak{e}^x + c) \\ &= (x^2 - 2x + 2)\mathfrak{e}^x + C \end{aligned}$$

34) Evaluate $\int \mathfrak{e}^{-x} \sin x dx$.



Usually we let $f(x)$ equal the polynomial function. But there is no polynomial function in this case. So we set $f(x) = \sin x$ and let $g(x)$

equal the non linear function, in this case, $g(x) = \mathfrak{e}^{-x}$.

$$\begin{aligned}
&\int \mathfrak{e}^{-x} \sin x dx \\
&= \int f(x) \cdot g(x) dx \\
&= f(x) \cdot \int g(x) dx - \int \left(\frac{d\left\{f(x)\right\}}{dx} \cdot \int g(x) dx \right) dx \\
&= \sin x \cdot (-\mathfrak{e}^{-x}) - \int \left(\cos x \cdot (-\mathfrak{e}^{-x})\right) dx \\
&= \sin x \cdot (-\mathfrak{e}^{-x}) + \int \left(\cos x \cdot \mathfrak{e}^{-x}\right) dx
\end{aligned}$$

In order to find $\int \left(\cos x \mathfrak{e}^{-x}\right) dx$, we apply "by-parts" again as follows: This time, $f(x)$ is changed to $f(x) = \cos x$ whilst $g(x)$ is the same as before.

$$\begin{aligned}
&\int \left(\cos x \cdot \mathfrak{e}^{-x}\right) dx \\
&= \int f(x) \cdot g(x) dx \\
&= f(x) \cdot \int g(x) dx - \int \left(\frac{d\left\{f(x)\right\}}{dx} \cdot \int g(x) dx \right) dx \\
&= \cos x \cdot \int \mathfrak{e}^{-x} dx - \int \left(\frac{d\left\{\cos x\right\}}{dx} \cdot \int \mathfrak{e}^{-x} dx \right) dx \\
&= \cos x \cdot (-\mathfrak{e}^{-x}) - \int \left(-\sin x \cdot (-\mathfrak{e}^{-x})\right) dx \\
&= -\cos x \mathfrak{e}^{-x} - \int \left(\sin x \cdot \mathfrak{e}^{-x}\right) dx
\end{aligned}$$

By putting
$\int \left(\cos x \cdot \mathfrak{e}^{-x}\right) dx = -\cos x \mathfrak{e}^{-x} - \int \left(\sin x \cdot \mathfrak{e}^{-x}\right) dx$ into
$\int \mathfrak{e}^{-x} \sin x dx = -\mathfrak{e}^{-x} \sin x + \int \left(\cos x \cdot \mathfrak{e}^{-x}\right) dx$
we get:

$$\begin{aligned}
\int \mathfrak{e}^{-x} \sin x dx &= -\mathfrak{e}^{-x} \sin x - \cos x \mathfrak{e}^{-x} - \int \mathfrak{e}^{-x} \sin x dx \\
\therefore 2 \int \mathfrak{e}^{-x} \sin x dx &= -\mathfrak{e}^{-x} \sin x - \cos x \mathfrak{e}^{-x} \\
\therefore \int \mathfrak{e}^{-x} \sin x dx &= \frac{-\mathfrak{e}^{-x} \sin x - \cos x \mathfrak{e}^{-x}}{2} + C
\end{aligned}$$

Alternatively, we can find a function whose differentiated function becomes $\mathfrak{e}^{-x} \sin x$.

$$\begin{aligned}
\frac{d\left\{\mathfrak{e}^{ax} \sin bx\right\}}{dx} &= a\mathfrak{e}^{ax} \sin bx + b\mathfrak{e}^{ax} \cos bx \qquad ① \\
\frac{d\left\{\mathfrak{e}^{ax} \cos bx\right\}}{dx} &= a\mathfrak{e}^{ax} \cos bx - b\mathfrak{e}^{ax} \sin bx \qquad ②
\end{aligned}$$

①$\times a$ - ②$\times b$ gives us

$$\begin{aligned}
a\frac{d\left\{\mathfrak{e}^{ax} \sin bx\right\}}{dx} - b\frac{d\left\{\mathfrak{e}^{ax} \cos bx\right\}}{dx} &= a^2 \mathfrak{e}^{ax} \sin bx + b^2 \mathfrak{e}^{ax} \sin bx \\
\therefore \frac{a\frac{d\left\{\mathfrak{e}^{ax} \sin bx\right\}}{dx} - b\frac{d\left\{\mathfrak{e}^{ax} \cos bx\right\}}{dx}}{a^2 + b^2} &= \mathfrak{e}^{ax} \sin bx
\end{aligned}$$

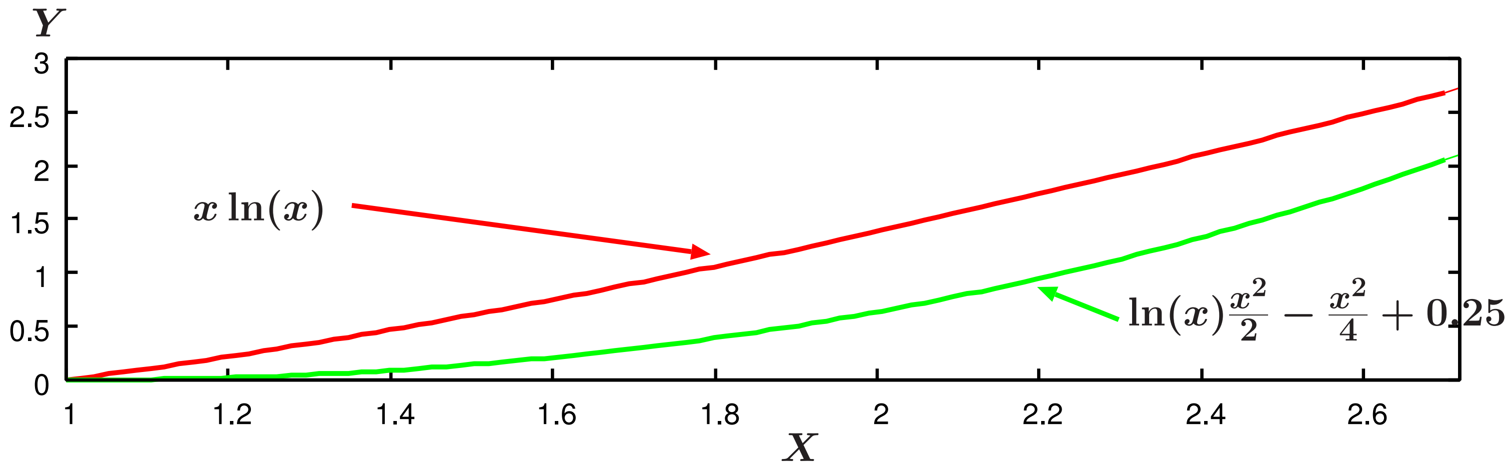
When we set $a = -1$ and $b = 1$ we obtain

$$\mathfrak{e}^{-x} \sin x = \frac{-\frac{d\left\{\mathfrak{e}^{-x} \sin x\right\}}{dx} - \frac{d\left\{\mathfrak{e}^{-x} \cos x\right\}}{dx}}{2}$$

By taking the integral of both sides,

$$\begin{aligned}
\int \mathfrak{e}^{-x} \sin x dx &= \int \frac{-\frac{d\left\{\mathfrak{e}^{-x} \sin x\right\}}{dx} - \frac{d\left\{\mathfrak{e}^{-x} \cos x\right\}}{dx}}{2} dx \\
&= \frac{-\mathfrak{e}^{-x} \sin x - \mathfrak{e}^{-x} \cos x}{2} + C
\end{aligned}$$

35) Evaluate $\int_1^{\mathfrak{e}} x \ln x dx$



When you see $\ln x$, it is safe to set $f(x)$ to $\ln x$ and $g(x)$ to the rest of the function. In this case, we set $f(x) = \ln x$ and $g(x) = x$.

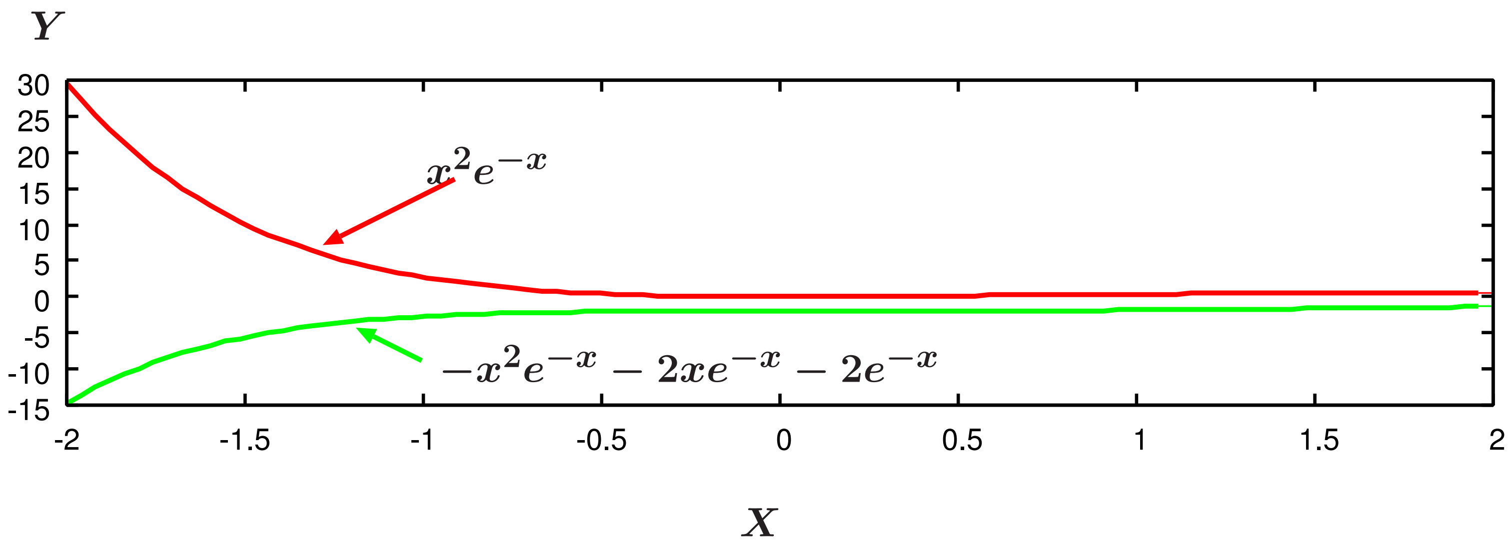
$$\int x \ln x dx = \int f(x) \cdot g(x) dx$$
$$= f(x) \cdot \int g(x) dx - \int \left(\frac{d\{f(x)\}}{dx} \cdot \int g(x) dx \right) dx$$
$$= \ln x \cdot \int x dx - \int \left(\frac{d\{\ln x\}}{dx} \cdot \int x dx \right) dx$$
$$= \ln x \cdot \frac{x^2}{2} - \int \left(\frac{1}{x} \cdot \frac{x^2}{2} \right) dx$$
$$= \ln x \cdot \frac{x^2}{2} - \int \left(\frac{x}{2} \right) dx$$
$$= \ln x \cdot \frac{x^2}{2} - \frac{x^2}{4} + C$$

Thus the definite integral is :

$$\int_1^{\mathfrak{e}} x \ln x dx$$
$$= [\ln x \cdot \frac{x^2}{2} - \frac{x^2}{4}]_1^{\mathfrak{e}}$$
$$= \frac{\mathfrak{e}^2}{2} - \frac{\mathfrak{e}^2}{4} + \frac{1}{4}$$
$$= \frac{\mathfrak{e}^2}{4} + \frac{1}{4}$$

36) Find

$$\int x^2 \mathfrak{e}^{-x} dx$$



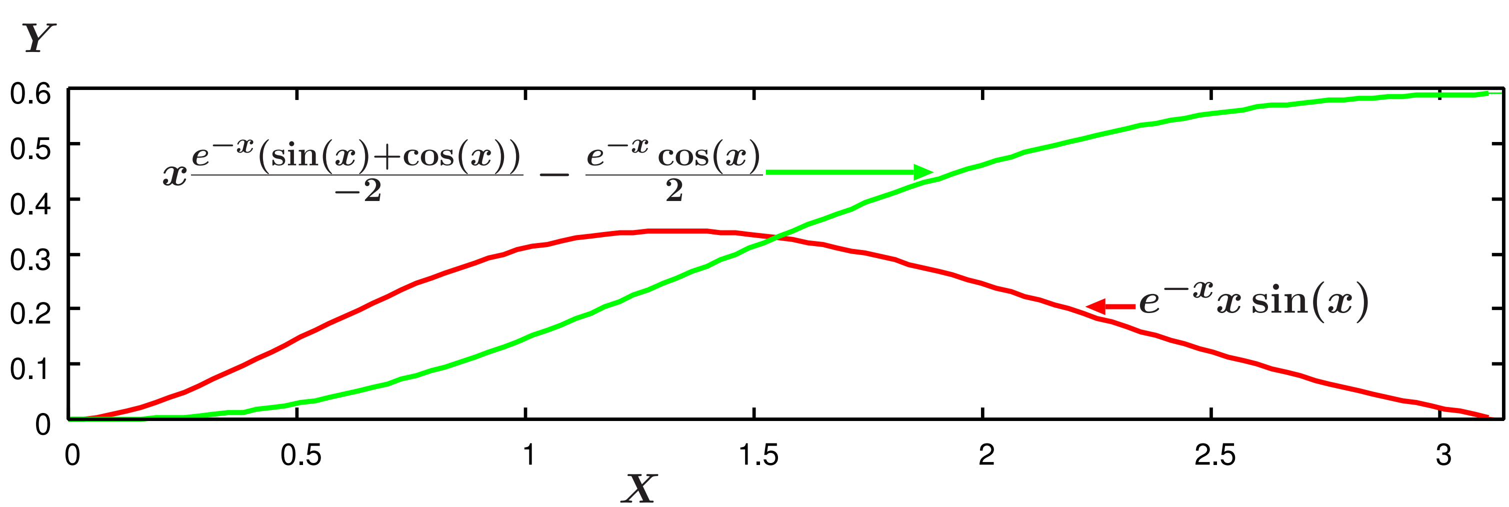
When you find a function which reduces its order by differentiation, then set the function to $f(x)$.
When you find a function which does not change its order by integration, then set the function to $g(x)$.
Since x^2 reduces its order by differentiation, we let $f(x) = x^2$ and $g(x) = \mathfrak{e}^{-x}$. Using Equation (55),

$$\int x^2\mathfrak{e}^{-x}dx = x^2\int \mathfrak{e}^{-x}dx - \int\left(\frac{d\left\{x^2\right\}}{dx}\int \mathfrak{e}^{-x}dx\right)dx$$
$$= x^2(-\mathfrak{e}^{-x}) - \int((2x)\cdot(-\mathfrak{e}^{-x}))\,dx$$

Since we can not get $\int 2x\mathfrak{e}^{-x}dx$ straightaway, we need to apply Equation (55) again for $\int 2x\mathfrak{e}^{-x}dx$. In this case, let $f(x) = 2x$ and $g(x) = \mathfrak{e}^{-x}$.

$$\int x^2\mathfrak{e}^{-x}dx$$
$$= -x^2\mathfrak{e}^{-x} + \int 2x\mathfrak{e}^{-x}dx$$
$$= -x^2\mathfrak{e}^{-x} + 2x\int \mathfrak{e}^{-x}dx - \int\left(\frac{d\left\{2x\right\}}{dx}\int \mathfrak{e}^{-x}dx\right)dx$$
$$= -x^2\mathfrak{e}^{-x} + 2x(-\mathfrak{e}^{-x}) - \int 2\cdot(-\mathfrak{e}^{-x})dx$$
$$= -x^2\mathfrak{e}^{-x} - 2x\mathfrak{e}^{-x} - 2(\mathfrak{e}^{-x}) + c$$
$$= -x^2\mathfrak{e}^{-x} - 2x\mathfrak{e}^{-x} - 2\mathfrak{e}^{-x} + c$$

37) Evaluate $\int_0^{\pi} \mathfrak{e}^{-x}x\sin x dx$



Usually we let $f(x)$ equal the polynomial function, in this case $f(x) = x$ and let $g(x)$ equal the non linear function, in this case, $g(x) = \mathfrak{e}^{-x}\sin x$

$$\begin{aligned}
&\int f(x)\cdot g(x)dx \\
&= f(x)\cdot\int g(x)dx - \int\left(\frac{d\{f(x)\}}{dx}\cdot\int g(x)dx\right)dx \\
&= x\cdot\int \mathfrak{e}^{-x}\sin x dx - \int\left(\frac{d\{x\}}{dx}\cdot\int \mathfrak{e}^{-x}\sin x dx\right)dx \\
&= x\cdot\int \mathfrak{e}^{-x}\sin x dx - \int\left(\int \mathfrak{e}^{-x}\sin x dx\right)dx \qquad ①
\end{aligned}$$

Now we need to find out $\int \mathfrak{e}^{-x}\sin x dx$. Rather than using Equation (55), this will demonstrate the alternative approach.

$$\begin{aligned}
\frac{d\left\{\mathfrak{e}^{-x}\sin x\right\}}{dx} &= -\mathfrak{e}^{-x}\sin x + \mathfrak{e}^{-x}\cos x \qquad ② \\
\frac{d\left\{\mathfrak{e}^{-x}\cos x\right\}}{dx} &= -\mathfrak{e}^{-x}\cos x - \mathfrak{e}^{-x}\sin x \qquad ③
\end{aligned}$$

②+ ③ gives us:

$$\begin{aligned}
\frac{d\left\{\mathfrak{e}^{-x}\sin x\right\}}{dx} + \frac{d\left\{\mathfrak{e}^{-x}\cos x\right\}}{dx} &= -2\mathfrak{e}^{-x}\sin x \\
\therefore \mathfrak{e}^{-x}\sin x &= \frac{\frac{d\left\{\mathfrak{e}^{-x}\sin x\right\}}{dx} + \frac{d\left\{\mathfrak{e}^{-x}\cos x\right\}}{dx}}{-2} \\
\therefore \int \mathfrak{e}^{-x}\sin x dx &= \frac{\mathfrak{e}^{-x}\sin x + \mathfrak{e}^{-x}\cos x}{-2} \qquad ④
\end{aligned}$$

Putting ④ into ①:

$$\begin{aligned}
&x\cdot\int \mathfrak{e}^{-x}\sin x dx - \int\left(\frac{d\{x\}}{dx}\cdot\int \mathfrak{e}^{-x}\sin x dx\right)dx \\
&= x\cdot\frac{\mathfrak{e}^{-x}\sin x + \mathfrak{e}^{-x}\cos x}{-2} - \int\left(\frac{\mathfrak{e}^{-x}\sin x + \mathfrak{e}^{-x}\cos x}{-2}\right)dx \\
&= x\cdot\frac{\mathfrak{e}^{-x}\sin x + \mathfrak{e}^{-x}\cos x}{-2} + \frac{1}{2}\int\left(\mathfrak{e}^{-x}\sin x + \mathfrak{e}^{-x}\cos x\right)dx \qquad ⑤
\end{aligned}$$

②- ③ gives us:

$$\begin{aligned}
\frac{d\left\{\mathfrak{e}^{-x}\sin x\right\}}{dx} - \frac{d\left\{\mathfrak{e}^{-x}\cos x\right\}}{dx} &= 2\mathfrak{e}^{-x}\cos x \\
\therefore \mathfrak{e}^{-x}\cos x &= \frac{1}{2}\frac{d\left\{\mathfrak{e}^{-x}\sin x - \mathfrak{e}^{-x}\cos x\right\}}{dx} \\
\therefore \int \mathfrak{e}^{-x}\cos x dx &= \frac{\mathfrak{e}^{-x}\sin x - \mathfrak{e}^{-x}\cos x}{2} \qquad ⑥
\end{aligned}$$

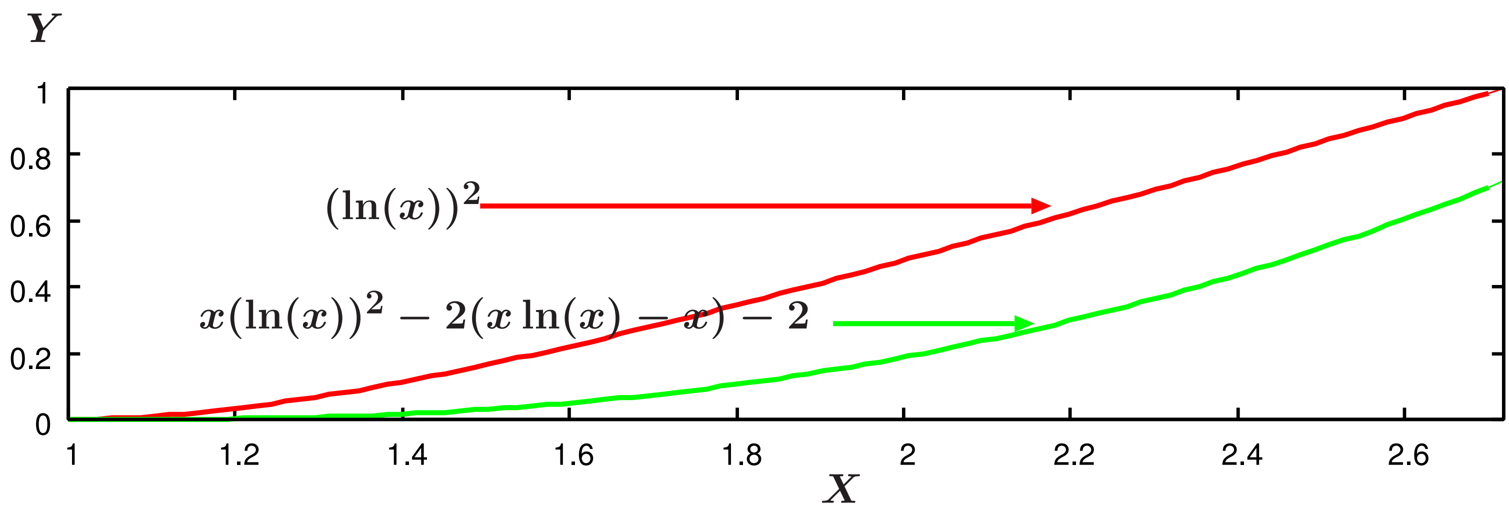
Putting ④ and ⑥ into ⑤:

$$\begin{aligned}
&x\cdot\frac{\mathfrak{e}^{-x}\sin x + \mathfrak{e}^{-x}\cos x}{-2} + \frac{1}{2}\int\left(\mathfrak{e}^{-x}\sin x + \mathfrak{e}^{-x}\cos x\right)dx \\
&= x\cdot\frac{\mathfrak{e}^{-x}\sin x + \mathfrak{e}^{-x}\cos x}{-2} - \frac{\mathfrak{e}^{-x}\sin x + \mathfrak{e}^{-x}\cos x}{4} + \frac{\mathfrak{e}^{-x}\sin x - \mathfrak{e}^{-x}\cos x}{4} \\
&= x\cdot\frac{\mathfrak{e}^{-x}\sin x + \mathfrak{e}^{-x}\cos x}{-2} - \frac{\mathfrak{e}^{-x}\cos x}{2}
\end{aligned}$$

Now we need to find the value of the definite integral :

$$\begin{aligned}
&[x\mathfrak{e}^{-x}\cdot\frac{\sin x + \cos x}{-2} - \frac{\mathfrak{e}^{-x}\cos x}{2}]_0^{\pi} \\
&= \pi\mathfrak{e}^{-\pi}\cdot\frac{-1}{-2} - \frac{\mathfrak{e}^{-\pi}\cdot(-1)}{2} + \frac{1}{2} \\
&= \mathfrak{e}^{-\pi}\frac{\pi+1}{2} + \frac{1}{2}
\end{aligned}$$

38) Evaluate $\int_1^{\mathfrak{e}}(\ln x)^2 dx$



When you see $\ln x$, it is safe to set $f(x)$ to $\ln x$ and $g(x)$ to the rest of the function. In this case, we set $f(x) = (\ln x)^2$ and $g(x) = 1$.

$$\begin{aligned}
&\int f(x) \cdot g(x)dx \\
&= f(x) \cdot \int g(x)dx - \int \left(\frac{d\{f(x)\}}{dx} \cdot \int g(x)dx \right) dx \\
&= (\ln x)^2 \cdot \int 1dx - \int \left(\frac{d\{(\ln x)^2\}}{dx} \cdot \int 1dx \right) dx \\
&= x(\ln x)^2 - \int \left(2\ln x \cdot \frac{1}{x} \cdot x \right) dx \\
&= x(\ln x)^2 - 2\int (\ln x)\, dx \\
&= x(\ln x)^2 - 2(x\ln x - x)(\because Equation(74))
\end{aligned}$$

Now we find the value of the definite integral :

$$\begin{aligned}
&[x(\ln x)^2 - 2(x\ln x - x)]_1^{\mathfrak{e}} \\
&= \mathfrak{e} - 2\mathfrak{e} + 2\mathfrak{e} - 2 \\
&= \mathfrak{e} - 2
\end{aligned}$$

39) Find $p(x)$ and $q(x)$ when

$$p(x) = \int_0^{\frac{\pi}{2}} q(t)\sin(x-t)dt,$$

$$q(x) = x + \int_0^{\frac{\pi}{2}} p(t)dt.$$

When you find a function which reduces its order by differentiation, then set the function to $f(x)$.
When you find a function which does not change its order by integration, then set the function to $g(x)$.
Let

$$a = \int_0^{\frac{\pi}{2}} p(t)dt,$$

then

$$q(x) = x + a.$$

Thus

$$\begin{aligned}
p(x) &= \int_0^{\frac{\pi}{2}} q(t)\sin(x-t)dt \\
&= \int_0^{\frac{\pi}{2}} (t+a)\sin(x-t)dt \\
&= [(t+a)\int \sin(x-t)dt]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (\frac{d\{t+a\}}{dt} \int \sin(x-t)dt)dt \\
&= [(t+a)(-\cos(x-t)) \cdot (-1)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} ((-\cos(x-t)) \cdot (-1))dt
\end{aligned}$$

$$= [(t+a)\cos(x-t)]_0^{\frac{\pi}{2}} - [-\sin(x-t)]_0^{\frac{\pi}{2}}$$
$$= [(t+a)\cos(x-t)+\sin(x-t)]_0^{\frac{\pi}{2}}$$
$$= [(\frac{\pi}{2}+a)\cos(x-\frac{\pi}{2})+\sin(x-\frac{\pi}{2})] - [(a)\cos(x)+\sin(x)]$$
$$= (\frac{\pi}{2}+a)\sin x - \cos x - a\cos(x) - \sin x$$
$$= (\frac{\pi}{2}+a-1)\sin x - (1+a)\cos x$$

We now put

$$p(x) = (\frac{\pi}{2}+a-1)\sin x - (1+a)\cos x$$

into

$$a = \int_0^{\frac{\pi}{2}} p(t)dt.$$

$$a = \int_0^{\frac{\pi}{2}} p(t)dt$$
$$= \int_0^{\frac{\pi}{2}} \left((\frac{\pi}{2}+a-1)\sin t - (1+a)\cos t\right) dt$$
$$= \left[(\frac{\pi}{2}+a-1)(-\cos t) - (1+a)\sin t\right]_0^{\frac{\pi}{2}}$$
$$= \left[(\frac{\pi}{2}+a-1)(-\cos\frac{\pi}{2}) - (1+a)\sin\frac{\pi}{2}\right]$$
$$- \left[(\frac{\pi}{2}+a-1)(-\cos 0) - (1+a)\sin 0\right]$$
$$= -(1+a) + \frac{\pi}{2} + a - 1 = \frac{\pi}{2} - 2$$

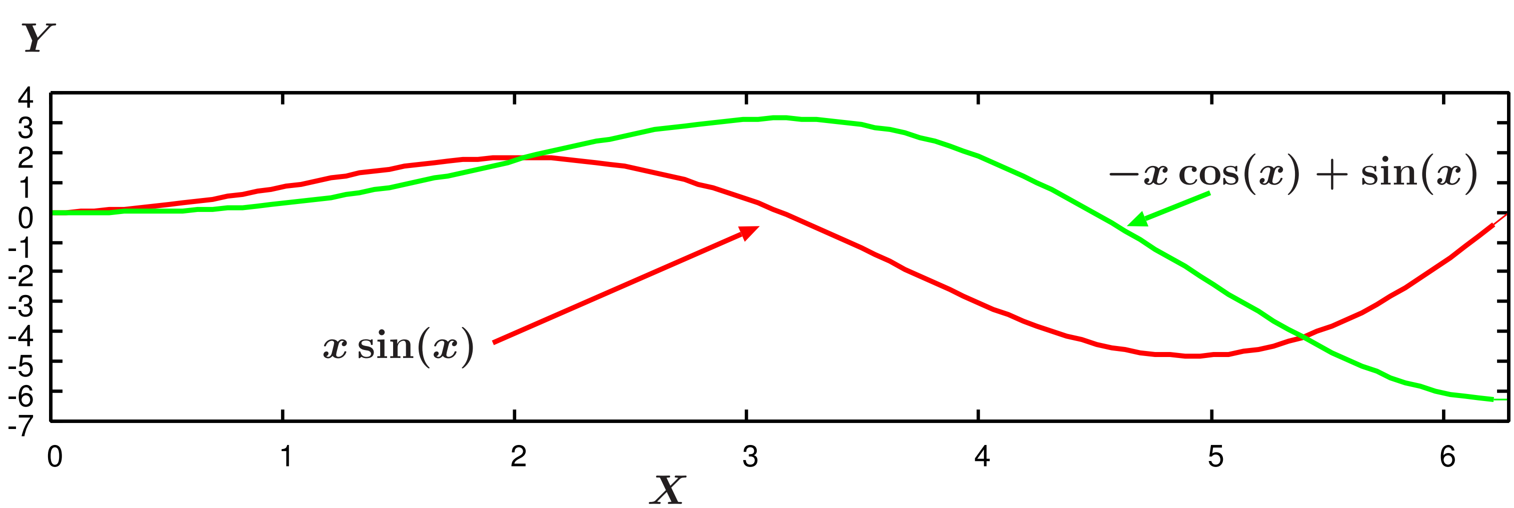
Therefore,

$$p(x) = (\frac{\pi}{2}+\frac{\pi}{2}-2-1)\sin x - (1+\frac{\pi}{2}-2)\cos x$$
$$= (\pi-3)\sin x - (\frac{\pi}{2}-1)\cos x$$

and

$$q(x) = x + a$$
$$= x + \frac{\pi}{2} - 2.$$

40) Find $\int x\sin x dx$



Compare $\int x\sin x dx$ with Equation (55) and we choose $f(x) = x$ and $g(x) = \sin x$. This is because

$$\frac{d\{f(x)\}}{dx} = 1$$

and

$$\int g(x)dx$$

$$= \int \sin x dx$$

$$= -\cos x.$$

Applying Equation (55) we obtain

$$\begin{aligned} &\int x \sin x dx \\ &= x \int \sin x dx - \int \left(1 \cdot \int \sin x dx\right) dx \\ &= x(-\cos x) - \int (-\cos x)\, dx \\ &= -x \cos x - (-\sin x) + c \\ &= -x \cos x + \sin x + c \end{aligned}$$

41) By writing $\ln x$ as $1 \times \ln x$ find

$$\int \ln x dx$$

Although we do not know the integral of $\ln x$, we do know the derivative of $\ln x$ which is $\frac{1}{x}$. Thus let $f(x) = \ln x$ and $g(x) = 1$. Using Equation (55),

$$\begin{aligned} \int \ln x dx = \ln x \int 1dx - \int \left(\frac{d\{\ln x\}}{dx} \int 1dx\right) dx \\ = \ln x \int 1dx - \int \left(\frac{1}{x} \cdot x\right) dx \\ = \ln x \cdot x - \int 1dx \\ = x \ln x - x + c \end{aligned}$$

42) Find

$$\int z^2 \cos kz dz \quad \text{and} \quad \int z^2 \cos kz dk$$

Let $f(x) = z^2$ and $g(x) = \cos kz$. Using Equation (55), Here, in order to find $\int z \sin kz dz$, let $f(z) = z$ and $g(z) = \sin kz$

$$\begin{aligned} &\int z^2 \cos kz dz \\ &= z^2 \int \cos kz dz - \int \left(\frac{\partial\{z^2\}}{\partial z} \int \cos kz dz\right) dz \\ &= \frac{z^2 \sin kz}{k} - \frac{2}{k} \int (z \sin kz)\, dz \\ &= z^2 \frac{\sin kz}{k} - \frac{2}{k} \times \left\{z \int \sin kz dz - \int \left(\frac{d\{z\}}{dz} \int \sin kz dz\right) dz\right\} \\ &= \frac{z^2 \sin kz}{k} - \frac{2}{k} \left\{z(-\frac{1}{k} \cos kz) - \int 1 \cdot \left(-\frac{1}{k} \cos kz\right) dz\right\} \\ &= \frac{z^2 \sin kz}{k} - \frac{2}{k} \left\{z(-\frac{1}{k} \cos kz) + \frac{1}{k^2} \sin kz\right\} + c \\ &= \frac{z^2 \sin kz}{k} + \frac{2z}{k^2} \cos kz - \frac{2}{k^3} \sin kz + c \end{aligned}$$

$$\int z^2 \cos kz dk = z^2 \int \cos kz dk = z^2 \frac{1}{z} \sin kz = z \sin kz + c$$

43) Find

$$\int_0^1 t q(t) dt$$

when $p(x)$ and $q(x)$ satisfy

$$p(x) = x\mathfrak{e}^x + \int_0^1 tq(t)dt$$

and

$$q(x) = \int_0^1 p(x-t)dt$$

Let $a = \int_0^1 tq(t)dt$. In this case, $p(x) = x\mathfrak{e}^x + \int_0^1 tq(t)dt = x\mathfrak{e}^x + a$. We now put $p(x) = x\mathfrak{e}^x + a$ into $q(x) = \int_0^1 p(x-t)dt$ as follows:

$$\begin{aligned}
q(x) &= \int_0^1 p(x-t)dt \\
&= \int_0^1 \left\{(x-t)\mathfrak{e}^{x-t} + a\right\} dt \\
&= \int_0^1 (x-t)\mathfrak{e}^{x-t}dt + \int_0^1 adt \\
&= \left[(x-t)\int \mathfrak{e}^{x-t}dt\right]_0^1 - \int_0^1 \left(\frac{d\{x-t\}}{dt}\int \mathfrak{e}^{x-t}dt\right) dt + \int_0^1 adt \\
&= \left[(x-t)\int \mathfrak{e}^{x-t}dt\right]_0^1 - \int_0^1 \left(-1\cdot(-\mathfrak{e}^{x-t})\right) dt + \int_0^1 adt \\
&= \left[(x-t)\int \mathfrak{e}^{x-t}dt\right]_0^1 - \int_0^1 \mathfrak{e}^{x-t}dt + \int_0^1 adt \\
&= \left[-(x-t)\mathfrak{e}^{x-t}\right]_0^1 - \left[-\mathfrak{e}^{x-t}\right] + a\,[t]_0^1 \\
&= \left[-(x-t)\mathfrak{e}^{x-t} + \mathfrak{e}^{x-t} + at\right]_0^1 \\
&= \left[-(x-1)\mathfrak{e}^{x-1} + \mathfrak{e}^{x-1} + a\right] - \left[-(x)\mathfrak{e}^{x} + \mathfrak{e}^{x}\right] \\
&= (2-x)\mathfrak{e}^{x-1} + a + (x-1)\mathfrak{e}^{x} \\
&= (2-x)\mathfrak{e}^{x-1} + a + (x\mathfrak{e}-\mathfrak{e})\mathfrak{e}^{x-1} \\
&= (x(\mathfrak{e}-1) + 2 - \mathfrak{e})\mathfrak{e}^{x-1} + a
\end{aligned}$$

We now put $q(x) = (x(\mathfrak{e}-1)+2-\mathfrak{e})\mathfrak{e}^{x-1} + a$ into $a = \int_0^1 tq(t)dt.$
Thus $a = \frac{1}{2}a + \mathfrak{e} - 4 + 4\mathfrak{e}^{-1}$ gives us $a = 2(\mathfrak{e} - 4 + 4\mathfrak{e}^{-1})$.

$$\begin{aligned}
a &= \int_0^1 tq(t)dt \\
&= \int_0^1 t((t(\mathfrak{e}-1) + 2 - \mathfrak{e})\mathfrak{e}^{t-1} + a)dt \\
&= \int_0^1 (t^2(\mathfrak{e}-1) + (2-\mathfrak{e})t)\mathfrak{e}^{t-1} + at)dt \\
&= \int_0^1 (\mathfrak{e}-1)t^2\mathfrak{e}^{t-1} + (2-\mathfrak{e})t\mathfrak{e}^{t-1} + atdt \\
&\text{simply split up the intergral} \\
\therefore \int_0^1 (\mathfrak{e}-1)t^2\mathfrak{e}^{t-1} &+ \int_0^1 (2-\mathfrak{e})t\mathfrak{e}^{t-1} + \int_0^1 atdt
\end{aligned}$$

Now you can integrate each bit of the integral above separately by performing "by parts" twice.

$$\begin{aligned}
\int_0^1 (\mathfrak{e}-1)t^2\mathfrak{e}^{t-1}dt &= \\
(\mathfrak{e}-1)\int_0^1 t^2\mathfrak{e}^{t-1}dt &\because (\mathfrak{e}-1) \text{ is constant}
\end{aligned}$$

Using by parts, letting $f(t) = t^2$ and $g(t) = \mathfrak{e}^{t-1}$
where $f'(t) = 2t$ (' means differential)and $\int g(t)dt = \mathfrak{e}^{t-1}$

$$
\begin{aligned}
(\mathfrak{e}-1)\int_0^1 t^2\mathfrak{e}^{t-1}dt &= \\
(\mathfrak{e}-1)\left(f(t)\cdot\int g(t)dt - \int(f'(t)\cdot\int(g(t))dt)\right) \\
(\mathfrak{e}-1)\left(t^2\cdot\mathfrak{e}^{t-1} - \int 2t\cdot\mathfrak{e}^{t-1}dt\right) \\
\int 2t\cdot\mathfrak{e}^{t-1}dt = 2t\cdot\mathfrak{e}^{t-1} - \int 2\cdot\mathfrak{e}^{t-1}dt \\
= 2t\cdot\mathfrak{e}^{t-1} - 2\cdot\mathfrak{e}^{t-1} \\
\therefore = (\mathfrak{e}-1)\left[\left(t^2\cdot\mathfrak{e}^{t-1} - 2t\cdot\mathfrak{e}^{t-1} + 2\cdot\mathfrak{e}^{t-1}\right)\right]_0^1 \\
= \frac{(\mathfrak{e}-2)(\mathfrak{e}-1)}{\mathfrak{e}}
\end{aligned}
$$

Using the same method as above we get

$$
\begin{aligned}
\int_0^1 (2-\mathfrak{e})t\mathfrak{e}^{t-1}dt &= (2-\mathfrak{e})\left(\mathfrak{e}^{t-1}\cdot(t-1)\right) \\
&= (2-\mathfrak{e})\left[\left(\mathfrak{e}^{t-1}\cdot(t-1)\right)\right]_0^1 \\
&= (2-\mathfrak{e})\left[\left((1\cdot 0)-(-\mathfrak{e}^{-1})\right)\right] \\
&= (2-\mathfrak{e})\cdot(\mathfrak{e}^{-1}) \\
&= \frac{2-\mathfrak{e}}{\mathfrak{e}}
\end{aligned}
$$

This is a simple intergral

$$
\begin{aligned}
\int_0^1 at\cdot dt &= \left[\frac{1}{2}at^2\right]_0^1 \\
&= \frac{1}{2}a - 0 = \frac{1}{2}a
\end{aligned}
$$

Therefore,

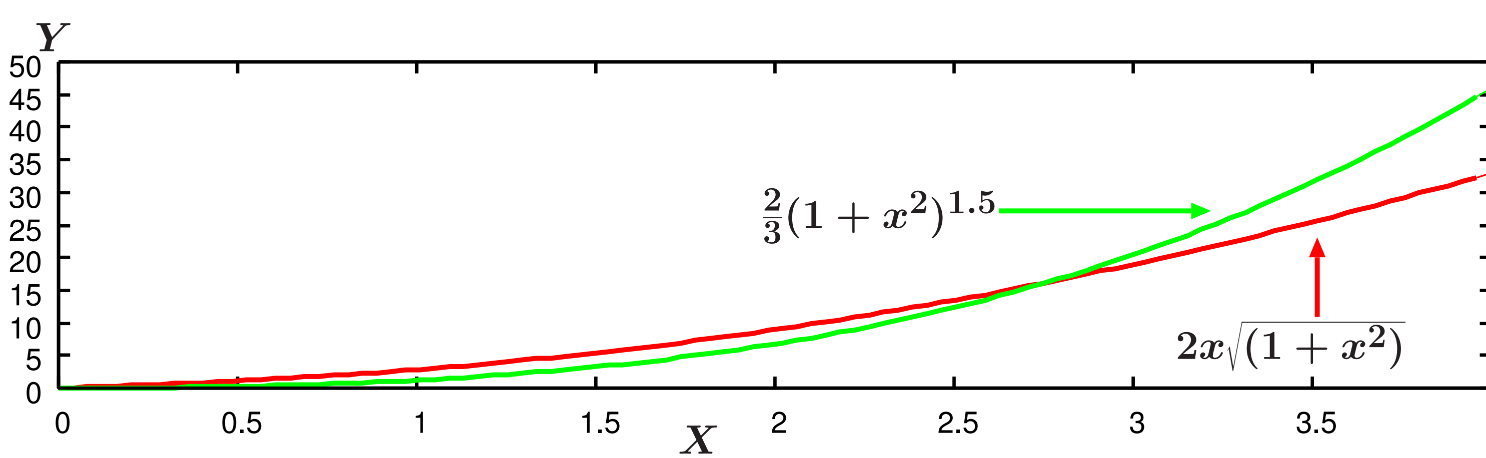
$$\int_0^1 (\mathfrak{e}-1)t^2\mathfrak{e}^{t-1} + \int_0^1 (2-\mathfrak{e})t\mathfrak{e}^{t-1} + \int_0^1 atdt$$

Becomes

$$= \frac{(\mathfrak{e}-2)\cdot(\mathfrak{e}-1)}{\mathfrak{e}} + \frac{(2-\mathfrak{e})}{\mathfrak{e}} + \frac{1}{2}a$$

Thus $a = \frac{1}{2}a + \mathfrak{e} - 4 + 4\mathfrak{e}^{-1}$ gives us $a = 2(\mathfrak{e} - 4 + 4\mathfrak{e}^{-1})$.

44) Evaluate $\int 2x\sqrt{1+x^2}dx$.



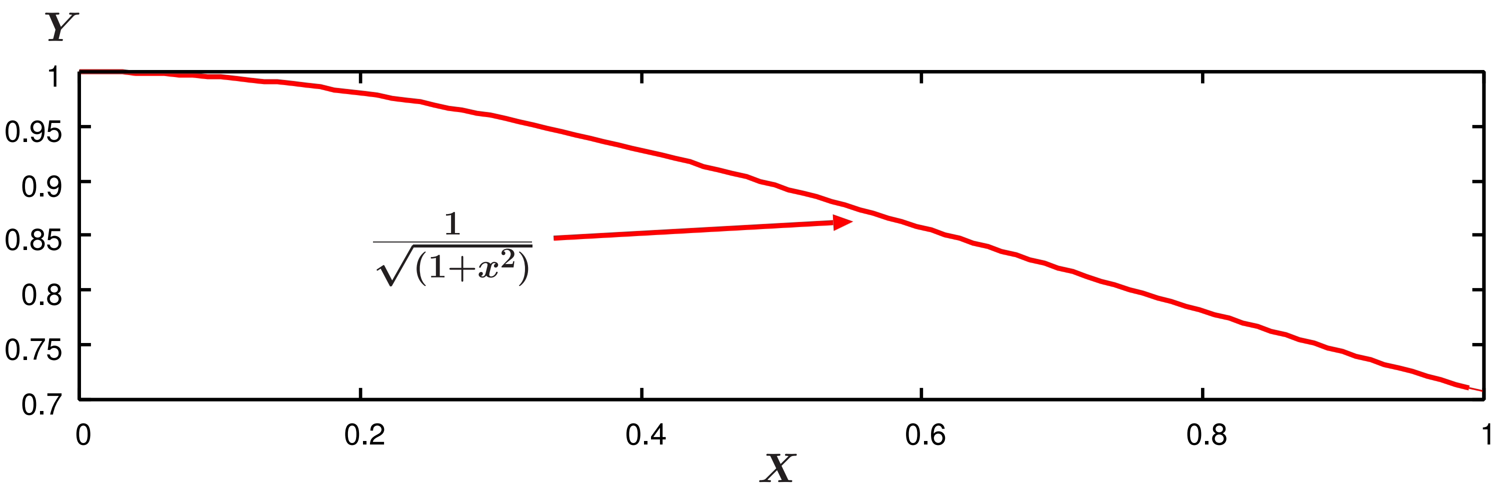
If we let $u = 1 + x^2$ the integral becomes $\int 2x\sqrt{u}dx$ but remember $\sqrt{u} \equiv u^{\frac{1}{2}}$ therefore $\int 2x\sqrt{u}dx$ is the same as $\int 2xu^{\frac{1}{2}}dx$.

$$\frac{d\{u\}}{dx} = 2x$$
$$\therefore du = 2xdx$$

The integral now just becomes

$$\int \sqrt{u}du$$
$$= \int u^{\frac{1}{2}}du$$
$$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]$$
$$\therefore \int 2x\sqrt{1+x^2}dx = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c$$

45) Evaluate $\int_0^1 \frac{1}{\sqrt{1+x^2}}dx$.



- First method
  The integral in question is the same as Equation (76) with $k = 1$. As $\int \frac{dx}{\sqrt{x^2+k^2}} = \sinh^{-1}\left(\frac{x}{k}\right)$,

$$\int_0^1 \frac{1}{\sqrt{1+x^2}}dx$$
$$= \left[\sinh^{-1}\left(\frac{x}{1}\right)\right]_0^1$$
$$= \sinh^{-1}(1) - \sinh^{-1}(0)$$

We need $\sinh^{-1}(1)$ and $\sinh^{-1}(0)$. As

$$\sinh(\theta) = \frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2}$$
$$\therefore \theta = \sinh^{-1}\left(\frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2}\right)$$

we need to find out θ which satisfies $\frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2} = 1$ for $\sinh^{-1}(1)$ and $\frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2} = 0$ for $\sinh^{-1}(0)$. When $\frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2} = 1$

$$\frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2} = 1$$
$$\therefore \mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta} = 2$$
$$\therefore \mathfrak{e}^{2\theta} - 1 = 2\mathfrak{e}^{\theta}$$
$$\therefore \mathfrak{e}^{2\theta} - 2\mathfrak{e}^{\theta} - 1 = 0$$

$$\therefore \mathfrak{e}^\theta = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$$

Since $\mathfrak{e}^\theta > 0$,

$$\mathfrak{e}^\theta = 1 + \sqrt{2}$$
$$\therefore \theta = \ln\left(1+\sqrt{2}\right)$$

When $\frac{\mathfrak{e}^\theta - \mathfrak{e}^{-\theta}}{2} = 0$

$$\frac{\mathfrak{e}^\theta - \mathfrak{e}^{-\theta}}{2} = 0$$
$$\therefore \mathfrak{e}^\theta - \mathfrak{e}^{-\theta} = 0$$
$$\therefore \mathfrak{e}^{2\theta} - 1 = 0$$
$$\therefore \mathfrak{e}^{2\theta} = 1 \equiv \mathfrak{e}^0$$
$$\therefore 2\theta = 0$$
$$\therefore \theta = 0$$

Thus the final answer is

$$\int_0^1 \frac{1}{\sqrt{1+x^2}} dx$$
$$= \sinh^{-1}(1) - \sinh^{-1}(0)$$
$$= \ln\left(1+\sqrt{2}\right) - 0$$
$$= \ln\left(1+\sqrt{2}\right)$$

- Second method
  When you see $\sqrt{a^2+b^2x^2}$, set $x = \frac{a}{b}\frac{\mathfrak{e}^\theta - \mathfrak{e}^{-\theta}}{2} \equiv \sinh(\theta)$. In general when you see $\sqrt{a^2+x^2}$, set $x = a\frac{\mathfrak{e}^\theta - \mathfrak{e}^{-\theta}}{2} \equiv \sinh(\theta)$.
  In this case we set $x = \frac{\mathfrak{e}^\theta - \mathfrak{e}^{-\theta}}{2} \equiv \sinh(\theta)$.

$$\sqrt{1+x^2}$$
$$= \sqrt{1 + (\frac{\mathfrak{e}^\theta - \mathfrak{e}^{-\theta}}{2})^2}$$
$$= \sqrt{1 + \frac{\mathfrak{e}^{2\theta} + \mathfrak{e}^{-2\theta} - 2}{4}}$$
$$= \sqrt{\frac{\mathfrak{e}^{2\theta} + \mathfrak{e}^{-2\theta} + 2}{4}}$$
$$= \sqrt{\frac{\mathfrak{e}^{2\theta} + \mathfrak{e}^{-2\theta} + 2}{4}}$$
$$= \sqrt{\frac{(\mathfrak{e}^\theta + \mathfrak{e}^{-\theta})^2}{4}}$$
$$= \frac{\mathfrak{e}^\theta + \mathfrak{e}^{-\theta}}{2}$$

$$x = \frac{\mathfrak{e}^\theta - \mathfrak{e}^{-\theta}}{2}$$
$$\therefore dx = \frac{\mathfrak{e}^\theta + \mathfrak{e}^{-\theta}}{2} d\theta$$

We also need to find out the range of θ based on $0 \le x \le 1$: When $x = 0$,

$$0 = \frac{\mathfrak{e}^\theta - \mathfrak{e}^{-\theta}}{2}$$
$$\therefore \mathfrak{e}^\theta = \mathfrak{e}^{-\theta}$$
$$\therefore \mathfrak{e}^{2\theta} = 1 = \mathfrak{e}^0$$
$$\therefore \theta = 0$$

When $x = 1$,

$$1 = \frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2}$$
$$\therefore 2 = \mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}$$
$$\therefore \mathfrak{e}^{2\theta} - 2\mathfrak{e}^{\theta} - 1 = 0$$
$$\therefore \mathfrak{e}^{\theta} = 1 \pm \sqrt{2}$$
$$\therefore \mathfrak{e}^{\theta} = 1 + \sqrt{2} (\because \mathfrak{e}^{\theta} > 0 \forall \theta)$$
$$\therefore \ln(\mathfrak{e}^{\theta}) = \ln(1 + \sqrt{2})$$
$$\therefore \theta = \ln(1 + \sqrt{2})$$

Thus

$$\int_0^1 \frac{1}{\sqrt{1 + x^2}} dx$$
$$= \int_0^{\ln(1+\sqrt{2})} \frac{2}{\mathfrak{e}^{\theta} + \mathfrak{e}^{-\theta}} \frac{\mathfrak{e}^{\theta} + \mathfrak{e}^{-\theta}}{2} d\theta$$
$$= \int_0^{\ln(1+\sqrt{2})} d\theta$$
$$= [\theta]_0^{\ln(1+\sqrt{2})}$$
$$= \ln(1 + \sqrt{2})$$

46) Evaluate $\int_0^1 \sqrt{1 + x^2} dx$.

When you see $\sqrt{a^2 + b^2 x^2}$, set $\frac{b}{a} x = \frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2} \equiv \sinh(\theta)$ In this case we set $x = \frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2} \equiv \sinh(\theta)$.

$$\sqrt{1 + x^2}$$
$$= \sqrt{1 + (\frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2})^2}$$
$$= \sqrt{1 + \frac{\mathfrak{e}^{2\theta} + \mathfrak{e}^{-2\theta} - 2}{4}}$$
$$= \sqrt{\frac{\mathfrak{e}^{2\theta} + \mathfrak{e}^{-2\theta} + 2}{4}}$$
$$= \sqrt{\frac{\mathfrak{e}^{2\theta} + \mathfrak{e}^{-2\theta} + 2}{4}}$$
$$= \sqrt{\frac{(\mathfrak{e}^{\theta} + \mathfrak{e}^{-\theta})^2}{4}}$$
$$= \frac{\mathfrak{e}^{\theta} + \mathfrak{e}^{-\theta}}{2}$$

$$x = \frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2}$$
$$\therefore dx = \frac{\mathfrak{e}^{\theta} + \mathfrak{e}^{-\theta}}{2} d\theta$$

We also need to find out the range of θ based on $0 \leq x \leq 1$: When $x = 0$,

$$0 = \frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2}$$
$$\therefore \mathfrak{e}^{\theta} = \mathfrak{e}^{-\theta}$$
$$\therefore \mathfrak{e}^{2\theta} = 1 = \mathfrak{e}^{0}$$
$$\therefore \theta = 0$$

When $x = 1$,

$$1 = \frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2}$$
$$\therefore 2 = \mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}$$
$$\therefore \mathfrak{e}^{2\theta} - 2\mathfrak{e}^{\theta} - 1 = 0$$
$$\therefore \mathfrak{e}^{\theta} = 1 \pm \sqrt{2}$$
$$\therefore \mathfrak{e}^{\theta} = 1 + \sqrt{2} (\because \mathfrak{e}^{\theta} > 0 \forall \theta)$$
$$\therefore \ln(\mathfrak{e}^{\theta}) = \ln(1 + \sqrt{2})$$
$$\therefore \theta = \ln(1 + \sqrt{2})$$

Thus

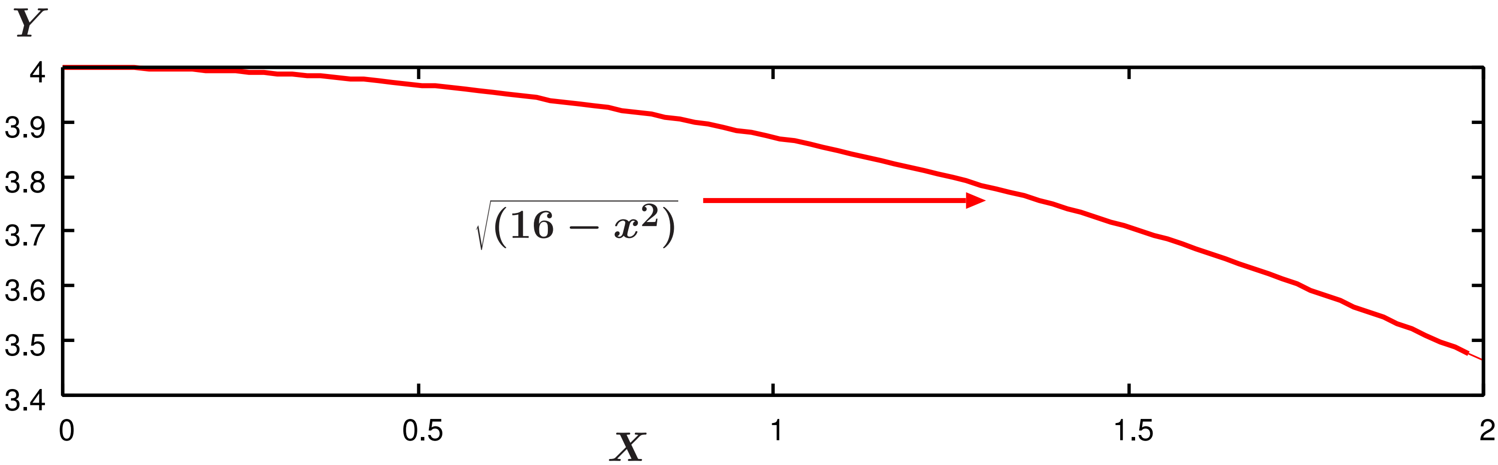
$$
\begin{aligned}
&\int_0^1 \sqrt{1+x^2}dx \\
&= \int_0^{\ln(1+\sqrt{2})} \frac{\mathfrak{e}^{\theta}+\mathfrak{e}^{-\theta}}{2}\frac{\mathfrak{e}^{\theta}+\mathfrak{e}^{-\theta}}{2}d\theta \\
&= \int_0^{\ln(1+\sqrt{2})} \frac{(\mathfrak{e}^{\theta}+\mathfrak{e}^{-\theta})^2}{4}d\theta \\
&= \int_0^{\ln(1+\sqrt{2})} \frac{\mathfrak{e}^{2\theta}+\mathfrak{e}^{-2\theta}+2}{4}d\theta \\
&= [\frac{\frac{1}{2}\mathfrak{e}^{2\theta}-\frac{1}{2}\mathfrak{e}^{-2\theta}+2\theta}{4}]_0^{\ln(1+\sqrt{2})} \\
&= [\frac{\mathfrak{e}^{2\theta}-\mathfrak{e}^{-2\theta}+4\theta}{8}]_0^{\ln(1+\sqrt{2})} \\
&= \frac{\mathfrak{e}^{2\ln(1+\sqrt{2})}-\mathfrak{e}^{-2\ln(1+\sqrt{2})}+4\ln(1+\sqrt{2})}{8} \\
&= \frac{\mathfrak{e}^{\ln(1+\sqrt{2})^2}-\mathfrak{e}^{\ln(1+\sqrt{2})^{-2}}+4\ln(1+\sqrt{2})}{8}
\end{aligned}
$$

Now that $\mathfrak{e}^{\ln a} = a$, we can simplify the terms above a bit more

$$
\begin{aligned}
&\frac{\mathfrak{e}^{\ln(1+\sqrt{2})^2}-\mathfrak{e}^{\ln(1+\sqrt{2})^{-2}}+4\ln(1+\sqrt{2})}{8} \\
&= \frac{(1+\sqrt{2})^2-(1+\sqrt{2})^{-2}+4\ln(1+\sqrt{2})}{8} \\
&= \frac{(1+\sqrt{2})^2-(1+\sqrt{2})^{-2}}{8}+\frac{\ln(1+\sqrt{2})}{2} \\
&= \frac{(1+\sqrt{2})^4-1}{8(1+\sqrt{2})^2}+\frac{\ln(1+\sqrt{2})}{2} \\
&= \frac{(1+\sqrt{2})^4-1}{8(1+\sqrt{2})^2}+\frac{\ln(1+\sqrt{2})}{2} \\
&= \frac{17+12\sqrt{2}-1}{8(1+\sqrt{2})^2}+\frac{\ln(1+\sqrt{2})}{2} \\
&= \frac{(16+12\sqrt{2})(\sqrt{2}-1)^2}{8(1+\sqrt{2})^2(\sqrt{2}-1)^2}+\frac{\ln(1+\sqrt{2})}{2} \\
&= \frac{(16+12\sqrt{2})(\sqrt{2}-1)^2}{8}+\frac{\ln(1+\sqrt{2})}{2} \\
&= \frac{(16+12\sqrt{2})(3-2\sqrt{2})}{8}+\frac{\ln(1+\sqrt{2})}{2} \\
&= \frac{4\sqrt{2}}{8}+\frac{\ln(1+\sqrt{2})}{2} \\
&= \frac{\sqrt{2}}{2}+\frac{\ln(1+\sqrt{2})}{2}
\end{aligned}
$$

47) Find

$$\int_0^2 \sqrt{16-x^2}dx$$



Let $x = 4\sin\theta$. Then

$$\frac{dx}{d\theta} = \frac{d4\sin\theta}{d\theta} = 4\cos\theta.$$

$$\therefore dx = 4\cos\theta d\theta.$$

When x varies from 0 to 2, $\sin\theta$ varies from 0 to $\frac{1}{2}(=\sin\frac{\pi}{6}$. Therefore θ varies from 0 to $\frac{\pi}{6}$.

$$\int_0^2 \sqrt{16-x^2}dx$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{16-(4\sin\theta)^2}(4\cos\theta d\theta)$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{16(1-\sin^2\theta)}(4\cos\theta d\theta)$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{16\cos^2\theta}(4\cos\theta d\theta)$$

$$= \int_0^{\frac{\pi}{6}} (4\cos\theta)\cdot(4\cos\theta d\theta)$$

$$= 16\int_0^{\frac{\pi}{6}} \cos^2\theta d\theta$$

$$= 16\int_0^{\frac{\pi}{6}} \frac{1+\cos 2\theta}{2} d\theta (\because \cos^2\theta = \frac{1+\cos 2\theta}{2})$$

$$= 8[\theta + \frac{1}{2}\sin 2\theta]_0^{\frac{\pi}{6}}$$

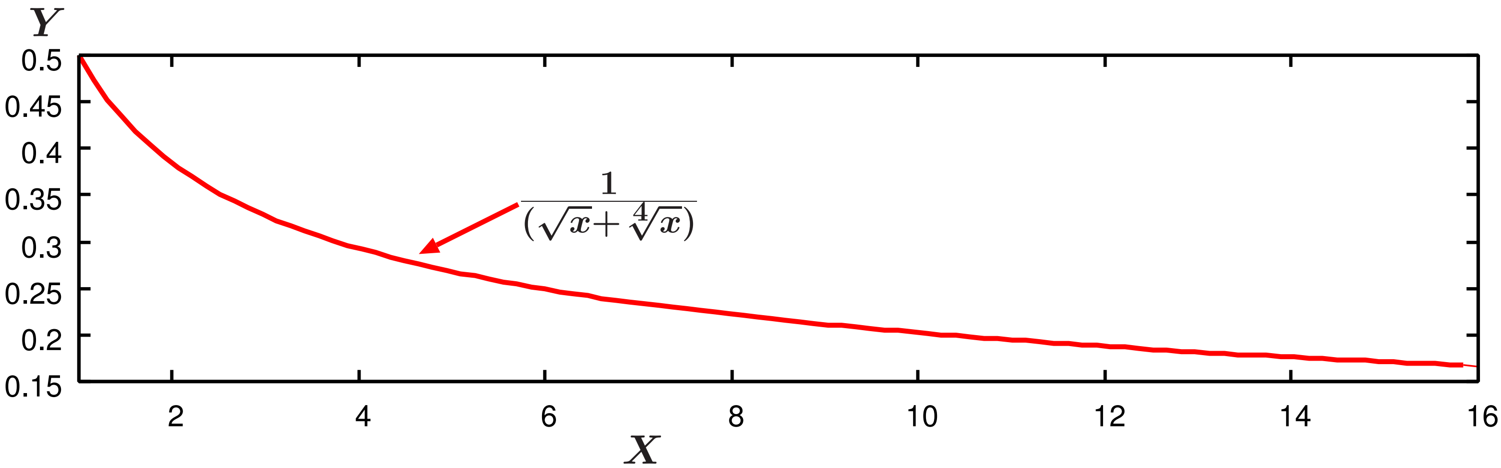
$$= 8[\frac{\pi}{6} + \frac{1}{2}\sin\frac{2\pi}{6}]$$

$$= 8[\frac{\pi}{6} + \frac{1}{2}\frac{\sqrt{3}}{2}]$$

$$= \frac{4\pi}{3} + 2\sqrt{3}$$

48) Find

$$\int_1^{16} \frac{dx}{\sqrt{x}+\sqrt[4]{x}}$$



Let

$$t = \sqrt[4]{x},$$

so

$$t^4 = x.$$

Thus

$$\frac{d\left\{t^4\right\}}{dx} = \frac{d\left\{x\right\}}{dx},$$

i.e.,

$$4t^3\frac{d\left\{t\right\}}{dx} = 1.$$

Therefore,

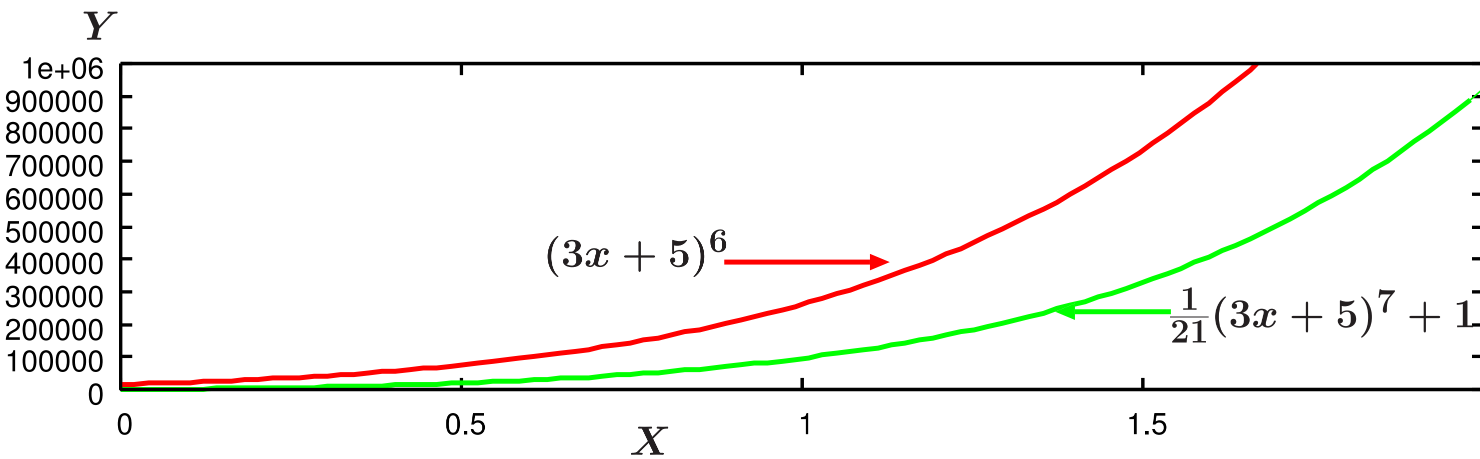
$$4t^3 dt = dx.$$

When x varies from 1 to 16, t varies from 1 to 2$(=\sqrt[4]{16})$. Thus Using Equation (56),

$$\int_1^{16} \frac{dx}{\sqrt{x}+\sqrt[4]{x}}$$
$$= \int_1^2 \frac{4t^3 dt}{t^2+t}$$
$$= 4\int_1^2 \frac{t^2}{t+1}dt$$
$$== 4\int_1^2 \frac{t^2-1+1}{t+1}dt \because t^2-1+1 \equiv t^2$$
$$= 4\times\int_1^2 \frac{(t-1)(t+1)+1}{t+1}dt$$
$$= 4\times\int_1^2 \left\{(t-1)+\frac{1}{t+1}\right\}dt$$
$$= 4\times\left[\frac{1}{2}t^2 - t + \ln|t+1|\right]_1^2$$
$$= 4\left\{(\frac{1}{2}2^2 - 2 + \ln|2+1|)\right.$$
$$\left. -(\frac{1}{2}1^2 - 1 + \ln|1+1|)\right\}$$
$$= 4\left\{\ln|3| - (-\frac{1}{2}+\ln|2|)\right\}$$
$$= 4\ln\frac{3}{2}+2$$

49) Find $\int (3x+5)^6 dx$
Hint: let

$$t = 3x+5.$$



Let

$$t = 3x + 5.$$

Thus

$$\frac{d\{t\}}{dx} = 3.$$

So

$$dt = 3dx.$$

$$\int (3x+5)^6 dx$$

$$= \int t^6 dx$$

$$= \int t^6 \frac{dt}{3}$$

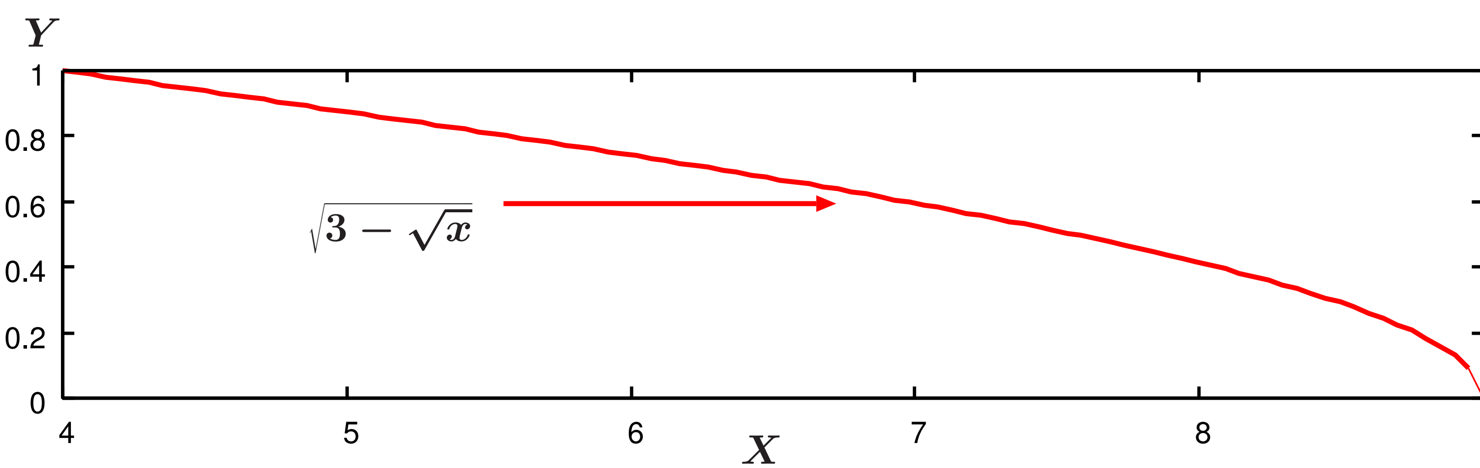
$$= \frac{1}{3} \cdot \frac{1}{7} t^7 + c$$

$$= \frac{1}{21}(3x+5)^7 + c$$

50) Find

$$\int_4^9 \frac{dx}{\sqrt{3-\sqrt{x}}}$$

Hint: $t = \sqrt{3-\sqrt{x}}$.



Let $t = \sqrt{3 - \sqrt{x}}$.

$$t = \sqrt{3 - \sqrt{x}}$$
$$\therefore t^2 = 3 - \sqrt{x}$$
$$\therefore 3 - t^2 = \sqrt{x}$$
$$\therefore (3 - t^2)^2 = x$$

We need to find the relationship between dx and dt. Let $u = 3 - t^2$. Then

$$\frac{d\left\{(3-t^2)^2\right\}}{dt} = \frac{d\{x\}}{dt}$$
$$\therefore \frac{d\left\{u^2\right\}}{dt} = \frac{d\{x\}}{dt}$$
$$\therefore \frac{d\{u\}}{dt}\frac{\partial\{u^2\}}{\partial u} = \frac{d\{x\}}{dt}$$
$$\therefore \frac{d\left\{3-t^2\right\}}{dt}2u = \frac{d\{x\}}{dt}$$
$$\therefore -2t \cdot 2(3-t^2) = \frac{d\{x\}}{dt}$$
$$\therefore -4t(3-t^2)dt = dx$$

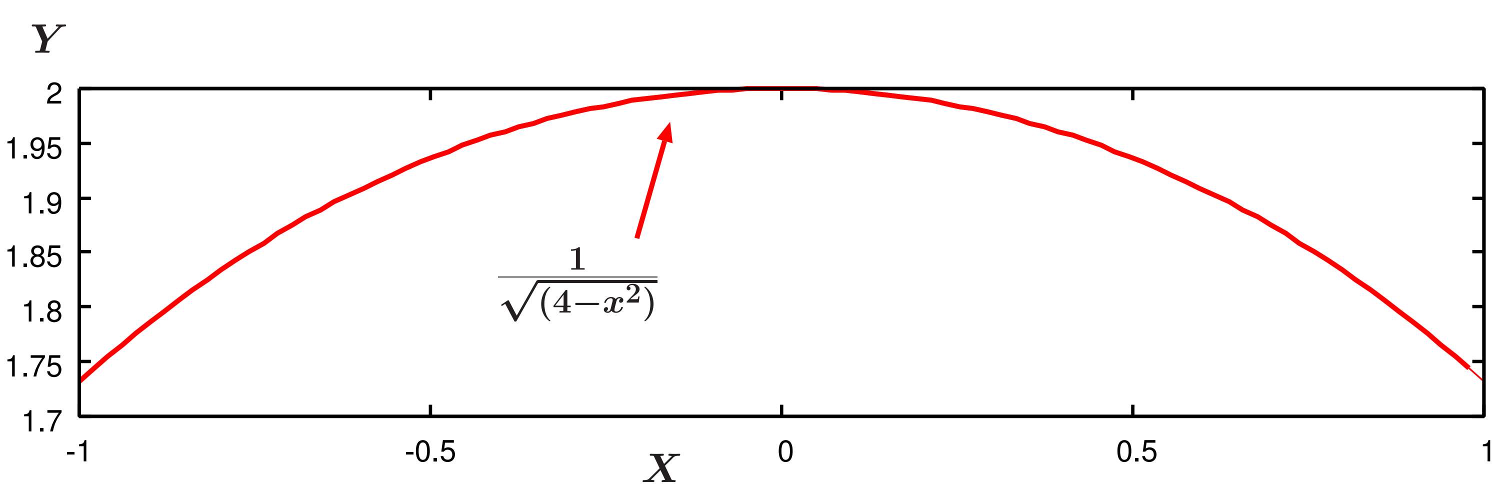
When x varies from 4 to 9, t varies from $1(= \sqrt{3 - \sqrt{4}})$ to $0(= \sqrt{3 - \sqrt{9}})$. Thus

$$\int_4^9 \frac{dx}{\sqrt{3 - \sqrt{x}}}$$
$$= \int_1^0 \frac{-4t(3-t^2)dt}{t}$$
$$= 4\int_1^0 (t^2 - 3)dt$$
$$= 4\left[\frac{1}{3}t^3 - 3t\right]_1^0$$
$$= -4\{\frac{1}{3}1^3 - 3\}$$
$$= \frac{32}{3}$$

51) Find

$$\int_{-1}^{1} \frac{dx}{\sqrt{4 - x^2}}$$

Hint: let $x = 2\sin\theta$.



- First method The integral in question is the same as Equation (77) with $k = 2$. As $\int \frac{dx}{\sqrt{k^2 - x^2}} = \sin^{-1}\left(\frac{x}{k}\right)$

$$\int_{-1}^{1} \frac{dx}{\sqrt{4 - x^2}}$$
$$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-1}^{1}$$
$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{-1}{2}\right)$$

We need $\sin^{-1}\left(\frac{1}{2}\right)$ and $\sin^{-1}\left(\frac{-1}{2}\right)$. As

$$\theta = \sin^{-1}\left(\frac{1}{2}\right)$$
$$\therefore \sin(\theta) = \frac{1}{2}$$
$$\therefore \theta = \frac{\pi}{6}$$

and

$$\theta = \sin^{-1}\left(\frac{-1}{2}\right)$$
$$\therefore \sin(\theta) = \frac{-1}{2}$$
$$\therefore \theta = -\frac{\pi}{6}$$

the final answer is

$$\int_{-1}^{1} \frac{dx}{\sqrt{4 - x^2}}$$
$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\frac{-1}{2}\right)$$
$$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{6} + \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$$

- Second method
Let

$$x = 2\sin\theta.$$

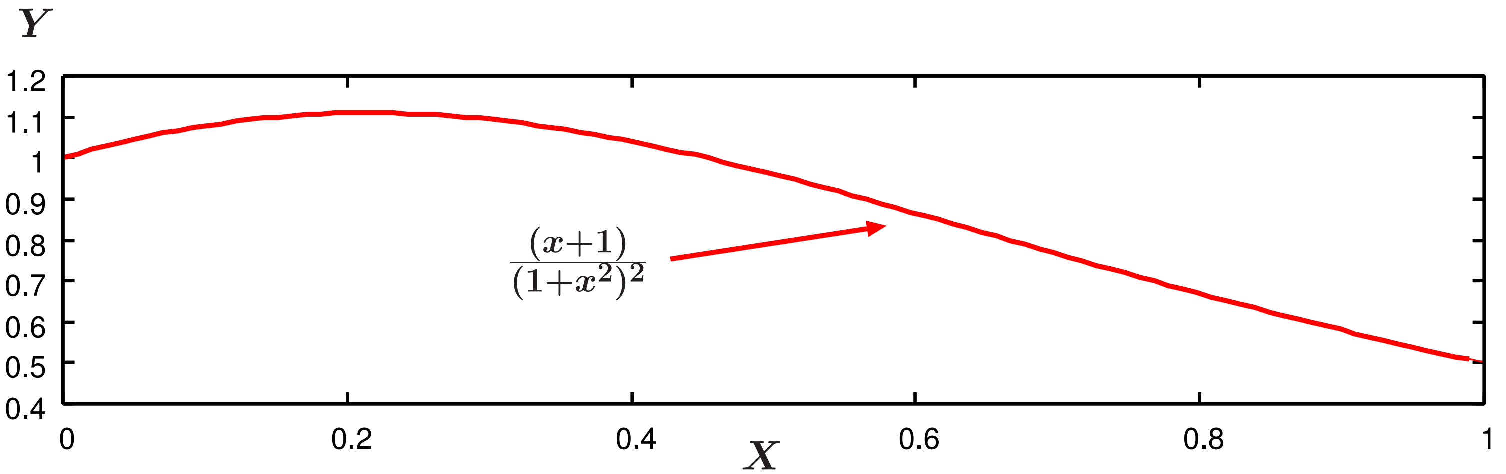
Then

$$\frac{dx}{d\theta} = \frac{d2\sin\theta}{d\theta} = 2\cos\theta.$$
$$\therefore dx = 2\cos\theta d\theta.$$

When x varies from -1 to 1, $\sin\theta$ varies from $-\frac{1}{2}$ to $\frac{1}{2}$. Therefore θ varies from $-\frac{\pi}{6}$ to $\frac{\pi}{6}$.

$$\int_{-1}^{1} \frac{dx}{\sqrt{4 - x^2}}$$
$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{2\cos\theta d\theta}{\sqrt{4 - 4\sin^2\theta}}$$
$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{2\cos\theta d\theta}{\sqrt{4(1 - \sin^2\theta)}}$$
$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{2\cos\theta d\theta}{\sqrt{4\cos^2\theta}}$$
$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{2\cos\theta d\theta}{2\cos\theta} (\because \cos\theta > 0 for -\frac{\pi}{6} \geq \theta \geq \frac{\pi}{6})$$
$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} d\theta$$
$$= [\theta]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$
$$= \frac{\pi}{6} - (-\frac{\pi}{6})$$
$$= \frac{\pi}{3}$$

52) Evaluate $\int_0^1 \frac{x+1}{(x^2+1)^2}dx$



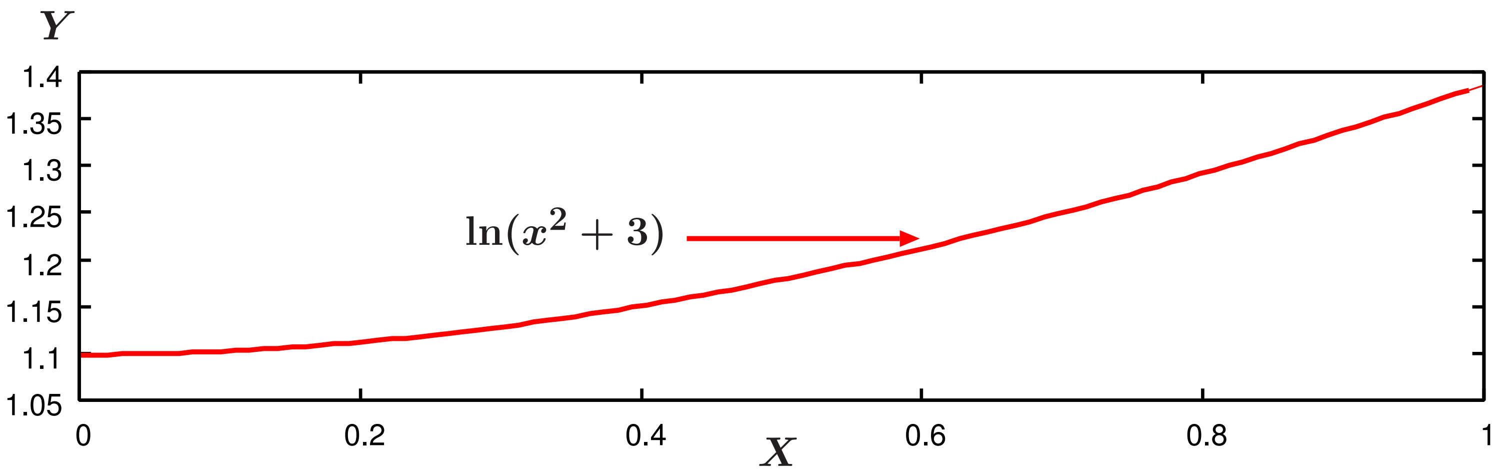
In general, if you find $\int \frac{1}{a^2x^2+b^2}dx$ it is better to set $x = \frac{b}{a}\tan\theta$. In this case, $x = \tan\theta$.

x	0	1
θ	0	$\frac{\pi}{4}$

$$x = \tan\theta$$
$$\therefore dx = \frac{1}{\cos^2\theta}d\theta$$

$$\begin{aligned}
&\int_0^1 \frac{x+1}{(x^2+1)^2}dx \\
&= \int_0^{\frac{\pi}{4}} \frac{\tan\theta+1}{(\tan^2\theta+1)^2}\frac{1}{\cos^2\theta}d\theta \\
&= \int_0^{\frac{\pi}{4}} (\tan\theta+1)\cos^4\theta\frac{1}{\cos^2\theta}d\theta \\
&= \int_0^{\frac{\pi}{4}} (\tan\theta+1)\cos^2\theta d\theta \\
&= \int_0^{\frac{\pi}{4}} (\tan\theta\cos^2\theta+\cos^2\theta)d\theta \\
&= \int_0^{\frac{\pi}{4}} (\sin\theta\cos\theta+\frac{1+\cos 2\theta}{2})d\theta \\
&= \int_0^{\frac{\pi}{4}} (\frac{1}{2}\sin 2\theta+\frac{1+\cos 2\theta}{2})d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sin 2\theta+1+\cos 2\theta)d\theta \\
&= \frac{1}{2}[-\frac{1}{2}\cos 2\theta+\theta+\frac{1}{2}\sin 2\theta]_0^{\frac{\pi}{4}} \\
&= \frac{1}{4}[-\cos 2\theta+2\theta+\sin 2\theta]_0^{\frac{\pi}{4}} \\
&= \frac{1}{4}\left\{2\frac{\pi}{4}+1+1\right\} \\
&= \frac{1}{4}\left\{2\frac{\pi}{4}+2\right\} \\
&= \frac{\pi}{8}+\frac{1}{2}
\end{aligned}$$

53) Evaluate $\int_0^1 \ln(x^2+3)dx$



When you see $\ln x$, it is safe to set $f(x)$ to $\ln x$ and $g(x)$ to the rest of the function. In this case, we set $f(x) = \ln(x^2+3)$ and $g(x) = 1$.

$$\begin{aligned}
&\int \ln(x^2+3)dx \\
&= \int f(x)\cdot g(x)dx \\
&= f(x)\cdot\int g(x)dx - \int\left(\frac{d\{f(x)\}}{dx}\cdot\int g(x)dx\right)dx \\
&= \ln(x^2+3)\cdot\int 1dx - \int\left(\frac{d\{\ln(x^2+3)\}}{dx}\cdot\int 1dx\right)dx \\
&= \ln(x^2+3)\cdot x - \int\left(\frac{2x}{x^2+3}\cdot x\right)dx \\
&= x\ln(x^2+3) - \int\left(\frac{2x^2}{x^2+3}\right)dx \\
&= x\ln(x^2+3) - \int\left(\frac{2x^2+6-6}{x^2+3}\right)dx \\
&= x\ln(x^2+3) - \int\left(2-\frac{6}{x^2+3}\right)dx \\
&= x\ln(x^2+3) - \int 2dx + \int\frac{6}{x^2+3}dx \\
&= x\ln(x^2+3) - 2x + \int\frac{6}{x^2+3}dx
\end{aligned}$$

We now need to find out $\int\frac{6}{x^2+3}dx$.

- First method
  Using Equation (78), we can modify $\int\frac{6}{x^2+3}dx$ as

$$\int\frac{6}{x^2+3}dx = 6\int\frac{1}{x^2+(\sqrt{3})^2}dx = \frac{6}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) = 2\sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

Thus

$$\int_0^1 \ln(x^2+3)dx = \left[x\ln(x^2+3)-2x\right]_0^1 + \int_0^1\frac{6}{x^2+3}dx$$

$$= \left[x\ln(x^2+3)-2x+2\sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_0^1 = \ln(1+3)-2+2\sqrt{3}\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\tan^{-1}\left(\frac{0}{\sqrt{3}}\right)\right)$$

Here we need to find out $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\tan^{-1}\left(\frac{0}{\sqrt{3}}\right)$.

$$\begin{aligned}
\theta &= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\
\therefore \tan\theta &= \frac{1}{\sqrt{3}} \\
\therefore \theta &= \frac{\pi}{6}
\end{aligned}$$

$$\theta = \tan^{-1}\left(\frac{0}{\sqrt{3}}\right)$$

$$\therefore \tan\theta = 0$$
$$\therefore \theta = 0$$

Finally

$$\int_0^1 \ln(x^2+3)dx = \ln(4) - 2 + 2\sqrt{3}\left(\frac{\pi}{6} - 0\right) = \ln(4) - 2 + \frac{\sqrt{3}\pi}{3}$$

- Second method

  In general, if you find $\int \frac{1}{a^2x^2+b^2}dx$ it is better to set $x = \frac{b}{a}\tan\theta$. In this case, $x = \sqrt{3}\tan\theta$.

x	0	1
θ	0	$\frac{\pi}{6}$

$$x = \sqrt{3}\tan\theta$$
$$dx = \frac{\sqrt{3}}{\cos^2\theta}d\theta$$

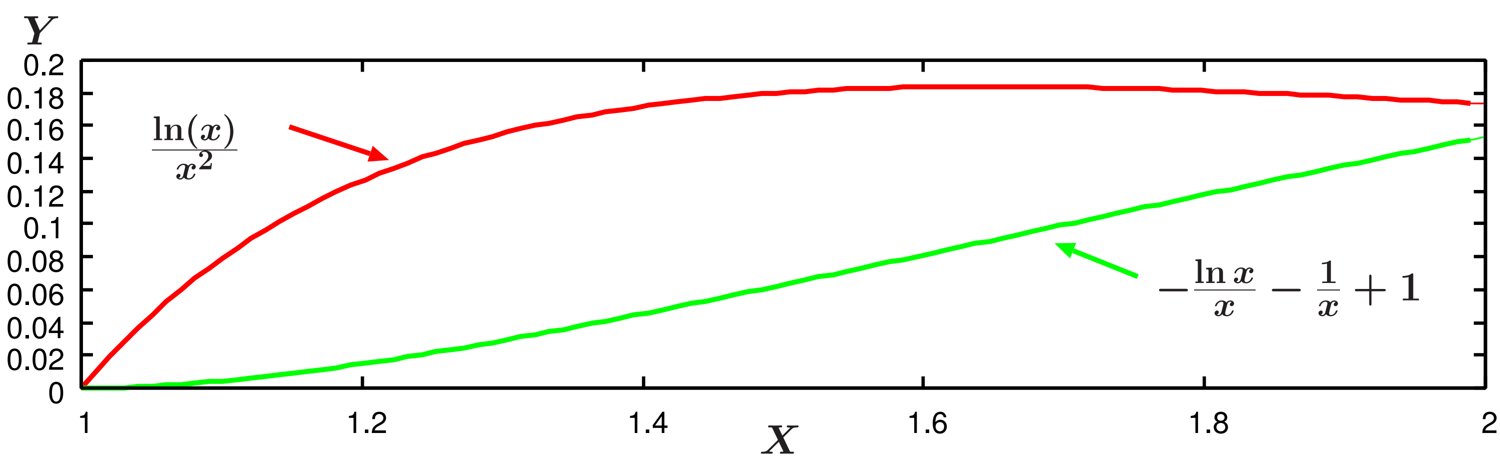
Thus

$$\int \frac{6}{x^2+3}dx$$
$$= \int \frac{6}{3\tan^2\theta + 3}\frac{\sqrt{3}}{\cos^2\theta}d\theta$$
$$= \int \frac{2}{\tan^2\theta + 1}\frac{\sqrt{3}}{\cos^2\theta}d\theta$$
$$= \int 2\cos^2\theta \frac{\sqrt{3}}{\cos^2\theta}d\theta$$
$$= \int 2\sqrt{3}d\theta$$
$$= 2\sqrt{3}\theta + C$$

Finally

$$\int_0^1 \ln(x^2+3)dx = [x\ln(x^2+3) - 2x]_0^1 + [2\sqrt{3}\theta]_0^{\frac{\pi}{6}}$$
$$= \ln 4 - 2 + 2\sqrt{3}\frac{\pi}{6} = \ln 4 - 2 + \sqrt{3}\frac{\pi}{3}$$

54) Evaluate $\int_1^2 \frac{\ln x}{x^2}dx$



When you see $\ln x$, it is safe to set $f(x)$ to $\ln x$ and $g(x)$ to the rest of the function. In this case, we set $f(x) = \ln x$ and $g(x) = x^{-2}$.

$$\int \frac{\ln x}{x^2}dx$$
$$= \int f(x)\cdot g(x)dx$$
$$= f(x)\cdot\int g(x)dx - \int\left(\frac{d\{f(x)\}}{dx}\cdot\int g(x)dx\right)dx$$
$$= \ln x\cdot\int x^{-2}dx - \int\left(\frac{d\{\ln x\}}{dx}\cdot\int x^{-2}dx\right)dx$$

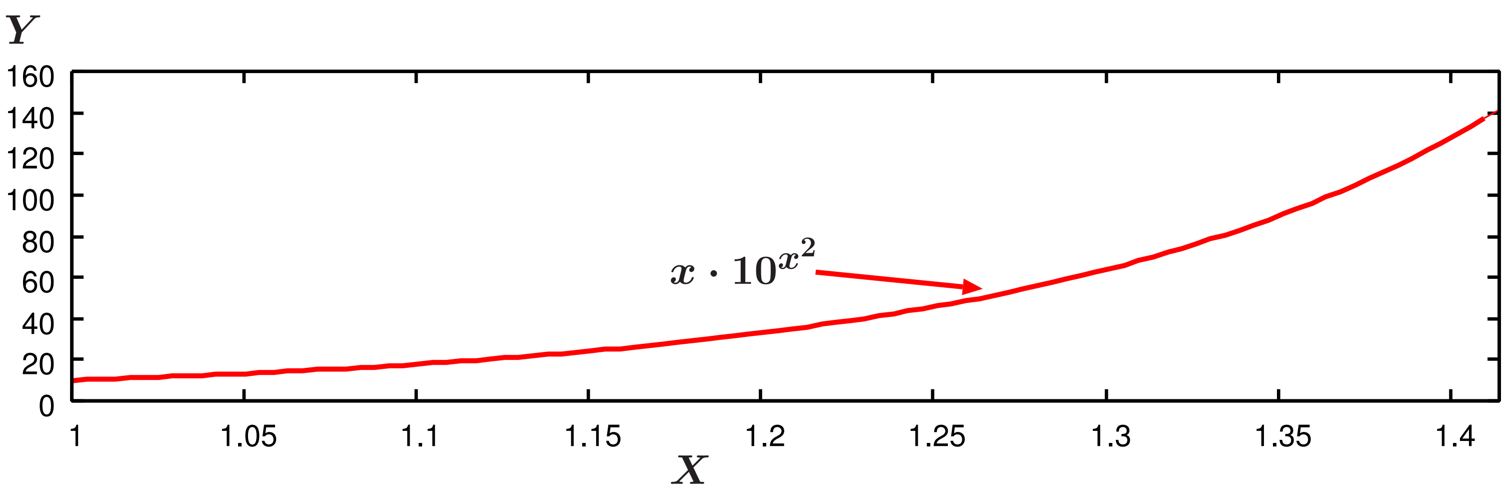
$$= \ln x \cdot (-x^{-1}) - \int \left(\frac{1}{x} \cdot (-x^{-1})\right) dx$$
$$= -\frac{\ln x}{x} + \int x^{-2} dx$$
$$= -\frac{\ln x}{x} - \frac{1}{x} + c$$

Thus the definite integral is

$$\int_1^2 \frac{\ln x}{x^2} dx$$
$$= [-\frac{\ln x}{x} - \frac{1}{x}]_1^2$$
$$= -\frac{\ln 2}{2} - \frac{1}{2} + 1$$
$$= -\frac{\ln 2}{2} + \frac{1}{2}$$

55) Find the definite integral

$$\int_1^{\sqrt{2}} x 10^{x^2} dx$$

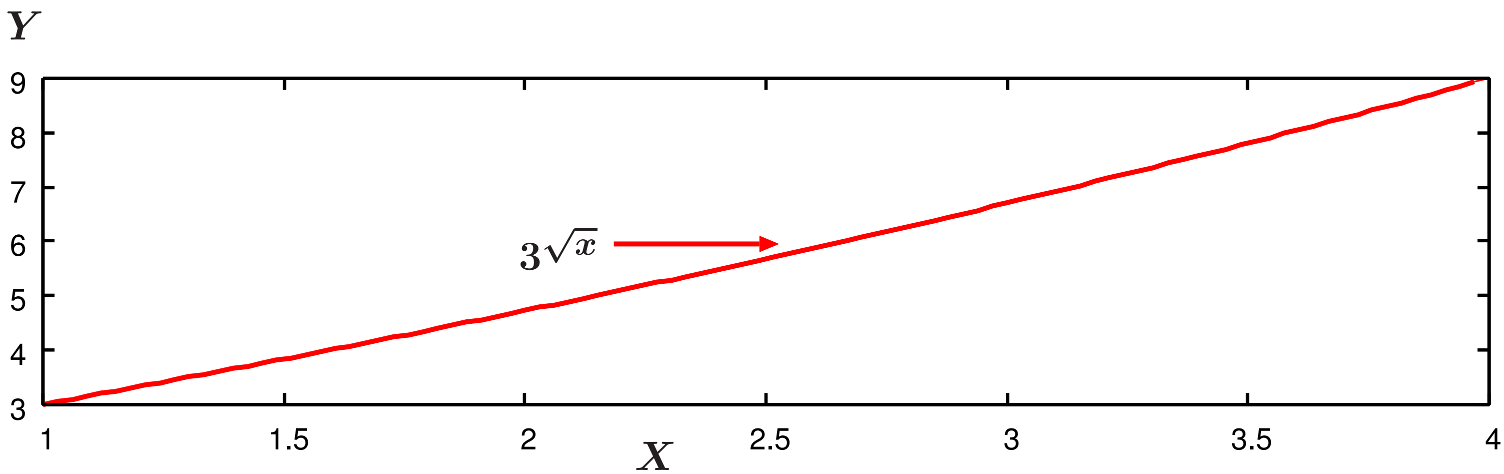


Let $t = x^2$. Since $\frac{d\{t\}}{dx} = 2x$, we obtain $dt = 2xdx$. When x varies from 1 to $\sqrt{2}$, t varies from $1(= 1^2) to 2(= \sqrt{2})^2$.

$$\int_1^{\sqrt{2}} x 10^{x^2} dx$$
$$= \int_1^{\sqrt{2}} 10^{x^2} \cdot \frac{1}{2} \cdot 2xdx$$
$$= \int_1^2 10^t \cdot \frac{1}{2} dt = \frac{1}{2} \int_1^2 10^t dt$$
$$= \left[\frac{1}{2} \frac{10^t}{\ln 10}\right]_1^2$$
$$= \frac{1}{2} \frac{10^2}{\ln 10} - \frac{1}{2} \frac{10^1}{\ln 10}$$
$$= \frac{1}{2} \frac{10^2 - 10}{\ln 10} = \frac{1}{2} \cdot \frac{90}{\ln 10}$$
$$= \frac{45}{\ln 10}$$

56) Find the definite integral

$$\int_0^4 3^{\sqrt{x}} dx$$



Let $t = \sqrt{x}$, so $t^2 = x$. Thus $2tdt = dx$. When x is varied from 0 to 4, t varies from $0(=\sqrt{0})$ to $2(=\sqrt{4})$.

x	0	4
t	0	2

Using Equation (56),

$$\int_0^4 3^{\sqrt{x}}dx = \int_0^2 3^t 2tdt$$

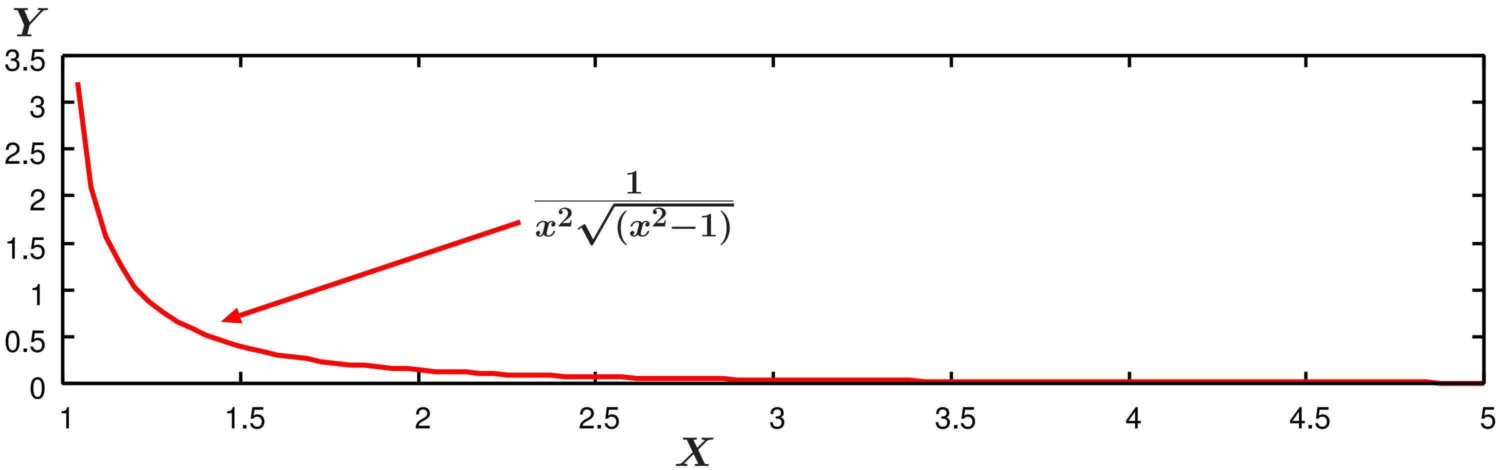
Let $f(t) = 2t$, $g(t) = 3^t$,

$$\int 3^t 2tdt = 2t\int 3^t dt - \int \left(\frac{d\{2t\}}{dt}\int 3^t dt\right) dt$$
$$= 2t\frac{3^t}{\ln 3} - \int \left(2\frac{3^t}{\ln 3}\right) dt$$
$$= \frac{2t\cdot 3^t}{\ln 3} - \frac{2}{\ln 3}\int 3^t dt$$
$$= \frac{2t\cdot 3^t}{\ln 3} - \frac{2}{\ln 3}\frac{3^t}{\ln 3}(\because Equation(69))$$
$$= \frac{2t\cdot 3^t}{\ln 3} - \frac{2\cdot 3^t}{(\ln 3)^2}$$

Thus

$$\int_0^2 3^t 2tdt$$
$$= \left[\frac{2t\cdot 3^t}{\ln 3} - \frac{2\cdot 3^t}{(\ln 3)^2}\right]_0^2$$
$$= (\frac{2\cdot 2\cdot 3^2}{\ln 3} - \frac{2\cdot 3^2}{(\ln 3)^2}) - (-\frac{2\cdot 3^0}{(\ln 3)^2})$$
$$= \frac{36}{\ln 3} - \frac{18}{(\ln 3)^2} + \frac{2}{(\ln 3)^2}$$
$$(\because a^0 = 1)$$
$$= \frac{36}{\ln 3} - \frac{16}{(\ln 3)^2}$$

57) Evaluate $\int \frac{1}{x^2\sqrt{x^2-1}}dx$ for $x > 1$ by setting $x = \frac{1}{\sin\theta}$ $(0 < \theta < \frac{\pi}{2})$



$$x = \frac{1}{\sin\theta}$$
$$\therefore dx = \frac{-\cos\theta}{\sin^2\theta}d\theta$$

Using this,

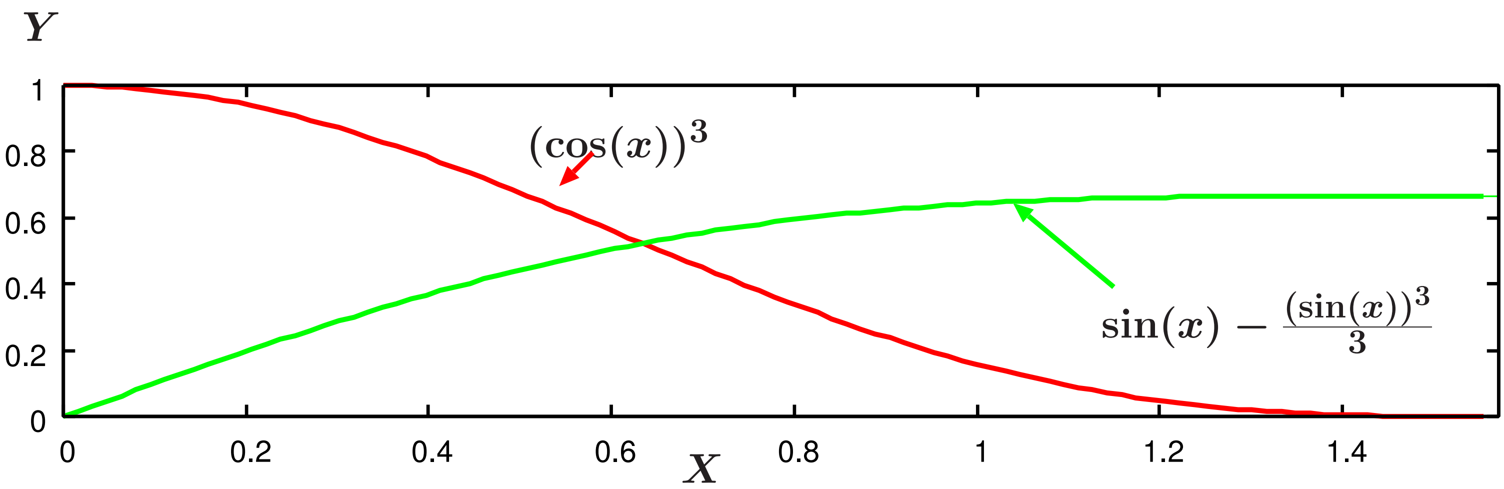
$$\int \frac{1}{x^2\sqrt{x^2-1}}dx$$
$$= \int \frac{1}{\frac{1}{\sin^2\theta}\sqrt{\frac{1}{\sin^2\theta}-1}}\frac{-\cos\theta}{\sin^2\theta}d\theta$$
$$= \int \frac{-\cos\theta}{\sqrt{\frac{1}{\sin^2\theta}-1}}d\theta$$
$$= \int \frac{-\cos\theta}{\sqrt{\frac{1-\sin^2\theta}{\sin^2\theta}}}d\theta$$
$$= \int \frac{-\cos\theta}{\sqrt{\frac{\cos^2\theta}{\sin^2\theta}}}d\theta$$
$$= \int \frac{-\cos\theta}{\frac{\cos\theta}{\sin\theta}}d\theta$$
$$= \int(-\sin\theta)d\theta$$
$$= \cos\theta + C$$

Since this is the indefinite integral, the answer has to be expressed using x. This we now express $\cos\theta$ using x:

$$x = \frac{1}{\sin\theta}$$
$$\therefore \sin\theta = \frac{1}{x}$$
$$\therefore \sin^2\theta = \frac{1}{x^2}$$
$$\therefore 1 - \sin^2\theta = 1 - \frac{1}{x^2}$$
$$\therefore \cos^2\theta = 1 - \frac{1}{x^2}$$
$$\therefore \cos\theta = \sqrt{1-\frac{1}{x^2}} = \frac{\sqrt{x^2-1}}{x}$$

58) Find the definite integral

$$\int_0^{\frac{\pi}{2}} \cos^3 x dx$$



If $a \cdot a^2 = a^3$ then $\cos^3 x = \cos^2 x \cdot \cos x$. Using this

$$\int_0^{\frac{\pi}{2}} \cos^3 x dx$$
$$= \int_0^{\frac{\pi}{2}} \cos^2 x \cos x dx$$
$$= \int_0^{\frac{\pi}{2}} (1 - \sin^2 x) \cos x dx$$
$$(\because \cos^2 x + \sin^2 x = 1, \cos^2 x = 1 - \sin^2 x)$$

When we let $u = \sin x$, we can get $\frac{d\{u\}}{dx} = \cos x$.

$$\begin{array}{cccc} x & 0 & \to & \frac{\pi}{2} \\ u = \sin x & 0 & \to & 1 \end{array}$$

Using this we get:

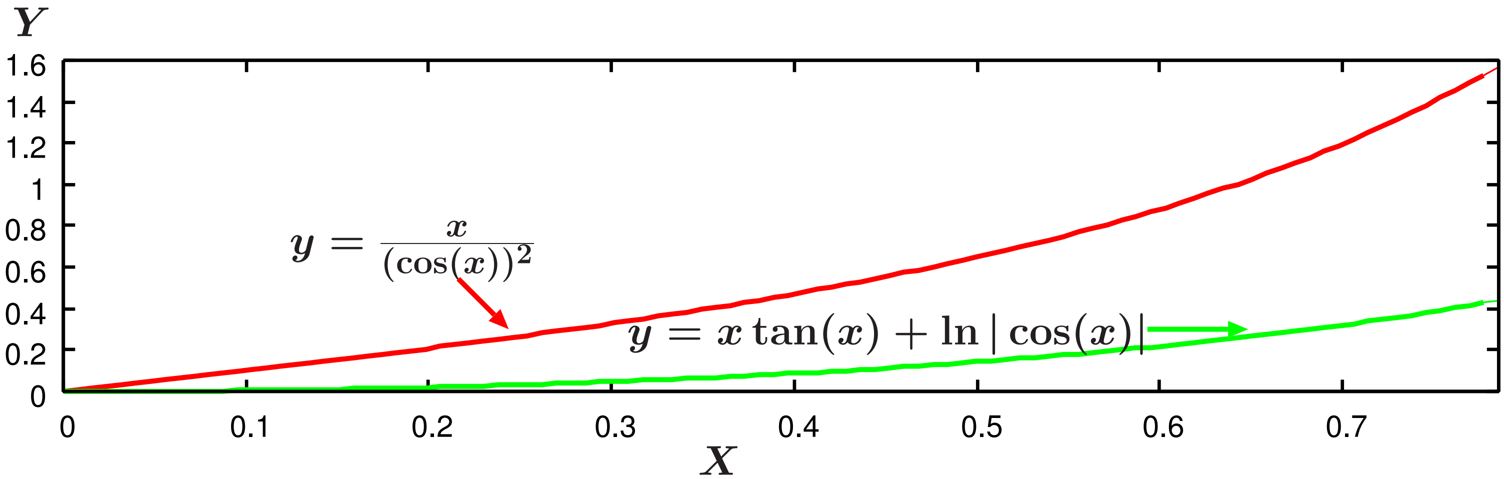
$$\int_0^{\frac{\pi}{2}} (1 - \sin^2 x) \cos x dx$$
$$= \int_{u=0}^{u=1} (1 - u^2) \cos x \cdot \frac{du}{\cos x}$$
$$= \int_0^1 (1 - u^2) du$$
$$= \left[u - \frac{1}{3} u^3 \right]_0^1$$
$$= \left[1 - \frac{1}{3} \right]$$
$$= \frac{2}{3}$$

Or alternatively;

$$\int_0^{\frac{\pi}{2}} \cos^3 x dx$$
$$= \int_0^{\frac{\pi}{2}} \cos^2 x \cos x dx$$
$$= \int_0^{\frac{\pi}{2}} (1 - \sin^2 x) \frac{d\{\sin x\}}{dx} dx$$
$$= \int_0^{\frac{\pi}{2}} \left(\frac{\partial \sin x}{\partial x} - \sin^2 x \frac{\partial \sin x}{\partial x} \right) dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{d\{\sin x\}}{dx} dx - \int_0^{\frac{\pi}{2}} \sin^2 x \frac{d\{\sin x\}}{dx} dx$$
$$= \left[\sin x - \frac{1}{3} \sin^3 x \right]_0^{\frac{\pi}{2}}$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

59) Evaluate $\int_0^{\frac{\pi}{4}} \frac{x}{\cos^2 x} dx$



Since Equation (71) tells us that $\frac{1}{\cos^2 x}$ is integrable, we set $g(x) = \frac{1}{\cos^2 x}$ and the rest is $f(x)$. In this case, we set $f(x) = x$ and $g(x) = \frac{1}{\cos^2 x}$.

$$\begin{aligned}
&\int \frac{x}{\cos^2 x} dx \\
&= \int f(x) \cdot g(x) dx \\
&= f(x) \cdot \int g(x) dx - \int \left(\frac{d\{f(x)\}}{dx} \cdot \int g(x) dx \right) dx \\
&= x \cdot \int \frac{1}{\cos^2 x} dx - \int \left(\frac{d\{x\}}{dx} \cdot \int \frac{1}{\cos^2 x} dx \right) dx \\
&= x \cdot \tan x - \int \tan x dx \\
&= x \tan x - (-\ln|\cos x|) \quad (\because Equation(67)) \\
&= x \tan x + \ln|\cos x| + c
\end{aligned}$$

Finally the definite integral is

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} \frac{x}{\cos^2 x} dx \\
&= [x \tan x + \ln|\cos x|]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{4} + \ln \frac{1}{\sqrt{2}} \\
&= \frac{\pi}{4} - \frac{1}{2} \ln 2
\end{aligned}$$

60) Given points $A(0,0)$ $B(2,2)$ and $C(9,3)$ Work out the equation of the lines AB and BC.
In general, an equation of a line whose gradient is m, going through (x_a, y_a)can be written as

$$y - y_a = m(x - x_a).$$

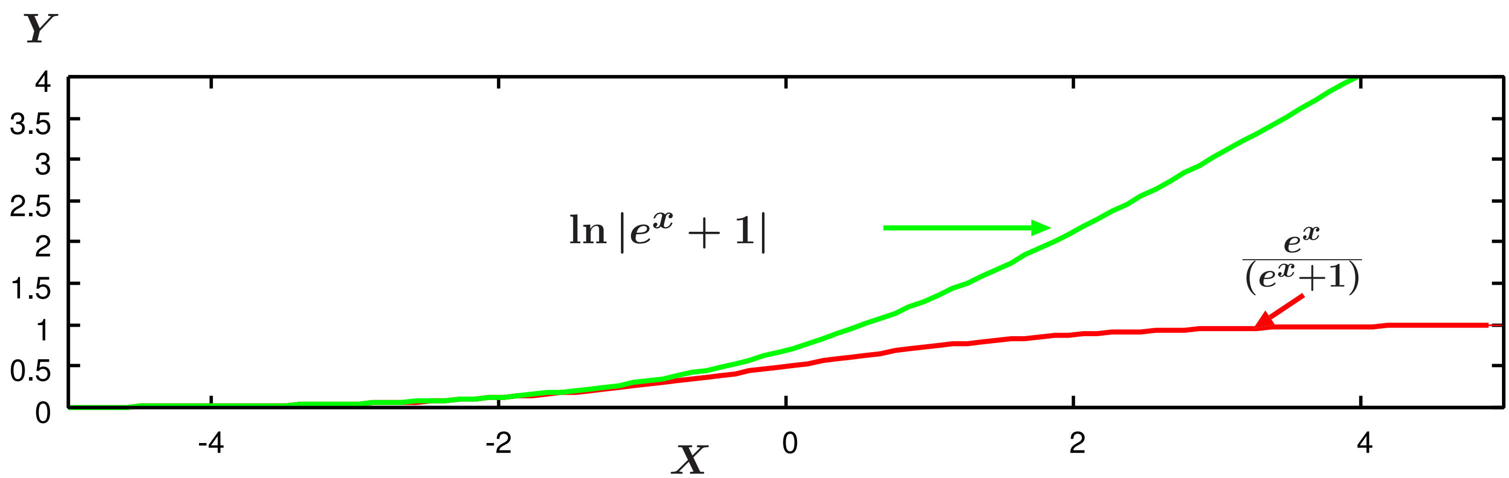
As for the line AB, since the line goes through the point A(0,0)

$$\begin{aligned}
y - y_a &= m(x - x_a) \\
\therefore y - 0 &= m(x - 0) \\
\therefore m &= \frac{2-0}{2-0} = 1 \\
\therefore y &= x
\end{aligned}$$

As for the line BC, since the line goes through the point B(2,2),

$$y - y_b = m(x - x_b)$$
$$\therefore y - 2 = m(x - 2)$$
$$\therefore y = m(x - 2) + 2$$
$$\therefore m = \frac{3-2}{9-2} = \frac{1}{7}$$
$$\therefore y = \frac{1}{7}x - \frac{2}{7} + 2$$
$$\therefore y = \frac{1}{7}x - \frac{2}{7} + \frac{14}{7}$$
$$\therefore y = \frac{1}{7}x + \frac{12}{7}$$

61) Find $\int \frac{\mathfrak{e}^x}{\mathfrak{e}^x + 1} dx$



Since

$$\frac{d\left\{\mathfrak{e}^x + 1\right\}}{dx} = \mathfrak{e}^x,$$

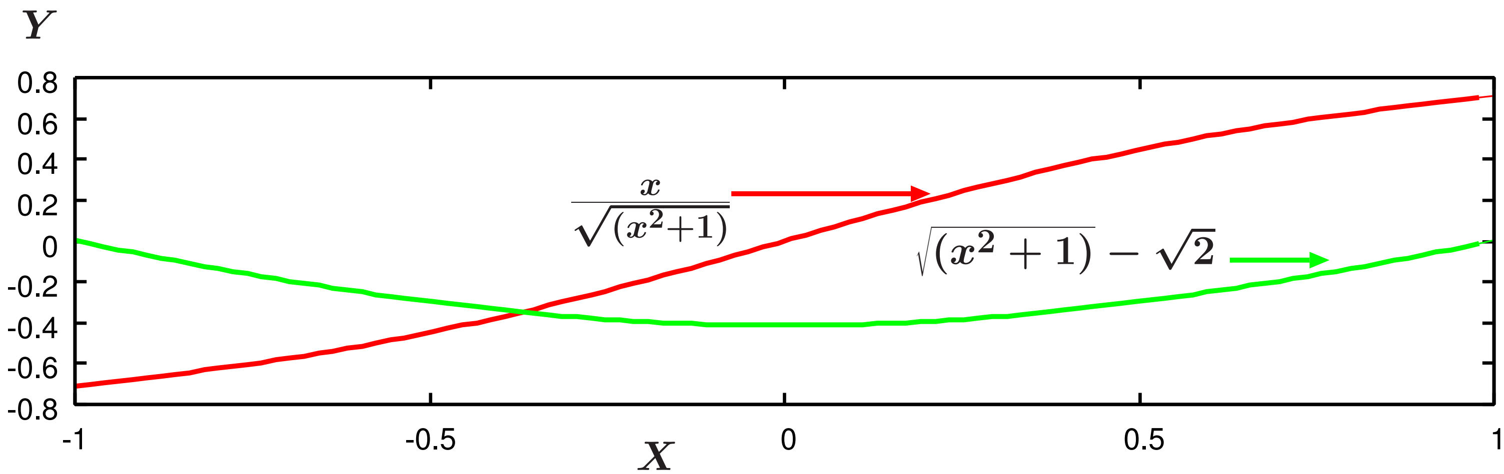
when we let

$$f(x) = \mathfrak{e}^x + 1,$$

then

$$\int \frac{\mathfrak{e}^x}{\mathfrak{e}^x + 1} dx$$
$$= \int \frac{\frac{d\{f(x)\}}{dx}}{f(x)} dx$$
$$= \int \frac{\partial f(x)}{f(x)}$$
$$= \ln |f(x)| + c$$
$$= \ln |\mathfrak{e}^x + 1| + c$$

62) Find $\int \frac{x}{\sqrt{x^2 + 1}} dx$

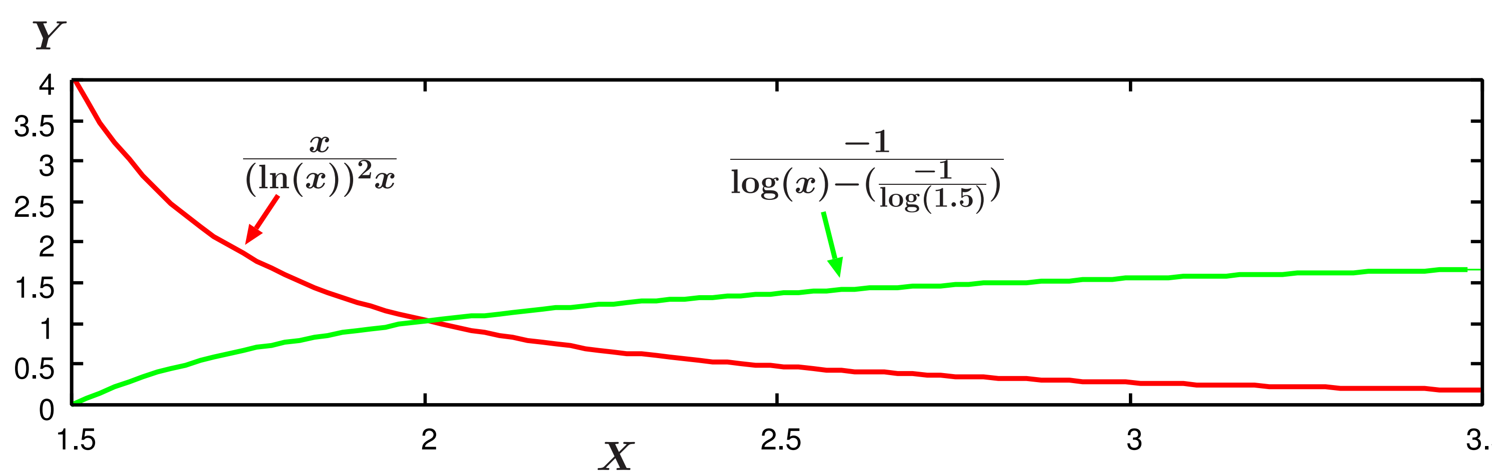


If we let $t = x^2 + 1$ Then $\frac{d\{t\}}{dx} = 2x$
Therefore $\partial t = \partial x \cdot 2x$ which equals $\frac{1}{2} \cdot \partial t = \partial x \cdot x$

$$\therefore \int \frac{x}{\sqrt{x^2+1}}dx$$
$$= \int \frac{1}{2} \cdot \frac{1}{\sqrt{t}}dt \because t = x^2 + 1 \text{ and } \partial x \cdot x = \frac{1}{2} \cdot \partial t$$
$$= \frac{1}{2} \int \frac{1}{\sqrt{t}}dt$$
$$= \frac{1}{2} \int t^{-\frac{1}{2}}dt$$
$$= \frac{1}{2} \cdot 2t^{\frac{1}{2}} + c$$
$$= t^{\frac{1}{2}} + c$$
$$= (x^2 + 1)^{\frac{1}{2}} + c (\because t = x^2 + 1)$$

63) Find the indefinite integral

$$\int \frac{dx}{x(\ln x)^2}$$



Let $t = \ln x$ therefore $\frac{d\{t\}}{dx} = \frac{1}{x}$.
$x \cdot \partial t = \partial x$
The integral now becomes

$$
\begin{aligned}
&\int \frac{dx}{x(\ln x)^2} \\
&= \int \frac{1}{xt^2} dx \\
&= \int \frac{1}{t^2} dt (\because \partial x = x \cdot \partial t) \\
&= \int t^{-2} dt \\
&= -t^{-1} + c \\
&= -\frac{1}{\ln x} + c \because t = \ln x
\end{aligned}
$$

DAY3

64) What is $\frac{\pi}{2}+\pi+\frac{4\pi}{5}+2$.

$$\begin{aligned}\frac{\pi}{2}+\pi+\frac{4\pi}{5}+2\\ \frac{\pi\times 5}{2\times 5}+\frac{10\pi}{10}+\frac{4\pi\times 2}{5\times 2}+\frac{2\times 10}{10}\\ \frac{5\pi}{10}+\frac{10\pi}{10}+\frac{8\pi}{10}+\frac{20}{10}\\ \frac{5\pi+10\pi+8\pi+20}{10}\\ \frac{23\pi+20}{10}\end{aligned}$$

65) Simplify

$$(-8a^2b^{-3})^2$$

$$\begin{aligned}&(-8a^2b^{-3})^2\\ =&(-8)^2a^{2\times2}b^{-3\times2}\\ =&64a^4b^{-6}\end{aligned}$$

66) Simplify $\frac{(16x^2y^{-3})^3(-2x^{-5}y^2)^0}{64(xy)^5}$

$$\begin{aligned}&\frac{(16x^2y^{-3})^3(-2x^{-5}y^2)^0}{64(xy)^5}\\ =&\frac{16^3x^{2\times3}y^{-3\times3}\cdot 1}{64x^5y^5}\\ =&\frac{16^3x^6y^{-9}}{64x^5y^5}\\ =&\frac{4096x^{6-5}y^{-9-5}}{64}\\ =&64xy^{-14}\end{aligned}$$

67) Solve the system of

$$\begin{aligned}2x+3y=4\\ x+2y=3\end{aligned}$$

$$\begin{aligned}\begin{pmatrix}2&3\\1&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}&=\begin{pmatrix}4\\3\end{pmatrix}\\ \therefore\begin{pmatrix}x\\y\end{pmatrix}&=\begin{pmatrix}2&3\\1&2\end{pmatrix}^{-1}\begin{pmatrix}4\\3\end{pmatrix}\\ &=\frac{1}{2\cdot2-3\cdot1}\begin{pmatrix}2&-3\\-1&2\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix}\\ &=\begin{pmatrix}2&-3\\-1&2\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix}\\ &=\begin{pmatrix}2\cdot4-3\cdot3\\-1\cdot4+2\cdot3\end{pmatrix}\\ &=\begin{pmatrix}8-9\\-4+6\end{pmatrix}\\ &=\begin{pmatrix}-1\\2\end{pmatrix}\end{aligned}$$

68) Find $\int\frac{5x^2-9x+4}{x(2-3x)(1-2x)}dx$

We assume

$$\frac{5x^2-9x+4}{x(2-3x)(1-2x)}=\frac{A}{x}+\frac{B}{2-3x}+\frac{C}{1-2x}\qquad ①$$

When we multiply ① with $x(2-3x)(1-2x)$ we obtain

$$(5x^2 - 9x + 4) = A(2-3x)(1-2x) + Bx(1-2x) + Cx(2-3x) \qquad ②$$

When we put $x = \frac{1}{2}$ into ②, we obtain

$$\begin{aligned}(5\left(\frac{1}{2}\right)^2 - 9\cdot\frac{1}{2} + 4) &= C\cdot\frac{1}{2}\cdot(2-3\cdot\frac{1}{2})\\ \therefore (\frac{5}{4} - \frac{18}{4} + \frac{16}{4}) &= C\cdot\frac{1}{2}\cdot(\frac{4}{2}-\frac{3}{2})\\ \therefore \frac{3}{4} &= C\cdot\frac{1}{4}\\ \therefore 3 &= C\end{aligned}$$

When we put $x = \frac{2}{3}$ into ②, we obtain

$$\begin{aligned}(5\left(\frac{2}{3}\right)^2 - 9\cdot\frac{2}{3} + 4) &= B\frac{2}{3}(1-2\cdot\frac{2}{3})\\ \therefore (\frac{20}{9} - \frac{54}{9} + \frac{36}{9}) &= B\frac{2}{3}(\frac{3}{3}-\frac{4}{3})\\ \therefore \frac{2}{9} &= -B\frac{2}{9}\\ \therefore -1 &= B\end{aligned}$$

When we put $x = 0$ into ②, we obtain

$$\begin{aligned}4 &= 2A\\ \therefore 2 &= A\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{5x^2-9x+4}{x(2-3x)(1-2x)}dx &= \int\left(\frac{2}{x} - \frac{1}{2-3x} + \frac{3}{1-2x}\right)dx\\ &= \int\left(\frac{2}{x} + \frac{1}{3}\cdot\frac{1}{x-\frac{2}{3}} - \frac{3}{2}\cdot\frac{1}{x-\frac{1}{2}}\right)dx\\ &= 2\ln|x| + \frac{1}{3}\ln|x-\frac{2}{3}| - \frac{3}{2}\ln|x-\frac{1}{2}| + c\end{aligned}$$

69) Find $\int \frac{23x^2+22x+2}{(2x-1)(x+2)(3x+1)}dx$

We assume

$$\frac{23x^2+22x+2}{(2x-1)(x+2)(3x+1)} = \frac{A}{2x-1} + \frac{B}{x+2} + \frac{C}{3x+1} \qquad ①$$

When we multiply ① with $(2x-1)(x+2)(3x+1)$ we obtain

$$(23x^2 + 22x + 2) = A(x+2)(3x+1) + B(2x-1)(3x+1) + C(2x-1)(x+2) \qquad ②$$

When we put $x = \frac{1}{2}$ into ②, we obtain

$$\begin{aligned}(23\left(\frac{1}{2}\right)^2 + 22\cdot\frac{1}{2} + 2) &= A(\frac{1}{2}+2)(3\cdot\frac{1}{2}+1)\\ \therefore (\frac{23}{4} + \frac{44}{4} + \frac{8}{4}) &= A(\frac{1}{2}+\frac{4}{2})(\frac{3}{2}+\frac{2}{2})\\ \therefore \frac{75}{4} &= A(\frac{5}{2})(\frac{5}{2})\\ \therefore \frac{75}{4} &= A\cdot\frac{25}{4}\\ \therefore 3 &= A\end{aligned}$$

When we put $x = -2$ into ②, we obtain

$$\begin{aligned}23\cdot(-2)^2 + 22\cdot(-2) + 2 &= B(2\cdot(-2)-1)(3\cdot(-2)+1)\\ \therefore 4\cdot 23 - 44 + 2 &= B(-4-1)(-6+1)\\ \therefore 92 - 44 + 2 &= B(-5)(-5)\\ \therefore 50 &= 25B\\ \therefore 2 &= B\end{aligned}$$

When we put $x = -\frac{1}{3}$ into ②, we obtain

$$\begin{aligned}(23\left(-\frac{1}{3}\right)^2 + 22\cdot\left(-\frac{1}{3}\right) + 2) &= C(2\left(-\frac{1}{3}\right) - 1)(-\frac{1}{3} + 2)\\ \therefore \frac{23}{9} - \frac{22}{3} + 2 &= C(-\frac{2}{3} - 1)(-\frac{1}{3} + \frac{6}{3})\\ \therefore \frac{23}{9} - \frac{66}{9} + \frac{18}{9} &= C(-\frac{2}{3} - \frac{3}{3})(\frac{5}{3})\\ \therefore -\frac{25}{9} &= C(-\frac{5}{3})(\frac{5}{3})\\ \therefore -\frac{25}{9} &= C(-\frac{25}{9})\\ \therefore 1 &= C\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{23x^2 + 22x + 2}{(2x-1)(x+2)(3x+1)}dx &= \int \left(\frac{3}{2x-1} + \frac{2}{x+2} + \frac{1}{3x+1}\right)dx\\ &= \int \left(\frac{3}{2}\cdot\frac{1}{x-\frac{1}{2}} + \frac{2}{x+2} + \frac{1}{3}\cdot\frac{1}{x+\frac{1}{3}}\right)dx\\ &= \frac{3}{2}\ln|x - \frac{1}{2}| + 2\ln|x+2| + \frac{1}{3}\ln|x + \frac{1}{3}| + c\end{aligned}$$

70) Find $\int \frac{4x^2 + 37x + 7}{(2x+1)(x-3)(4x-1)}dx$
We assume

$$\frac{4x^2 + 37x + 7}{(2x+1)(x-3)(4x-1)} = \frac{A}{2x+1} + \frac{B}{x-3} + \frac{C}{4x-1} \qquad ①$$

When we multiply ① with $(2x+1)(x-3)(4x-1)$ we obtain

$$(4x^2 + 37x + 7) = A(x-3)(4x-1) + B(2x+1)(4x-1) + C(2x+1)(x-3) \qquad ②$$

When we put $x = -\frac{1}{2}$ into ②, we obtain

$$\begin{aligned}(4\left(-\frac{1}{2}\right)^2 + 37\cdot\left(-\frac{1}{2}\right) + 7) &= A(-\frac{1}{2} - 3)(-4\cdot\frac{1}{2} - 1)\\ \therefore (\frac{4}{4} - \frac{74}{4} + \frac{28}{4}) &= A(-\frac{1}{2} - \frac{6}{2})(-2 - 1)\\ \therefore -\frac{42}{4} &= A(-\frac{7}{2})(-3)\\ \therefore -\frac{21}{2} &= A\cdot\frac{21}{2}\\ \therefore -1 &= A\end{aligned}$$

When we put $x = 3$ into ②, we obtain

$$\begin{aligned}4\cdot(3)^2 + 37\cdot 3 + 7 &= B(2\cdot 3 + 1)(4\cdot 3 - 1)\\ \therefore 4\cdot 9 + 111 + 7 &= B(6+1)(12-1)\\ \therefore 36 + 111 + 7 &= B(7)(11)\\ \therefore 154 &= 77B\\ \therefore 2 &= B\end{aligned}$$

When we put $x = \frac{1}{4}$ into ②, we obtain

$$\begin{aligned}(4\left(\frac{1}{4}\right)^2 + 37\cdot\frac{1}{4} + 7) &= C(2\cdot\frac{1}{4} + 1)(\frac{1}{4} - 3)\\ \therefore (\frac{4}{16} + \frac{148}{16} + \frac{112}{16}) &= C(\frac{1}{2} + \frac{2}{2})(\frac{1}{4} - \frac{12}{4})\\ \therefore \frac{264}{16} &= C(\frac{3}{2})(-\frac{11}{4})\\ \therefore \frac{33}{2} &= C\cdot(-\frac{33}{8})\\ \therefore -4 &= C\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{4x^2+37x+7}{(2x+1)(x-3)(4x-1)}dx &= \int \left(-\frac{1}{2x+1}+\frac{2}{x-3}-\frac{4}{4x-1}\right)dx \\ &= \int \left(-\frac{1}{2}\cdot\frac{1}{x+\frac{1}{2}}+\frac{2}{x-3}-\frac{1}{x-\frac{1}{4}}\right)dx \\ &= -\frac{1}{2}\ln|x+\frac{1}{2}|+2\ln|x-3|-\ln|x-\frac{1}{4}|+c\end{aligned}$$

71) Find $\int \frac{-x^2+13x+6}{x(x-2)(2x+3)}dx$

We assume

$$\frac{-x^2+13x+6}{x(x-2)(2x+3)} = \frac{A}{x}+\frac{B}{x-2}+\frac{C}{2x+3} \qquad ①$$

When we multiply ① with $x(x-2)(2x+3)$ we obtain

$$(-x^2+13x+6) = A(x-2)(2x+3)+Bx(2x+3)+Cx(x-2) \qquad ②$$

When we put $x=0$ into ②, we obtain

$$\begin{aligned}6 &= A\cdot(-2)\cdot 3 \\ \therefore 6 &= -6A \\ \therefore -1 &= A\end{aligned}$$

When we put $x=2$ into ②, we obtain

$$\begin{aligned}-2^2+13\cdot 2+6 &= B\cdot 2\cdot(2\cdot 2+3) \\ \therefore -4+26+6 &= B\cdot 2(4+3) \\ \therefore 28 &= 14B \\ \therefore 2 &= B\end{aligned}$$

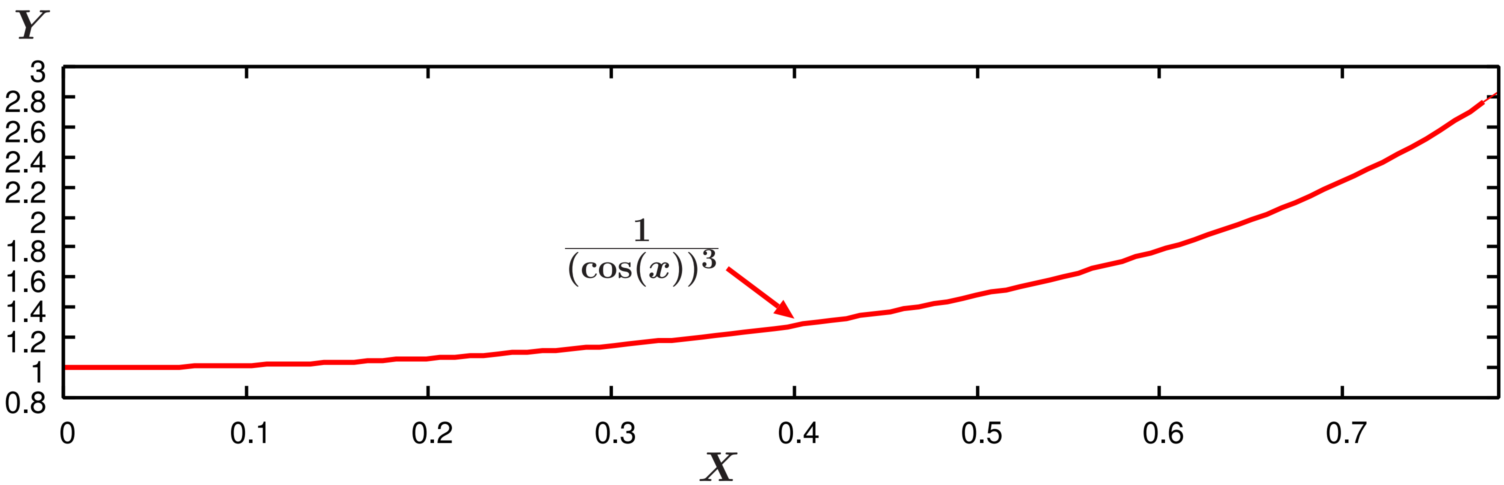
When we put $x=-\frac{3}{2}$ into ②, we obtain

$$\begin{aligned}-\left(-\frac{3}{2}\right)^2+13\cdot\left(-\frac{3}{2}\right)+6 &= C\cdot\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-2\right) \\ \therefore -\frac{9}{4}-\frac{39}{2}+6 &= C\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-\frac{4}{2}\right) \\ \therefore -\frac{9}{4}-\frac{78}{4}+\frac{24}{4} &= C\frac{3}{2}\cdot\frac{7}{2} \\ \therefore -\frac{63}{4} &= C\frac{21}{4} \\ \therefore -3 &= C\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{-x^2+13x+6}{x(x-2)(2x+3)}dx &= \int \left(-\frac{1}{x}+\frac{2}{x-2}-\frac{3}{2x+3}\right)dx \\ &= \int \left(-\frac{1}{x}+\frac{2}{x-2}-\frac{3}{2}\cdot\frac{1}{x+\frac{3}{2}}\right)dx \\ &= -\ln|x|+2\ln|x-2|-\frac{3}{2}\ln|x+\frac{3}{2}|+c\end{aligned}$$

72) Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{\cos^3 x}dx$



When we set $t = \sin x$, $dt = \cos x dx$ and the function to be integrated will have $\cos^4 x = (\cos^2 x)^2 = (1 - \sin^2 x)^2$.

x	0	$\frac{\pi}{4}$
t	0	$\frac{1}{\sqrt{2}}$

$$\begin{aligned} &\int \frac{1}{\cos^3 x} dx \\ =& \int \frac{1}{\cos^3 x} \frac{dt}{\cos x} \\ =& \int \frac{1}{\cos^4 x} dt \\ =& \int \frac{1}{(\cos^2 x)^2} dt \\ =& \int \frac{1}{(1 - \sin^2 x)^2} dt \\ =& \int \frac{1}{(1 - t^2)^2} dt \\ =& \int \frac{1}{\{(1 - t)(1 + t)\}^2} dt \\ =& \int \frac{1}{(1 - t)^2 (1 + t)^2} dt \end{aligned}$$

We now assume $\frac{1}{(t-1)^2(1+t)^2}$ can be separated into four terms as :

$$\frac{1}{(t-1)^2(1+t)^2} = \frac{A}{(1+t)^2} + \frac{B}{(t-1)^2} + \frac{C}{1+t} + \frac{D}{t-1} \qquad ①$$

$$\therefore 1 = (t-1)^2 A + (t+1)^2 B + (1+t)(t-1)^2 C + (t-1)(t+1)^2 D \qquad ②$$

When we put $t = 1$ into ②

$$1 = 4B$$
$$\therefore B = 1/4 \qquad ③$$

When we put $t = -1$ into ②

$$1 = 4A$$
$$\therefore A = 1/4 \qquad ④$$

When we put $t = 0$ and ③ and ④ into ②,

$$1 = \frac{1}{4} + \frac{1}{4} + C - D$$
$$\therefore \frac{1}{2} = C - D \qquad ⑤$$

When we put $t = 2$ and ③ and ④ into ②,

$$1 = \frac{10}{4} + 3C + 9D$$
$$\therefore \frac{-6}{4} = 3C + 9D$$
$$\therefore \frac{-1}{2} = C + 3D \qquad ⑥$$

⑤$\times 3$ + ⑥ gives

$$1 = 4C$$
$$\therefore C = \frac{1}{4} \qquad ⑦$$

By putting ⑦ into ⑤, we obtain

$$\frac{1}{2} = \frac{1}{4} - D$$
$$\therefore -\frac{1}{4} = D \qquad ⑧$$

By putting ③,④,⑦, and ⑧ into ①, we obtain

$$\int \frac{1}{(1-t)^2(1+t)^2} dt$$
$$= \frac{1}{4} \int \left(\frac{1}{(1+t)^2} + \frac{1}{(t-1)^2} + \frac{1}{1+t} - \frac{1}{t-1} \right) dt$$
$$= \frac{1}{4} \int \left((1+t)^{-2} + (t-1)^{-2} + (1+t)^{-1} - (t-1)^{-1} \right) dt$$
$$= \frac{1}{4} \Bigg(-(1+t)^{-1} - (t-1)^{-1} + \ln|1+t| - \ln|t-1| \Bigg)$$

Thus the definite integral is

$$\left[\frac{-(1+t)^{-1} - (t-1)^{-1} + \ln|1+t| - \ln|t-1|}{4} \right]_0^{\frac{1}{\sqrt{2}}}$$
$$= \frac{-(1+\frac{1}{\sqrt{2}})^{-1} - (\frac{1}{\sqrt{2}}-1)^{-1} + \ln|1+\frac{1}{\sqrt{2}}| - \ln|\frac{1}{\sqrt{2}}-1| + 2}{4} = \frac{-\frac{1}{1+\frac{1}{\sqrt{2}}} - \frac{1}{\frac{1}{\sqrt{2}}-1} + \ln|\frac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}-1}| + 2}{4}$$
$$= \frac{-\frac{1}{1+\frac{1}{\sqrt{2}}} - \frac{1}{\frac{1}{\sqrt{2}}-1} + \ln|\frac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}-1}| + 2}{4} = \frac{-\frac{\frac{2}{\sqrt{2}}}{(1+\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}-1)} + \ln|\frac{\sqrt{2}+1}{1-\sqrt{2}}| + 2}{4}$$
$$= \frac{-\frac{\frac{2}{\sqrt{2}}}{\frac{-1}{2}} + \ln|\frac{(\sqrt{2}+1)^2}{(1-\sqrt{2})(1+\sqrt{2})}| + 2}{4} = \frac{\frac{4}{\sqrt{2}} + \ln|(\sqrt{2}+1)^2| + 2}{4}$$
$$= \frac{\frac{4}{\sqrt{2}} + 2\ln|\sqrt{2}+1| + 2}{4} = \frac{\frac{2}{\sqrt{2}} + \ln|\sqrt{2}+1| + 1}{2} = \frac{\sqrt{2}}{2} + \frac{1}{2}\ln|\sqrt{2}+1| + \frac{1}{2}$$

DAY4

73) Find $\int \frac{dx}{\sqrt{x^2-2x}}$

When we complete the square of x^2-2x

$$x^2-2x=(x-1)^2-1=(x-1)^2-1^2$$

Thus

$$\int \frac{dx}{\sqrt{x^2-2x}}=\int \frac{dx}{\sqrt{(x-1)^2-1^2}}$$

Using Equation (75) with the replacement of x to $x-1$ and k to 1, the answer is

$$\int \frac{dx}{\sqrt{x^2-2x}}=\cosh^{-1}\left(\frac{x-1}{1}\right)+c=\cosh^{-1}(x-1)+c$$

74) Find $\int \frac{dx}{\sqrt{x^2-6x+13}}$

When we complete the square of $x^2-6x+13$

$$x^2-6x+13=(x-3)^2-9+13=(x-3)^2+4=(x-3)^2+2^2$$

Thus

$$\int \frac{dx}{\sqrt{x^2-6x+13}}=\int \frac{dx}{\sqrt{(x-3)^2+2^2}}$$

Using Equation (76) with the replacement of x to $x-3$ and k to 2, the answer is

$$\int \frac{dx}{\sqrt{x^2-6x+13}}=\sinh^{-1}\left(\frac{x-3}{2}\right)+c$$

75) Find $\int \frac{dx}{\sqrt{2x^2+4x+3}}$

When we complete the square of $2x^2+4x+3$

$$\begin{aligned}2x^2+4x+3&=2(x^2+2x)+3\\&=2((x+1)^2-1)+3\\&=2(x+1)^2-2+3\\&=2(x+1)^2+1\\&=2\left((x+1)^2+\frac{1}{2}\right)\\&=2\left((x+1)^2+\left(\frac{1}{\sqrt{2}}\right)^2\right)\end{aligned}$$

Thus the original integral can be re-written as

$$\begin{aligned}&\int \frac{dx}{\sqrt{2x^2+4x+3}}\\&=\int \frac{dx}{\sqrt{2\left\{(x+1)^2+\left(\frac{1}{\sqrt{2}}\right)^2\right\}}}\\&=\int \frac{1}{\sqrt{2}}\frac{dx}{\sqrt{(x+1)^2+\left(\frac{1}{\sqrt{2}}\right)^2}}\\&=\frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{(x+1)^2+\left(\frac{1}{\sqrt{2}}\right)^2}}\end{aligned}$$

Using Equation (76) with the replacement of x to $x+1$ and k to $\frac{1}{\sqrt{2}}$, the answer is

$$\begin{aligned}\int \frac{dx}{\sqrt{2x^2+4x+3}}&=\frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{(x+1)^2+\left(\frac{1}{\sqrt{2}}\right)^2}}\\&=\frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{x+1}{\frac{1}{\sqrt{2}}}\right)+c\\&=\frac{1}{\sqrt{2}}\sinh^{-1}\left(\sqrt{2}(x+1)\right)+c\end{aligned}$$

76) Find $\int \frac{dx}{\sqrt{3x^2+12x+8}}$
When we complete the square of $3x^2+12x+8$

$$\begin{aligned} & 3x^2+12x+8 \\ &= 3(x^2+4x)+8 \\ &= 3\left((x+2)^2-4\right)+8 \\ &= 3(x+2)^2-12+8 \\ &= 3(x+2)^2-4 \\ &= 3\left\{(x+2)^2-\frac{4}{3}\right\} \\ &= 3\left\{(x+2)^2-\left(\sqrt{\frac{4}{3}}\right)^2\right\} \end{aligned}$$

Thus the original integral can be re-written as

$$\begin{aligned} & \int \frac{dx}{\sqrt{3x^2+12x+8}} \\ &= \int \frac{dx}{\sqrt{3\left\{(x+2)^2-\left(\sqrt{\frac{4}{3}}\right)^2\right\}}} \\ &= \int \frac{dx}{\sqrt{3}\sqrt{(x+2)^2-\left(\sqrt{\frac{4}{3}}\right)^2}} \\ &= \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{(x+2)^2-\left(\sqrt{\frac{4}{3}}\right)^2}} \end{aligned}$$

Using Equation (75) with the replacement of x to $x+2$ and k to $\sqrt{\frac{4}{3}}$, the answer is

$$\begin{aligned} \int \frac{dx}{\sqrt{3x^2+12x+8}} &= \frac{1}{\sqrt{3}}\cosh^{-1}\left(\frac{x+2}{\sqrt{\frac{4}{3}}}\right)+c \\ &= \frac{1}{\sqrt{3}}\cosh^{-1}\left(\frac{\sqrt{3}(x+2)}{2}\right)+c \end{aligned}$$

77) Find $\int \frac{5x^3-10x^2+31x-13}{x^2-2x+5}dx$
As the order of numerator is higher than the denominator we perform the following polynomial division

$$\begin{array}{r} 5x \\ x^2-2x+5\ \overline{)5x^3-10x^2+31x-13} \\ -)5x^3-10x^2+25x \\ \hline 6x-13 \end{array}$$

Thus we obtain

$$\frac{5x^3-10x^2+31x-13}{x^2-2x+5} = 5x+\frac{6x-13}{x^2-2x+5}$$

As $\frac{d\left\{x^2-2x+5\right\}}{dx} = 2x-2$, by doing the following division

$$\begin{array}{r} 3 \\ 2x-2\ \overline{)6x-13} \\ -)6x\ \ -6 \\ \hline -7 \end{array}$$

we manipulate the original equation as

$$\begin{aligned} \frac{5x^3-10x^2+31x-13}{x^2-2x+5} &= 5x+\frac{3(2x-2)}{x^2-2x+5}-\frac{7}{x^2-2x+5} \\ &= 5x+\frac{3\frac{d\left\{x^2-2x+5\right\}}{dx}}{x^2-2x+5}-\frac{7}{x^2-2x+5} \end{aligned}$$

When we complete the square of x^2-2x+5, we obtain

$$\begin{aligned}&x^2-2x+5\\&=(x-1)^2-1+5\\&=(x-1)^2+4\\&=(x-1)^2+2^2\end{aligned}$$

Therefore the original integral can be re-written as

$$\int\left(5x+\frac{3\frac{d\left\{x^2-2x+5\right\}}{dx}}{x^2-2x+5}-\frac{7}{(x-1)^2+2^2}\right)dx$$

Using Equation (57) and Equation (78) with the replacement of x with $x-1$ and k with 2, the answer is

$$\frac{5}{2}x^2+3\ln|x^2-2x+5|-\frac{7}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+c$$

78) Find $\int\frac{8x^4-4x^3-10x^2}{4x^2+4x+3}dx$

As the order of numerator is higher than the denominator we perform the following polynomial division

$$\begin{array}{r|l}
 & 2x^2-3x-1\\
4x^2+4x+3 & 8x^4\ -4x^3-10x^2\\
 & -)8x^4\ +8x^3\ +6x^2\\ \hline
 & -12x^3-16x^2\\
 & -)-12x^3-12x^2-9x\\ \hline
 & -4x^2+9x\\
 & -)\ -4x^2-4x-3\\ \hline
 & 13x+3
\end{array}$$

Thus we obtain

$$\frac{8x^4-4x^3-10x^2}{4x^2+4x+3}=2x^2-3x-1+\frac{13x+3}{4x^2+4x+3}$$

As $\frac{d\left\{4x^2+4x+3\right\}}{dx}=8x+4$, by doing the following division

$$\begin{array}{r|l}
 & \frac{13}{8}\\
8x+4 & 13x\ \ +3\\
 & -)13x+6.5\\ \hline
 & -3.5
\end{array}$$

we manipulate the original equation as

$$\frac{8x^4-4x^3-10x}{4x^2+4x+3}=2x^2-3x-1+\frac{13}{8}\cdot\frac{8x+4}{4x^2+4x+3}-\frac{3.5}{4x^2+4x+3}$$

When we complete the square of $4x^2+4x+3$, we obtain

$$\begin{aligned}&4x^2+4x+3\\&=4(x^2+x)+3\\&=4((x+0.5)^2-0.25)+3\\&=4(x+0.5)^2-1+3\\&=4(x+0.5)^2+2\\&=4\left\{(x+0.5)^2+(\sqrt{0.5})^2\right\}\end{aligned}$$

Therefore the original integral can be re-written as

$$\int\left(2x^2-3x-1+\frac{13}{8}\frac{\frac{d\left\{4x^2+4x+3\right\}}{dx}}{4x^2+4x+3}-\frac{3.5}{4}\cdot\frac{1}{(x+0.5)^2+(\sqrt{0.5})^2}\right)dx$$

Using Equation (57) and Equation (78) with the replacement of x with $x+0.5$ and k with $\sqrt{0.5}$, the answer is

$$\frac{2}{3}x^3-\frac{3}{2}x^2-x+\frac{13}{8}\ln|4x^2+4x+3|-\frac{3.5}{4\sqrt{0.5}}\tan^{-1}\left(\frac{x+0.5}{\sqrt{0.5}}\right)+c$$

79) Find $\int\frac{-x^3-x^2-2x+1}{x^2+2x+2}dx$

As the order of numerator is higher than the denominator we perform the following polynomial division

$$\begin{array}{rl} & -x+1 \\ x^2+2x+2\,) & \overline{-x^3\ \ -x^2-2x+1} \\ -) & \underline{-x^3-2x^2-2x} \\ & x^2\qquad +1 \\ -) & \underline{x^2+2x+2} \\ & -2x-1 \end{array}$$

Thus we obtain

$$\frac{-x^3-x^2-2x+1}{x^2+2x+2}=-x+1+\frac{-2x-1}{x^2+2x+2}$$

As $\dfrac{d\left\{x^2+2x+2\right\}}{dx}=2x+2$, by doing the following division

$$\begin{array}{rl} & -1 \\ 2x+2\,) & \overline{-2x-1} \\ -) & \underline{-2x-2} \\ & 1 \end{array}$$

we manipulate the original equation as

$$\begin{aligned}\frac{-x^3-x^2-2x+1}{x^2+2x+2}&=-x+1+\frac{-(2x+2)}{x^2+2x+2}+\frac{1}{x^2+2x+2}\\ &=-x+1-\frac{\frac{d\left\{x^2+2x+2\right\}}{dx}}{x^2+2x+2}+\frac{1}{x^2+2x+2}\end{aligned}$$

When we complete the square of x^2+2x+2, we obtain

$$\begin{aligned}&x^2+2x+2\\ &=(x+1)^2-1+2\\ &=(x+1)^2+1\\ &=(x+1)^2+1^2\end{aligned}$$

Therefore the original integral can be re-written as

$$\int\left(-x+1+-\frac{\frac{d\left\{x^2+2x+2\right\}}{dx}}{x^2+2x+2}+\frac{1}{(x+1)^2+1^2}\right)dx$$

Using Equation (57) and Equation (78) with the replacement of x with $x+1$ and k with 1, the answer is

$$-\frac{1}{2}x^2+x-\ln\left|x^2+2x+2\right|+\tan^{-1}(x+1)+c$$

80) Find $\displaystyle\int\frac{-10x^3-259}{2x^2-4x+20}dx$

As the order of numerator is higher than the denominator we perform the following polynomial division

$$\begin{array}{rl} & -5x\ \ -10 \\ 2x^2-4x+20\,) & \overline{-10x^3\qquad\qquad\qquad -259} \\ -) & \underline{-10x^3+20x^2-100x} \\ & -20x^2\ \ 100x-259 \\ -) & \underline{-20x^2\ \ +40x-200} \\ & 60x\ \ -59 \end{array}$$

Thus we obtain

$$\frac{-10x^3-259}{2x^2-4x+20}=-5x-10+\frac{60x-59}{2x^2-4x+20}$$

As $\dfrac{d\left\{2x^2-4x+20\right\}}{dx}=4x-4$, by doing the following division

$$\begin{array}{rl} & 15 \\ 4x-4\,) & \overline{60x-59} \\ -) & \underline{60x-60} \\ & 1 \end{array}$$

we manipulate the original equation as

$$\begin{aligned}\frac{-10x^3-259}{2x^2-4x+20}&=-5x-10+\frac{15(4x-4)}{2x^2-4x+20}+\frac{1}{2x^2-4x+20}\\ &=-5x-10+\frac{15\frac{d\left\{2x^2-4x+20\right\}}{dx}}{2x^2-4x+20}+\frac{1}{2x^2-4x+20}\end{aligned}$$

When we complete the square of $2x^2 - 4x + 20$, we obtain

$$\begin{aligned} & 2x^2 - 4x + 20 \\ &= 2(x^2 - 2x + 10) \\ &= 2((x-1)^2 - 1 + 10) \\ &= 2((x-1)^2 + 9) \\ &= 2((x-1)^2 + 3^2) \end{aligned}$$

Therefore the original integral can be re-written as

$$\int \left(-5x - 10 + \frac{15 \dfrac{d\left\{2x^2 - 4x + 20\right\}}{dx}}{2x^2 - 4x + 20} + \frac{1}{2\left((x-1)^2 + 3^2\right)} \right) dx$$

Using Equation (57) and Equation (78) with the replacement of x with $x - 1$ and k with 2, the answer is

$$\begin{aligned} & -\frac{5}{2}x^2 - 10x + 15 \ln |2x^2 - 4x + 20| + \frac{1}{2} \cdot \frac{1}{3} \cdot \tan^{-1}\left(\frac{x-1}{3}\right) + c \\ &= -\frac{5}{2}x^2 - 10x + 15 \ln |2x^2 - 4x + 20| + \frac{1}{6} \tan^{-1}\left(\frac{x-1}{3}\right) + c \end{aligned}$$