Integral by parts

Multiplication of $f(x)$ and $g(x)$

$$\int f(x)g(x)dx$$

$$= \left[f(x)\int g(x)dx\right] - \int \left(\frac{d\{f(x)\}}{dx} \cdot \int g(x)dx\right) dx$$

$f(x)$ **is**

- **logarithmic (highest priority)**
- **polynomial**

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$

$$\int_a^b x^2 \mathfrak{e}^x dx$$

Question: What is $f(x)$? and What is $g(x)$?

$f(x)$ **is**

- **polynomial**
- **logarithmic**

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$

$$\int_a^b x^2 \mathfrak{e}^x dx$$

Answer:

$$f(x) = x^2$$
$$g(x) = \mathfrak{e}^x$$
$$\left[x^2 \cdot \int \mathfrak{e}^x dx\right]_a^b - \int_a^b \left(\frac{d\left\{x^2\right\}}{dx} \cdot \int \mathfrak{e}^x dx\right) dx$$

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$

$$\int_a^b x \ln x dx$$

Question: What is $f(x)$? and What is $g(x)$?

$f(x)$ **is**

- **polynomial**
- **logarithmic**

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$

$$\int_a^b x \ln x dx$$

Answer: a logarithmic fuction should have the higher priority for $f(x)$ than the polynomial function

$$f(x) = \ln x$$
$$g(x) = x$$
$$\left[\ln x \cdot \int x dx\right]_a^b - \int_a^b \left(\frac{d\{\ln x\}}{dx} \cdot \int x dx\right) dx$$

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$

$$\int_a^b (\ln x)^2 \, dx$$

Question: What is $f(x)$? and What is $g(x)$?

$f(x)$ **is**

- **polynomial**
- **logarithmic**

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$

$$\int_a^b (\ln x)^2 \, dx$$

Answer: When you see $\ln x$**, always see an** **<u>invisible 1</u> !! as** $1 \cdot (\ln x)^2$

$$f(x) = (\ln x)^2$$
$$g(x) = 1$$

$$\left[(\ln x)^2 \cdot \int 1 dx\right]_a^b - \int_a^b \left(\frac{d\left\{(\ln x)^2\right\}}{dx} \cdot \int 1 dx \right) dx$$

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$

$$\int_a^b \mathfrak{e}^{-x} \sin x dx$$

Question: What is $f(x)$? and What is $g(x)$?

$f(x)$ **is**

- **polynomial**
- **logarithmic**

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$

$$\int_a^b \mathfrak{e}^{-x} \sin x dx$$

Answer: We can not choose a suitable function for $f(x) \longrightarrow$ randome allocation.

$$f(x) = \sin x$$
$$g(x) = \mathfrak{e}^{-x}$$

$$\left[\sin x \cdot \int \mathfrak{e}^{-x} dx\right]_a^b - \int_a^b \left(\frac{d\{\sin x\}}{dx} \cdot \int \mathfrak{e}^{-x} dx\right) dx$$

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$: $\int_a^b \mathfrak{e}^{-x} \sin x dx$

$$\left[\sin x \cdot \int \mathfrak{e}^{-x} dx\right]_a^b - \int_a^b \left(\frac{d\{\sin x\}}{dx} \cdot \int \mathfrak{e}^{-x} dx\right) dx$$

$$= \left[\sin x(-\mathfrak{e}^{-x})\right]_a^b - \int_a^b \left(\cos x \cdot (-\mathfrak{e}^{-x})\right) dx$$

$$= -\left[\sin x \cdot \mathfrak{e}^{-x}\right]_a^b + \int_a^b \left(\cos x \cdot \mathfrak{e}^{-x}\right) dx$$

Question: You can not calculate the second term. What is the next step ?

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$: $\int_a^b \mathfrak{e}^{-x} \sin x dx$

$$\int_a^b \mathfrak{e}^{-x} \cos x dx$$

Answer : Another "Integral by parts" for $\int_a^b \left(\mathfrak{e}^{-x} \cdot \cos x\right) dx$

Question: How about the allocation of $f(x)$ and $g(x)$ for this integral ?

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$: $\int_a^b \mathfrak{e}^{-x} \sin x dx$

$$\int_a^b \mathfrak{e}^{-x} \cos x dx$$

Answer : Try to keep $f(x)$ and $g(x)$ from the first "integral by parts" as much as possible

$f(x) = \cos x$

$g(x) = \mathfrak{e}^{-x}$

$$\left[\cos x \cdot \int \mathfrak{e}^{-x} dx\right]_a^b - \int_a^b \left(\frac{d\{\cos x\}}{dx} \cdot \int \mathfrak{e}^{-x} dx\right) dx$$

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$: $\int_a^b \mathfrak{e}^{-x} \sin x dx$

$$\left[\cos x \cdot \int \mathfrak{e}^{-x} dx\right]_a^b - \int_a^b \left(\frac{d\left\{\cos x\right\}}{dx} \cdot \int \mathfrak{e}^{-x} dx\right) dx$$

$$= \left[\cos x \cdot (-\mathfrak{e}^{-x})\right]_a^b - \int_a^b \left(-\sin x \cdot (-\mathfrak{e}^{-x})\right) dx$$

$$= -\left[\cos x \cdot \mathfrak{e}^{-x}\right]_a^b - \int_a^b \left(\sin x \cdot \mathfrak{e}^{-x}\right) dx$$

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$: $\int_a^b e^{-x} \sin x dx$

In the end we've got

$$\int_a^b \cos x e^{-x} dx = -\left[\cos x \cdot e^{-x}\right]_a^b - \int_a^b \left(\sin x \cdot e^{-x}\right) dx$$

We've started with

$$\int_a^b e^{-x} \sin x dx = -\left[\sin x \cdot e^{-x}\right]_a^b + \int_a^b \left(\cos x \cdot e^{-x}\right) dx$$

By adding these two equations we get:

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$: $\int_a^b \mathfrak{e}^{-x} \sin x dx$

By adding these two equations we get:

$$\int_a^b \mathfrak{e}^{-x} \sin x dx$$

$$= -\left[\sin x \cdot \mathfrak{e}^{-x}\right]_a^b - \left[\cos x \cdot \mathfrak{e}^{-x}\right]_a^b - \int_a^b \sin x \cdot \mathfrak{e}^{-x} dx$$

We've got the original integral of $\int_a^b \left(\sin x \cdot \mathfrak{e}^{-x}\right) dx$ at the right hand side . Always expect this happens when random allocation is necessary.

Integral by parts : Example

Multiplication of $f(x)$ and $g(x)$: $\int_a^b \mathfrak{e}^{-x} \sin x dx$

Finally we get

$$\therefore 2\int_a^b \mathfrak{e}^{-x} \sin x dx = -\left[\sin x \cdot \mathfrak{e}^{-x}\right]_a^b - \left[\cos x \cdot \mathfrak{e}^{-x}\right]_a^b$$

$$\therefore \int_a^b \mathfrak{e}^{-x} \sin x dx = -\frac{1}{2}\left[\sin x \cdot \mathfrak{e}^{-x} + \cos x \cdot \mathfrak{e}^{-x}\right]_a^b$$

Integral by substitution $t = g(x)$

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b f(x)dx = \int_\alpha^\beta h(t)dt$$

1. **simplify $f(x)$ by setting $t = g(x)$**
2. **find the relationship between dx and dt such as** $\frac{dt}{dx} = \frac{dg(x)}{dx} \rightarrow \frac{dx}{dt} = \frac{1}{\frac{dg(x)}{dx}} \rightarrow dx = \frac{1}{\frac{dg(x)}{dx}}dt$

Integral by substitution $t = g(x)$

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b f(x)dx = \int_\alpha^\beta h(t)dt$$

3. **manipulate the original function $f(x)dx$ to remove x to achieve $f(x)dx = h(t)dt$**
4. **find the range of t, $i.e.,\alpha$ and β**
5. **solve $\int_\alpha^\beta h(t)dt$**

Integral by substitution $t = g(x)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b 2x\sqrt{1+x^2}dx$$

Question: How do you set t to simplify $2x\sqrt{1+x^2}$

Integral by substitution $t = g(x)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b 2x\sqrt{1+x^2}dx$$

Answer : $t = 1 + x^2$

And

$$\frac{d\{t\}}{dx} = \frac{d\{1+x^2\}}{dx} = 2x$$

$$\therefore dt = 2xdx$$

$$\therefore dx = \frac{1}{2x}dt$$

Integral by substitution $t = g(x)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \frac{dx}{\sqrt{x} + \sqrt[4]{x}}$$

Question: How do you set t to simplify $\dfrac{1}{\sqrt{x} + \sqrt[4]{x}}$

Integral by substitution $t = g(x)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \frac{dx}{\sqrt{x} + \sqrt[4]{x}}$$

Answer : $t = \sqrt[4]{x}$

And

$$t^4 = x$$
$$\therefore \frac{d\left\{t^4\right\}}{dx} = \frac{d\left\{x\right\}}{dx}$$
$$\therefore 4t^3 \frac{d\left\{t\right\}}{dx} = 1$$
$$\therefore dx = 4t^3 dt$$

Integral by substitution $t = g(x)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b (3x+5)^6 dx$$

Question: How do you set t to simplify $(3x+5)^6$

Integral by substitution $t = g(x)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b (3x+5)^6 dx$$

Answer : $t = 3x + 5$

And

$$t = 3x + 5$$

$$\therefore \frac{d\{t\}}{dx} = \frac{d\{3x+5\}}{dx}$$

$$\therefore \frac{d\{t\}}{dx} = 3 \longrightarrow dt = 3dx$$

$$\therefore dx = \frac{1}{3}dt$$

Integral by substitution $t = g(x)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \frac{dx}{\sqrt{3-\sqrt{x}}}$$

Question: How do you set t to simplify $\dfrac{1}{\sqrt{3-\sqrt{x}}}$

Integral by substitution $t = g(x)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \frac{dx}{\sqrt{3-\sqrt{x}}}$$

Answer : $t = \sqrt{3-\sqrt{x}}$

$$t^2 = 3 - \sqrt{x}$$

$$\therefore \sqrt{x} = 3 - t^2 \longrightarrow x = (3-t^2)^2$$

$$\therefore \frac{d\{x\}}{dt} = \frac{d\{(3-t^2)^2\}}{dt} = 2(3-t^2)(-2t)$$

$$\therefore dx = -4t(3-t^2)dt$$

Integral by substitution $t = g(x)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \frac{e^x}{e^x + 1} dx$$

Question: How do you set t to simplify $\frac{e^x}{e^x + 1}$

Integral by substitution $t = g(x)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \frac{\mathfrak{e}^x}{\mathfrak{e}^x + 1} dx$$

Answer : $t = \mathfrak{e}^x + 1$

And

$$\frac{d\{t\}}{dx} = \frac{d\{\mathfrak{e}^x + 1\}}{dx} = \mathfrak{e}^x$$

$$\therefore dt = \mathfrak{e}^x dx$$

$$\therefore dx = \frac{1}{\mathfrak{e}^x} dt = \mathfrak{e}^{-x} dt$$

Integral by substitution $t = g(x)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \frac{x}{\sqrt{x^2+1}}dx$$

Question: How do you set t to simplify $\frac{x}{\sqrt{x^2+1}}$

Note: this is NOT $\frac{1}{\sqrt{x^2+1}}$

Integral by substitution $t = g(x)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \frac{x}{\sqrt{x^2+1}} dx$$

Answer : $t = x^2 + 1$

And

$$\frac{d\{t\}}{dx} = \frac{d\{x^2+1\}}{dx} = 2x$$

$$\therefore dt = 2xdx \longrightarrow dx = \frac{1}{2x} dt$$

Integral by substitution $x = k(t)$

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b f(x)dx = \int_\alpha^\beta h(t)dt$$

1. **simplify $f(x)$ by setting $x = k(t)$**
2. **find the relationship between dx and dt such as** $\frac{dx}{dt} = \frac{dk(t)}{dt} \rightarrow dx = \left(\frac{dk(t)}{dt}\right) dt$

Integral by substitution $x = k(t)$

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b f(x)dx = \int_\alpha^\beta h(t)dt$$

3. **manipulate the original function** $f(x)dx$ **to remove x to achieve** $f(x)dx = h(t)dt$
4. **find the range of** t, *i.e.*,α **and** β
5. **solve** $\int_\alpha^\beta h(t)dt$

Integral by substitution $x = k(t)$

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b f(x)dx = \int_\alpha^\beta h(t)dt$$

$f(x)$ **involves**	$x = k(t)$
$\sqrt{x^2 - a^2}$	$x = a\frac{\mathfrak{e}^t + \mathfrak{e}^{-t}}{2} \equiv \cosh(t)$
$\sqrt{x^2 + a^2}$	$x = a\frac{\mathfrak{e}^t - \mathfrak{e}^{-t}}{2} \equiv \sinh(t)$
$\sqrt{a^2 - x^2}$	$x = a\sin t$
$\frac{1}{x^2 + a^2}$	$x = a\tan t$

Note: The bottom 4 lines in the Yellow card may save your time and effort

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \frac{1}{\sqrt{3+x^2}}dx$$

Question: How do you set x to simplify $\dfrac{1}{\sqrt{3+x^2}}$

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \frac{1}{\sqrt{3+x^2}} dx$$

Answer : $x = \sqrt{3}\frac{\mathfrak{e}^t - \mathfrak{e}^{-t}}{2}$

Note: this is not necessary if you remember the one in yellow card:

$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$ **where** $a = \sqrt{3}$

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \sqrt{5 + x^2} dx$$

Question: How do you set x to simplify $\sqrt{5 + x^2}$

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \sqrt{5+x^2}\,dx$$

Answer : $x = \sqrt{5}\frac{\mathfrak{e}^{t} - \mathfrak{e}^{-t}}{2}$

Note: You can NOT use the one in yellow card:

$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)$

$$\frac{d\{x\}}{dt} = \frac{d\left\{\sqrt{5}\frac{\mathfrak{e}^{t}-\mathfrak{e}^{-t}}{2}\right\}}{dt} = \sqrt{5}\frac{\mathfrak{e}^{t} + \mathfrak{e}^{-t}}{2}$$

$$\therefore dx = \sqrt{5}\frac{\mathfrak{e}^{t} + \mathfrak{e}^{-t}}{2}dt$$

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \sqrt{3 - 2x^2}dx$$

Question: How do you set x to simplify $\sqrt{3 - 2x^2}$

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \sqrt{3-2x^2}dx$$

Answer : Modify

$\sqrt{3-2x^2} = \sqrt{2(3/2-x^2)} = \sqrt{2}\sqrt{3/2-x^2}$ **and then set** $x = \sqrt{3/2}\sin t$

Note: You can NOT use the one in yellow card:

$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

$$\frac{d\{x\}}{dt} = \frac{d\left\{\sqrt{3}\sin t\right\}}{dt} = \sqrt{3}\cos t; \therefore dx = \sqrt{3}\cos t \cdot dt$$

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \frac{1}{\sqrt{3 - x^2}} dx$$

Question: How do you set x to simplify $\frac{1}{\sqrt{3 - x^2}}$

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \frac{1}{\sqrt{3-x^2}}dx$$

Answer : $x = \sqrt{3}\sin t$

Note: This is not necessary if you remember the one in the yellow card : $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$ **where** $a = \sqrt{3}$

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \frac{1}{2x^2+6} dx$$

Question: How do you set x to simplify $\frac{1}{2x^2+6}$

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_a^b \frac{1}{2x^2+6}dx$$

Answer : $x = \sqrt{3}\tan t$

Note: This is not necessary if you remember the one in the yellow card

$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ **where** $a = \sqrt{3}$

because $\frac{1}{2x^2+6} = \frac{1}{2(x^2+3)} = \frac{1}{2}\cdot\frac{1}{x^2+3}$

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_0^1 \frac{x+1}{\left(x^2+1\right)^2} dx$$

Question: How do you set x to simplify $\dfrac{x+1}{\left(x^2+1\right)^2}$

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_0^1 \frac{x+1}{\left(x^2+1\right)^2} dx$$

Answer : $x = \tan t$

Note: You can NOT use the one in the yellow card: $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

$$\frac{d\{x\}}{dt} = \frac{d\{\tan t\}}{dt} = \frac{1}{\cos^2 t}$$

$$\therefore dx = \frac{dt}{\cos^2 t}$$

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_0^1 \frac{x+1}{\left(x^2+1\right)^2}dx$$

Manipulate $f(x)dx$ **with** $x = \tan t$

$$\frac{x+1}{\left(x^2+1\right)^2}dx = \frac{\tan t + 1}{\left(\tan^2 t + 1\right)^2}\frac{dt}{\cos^2 t}$$

$$= (\tan t + 1)(\cos^2 t)^2 \frac{dt}{\cos^2 t} = (\tan t + 1)\cos^2 t dt$$

$$= (\cos t \sin t + \cos^2 t)dt = (\frac{1}{2}\sin 2t + \frac{1 + \cos 2t}{2})dt$$

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

$$\int_0^1 \frac{x+1}{\left(x^2+1\right)^2} dx$$

Find the range of t with $x = \tan t$

$$0 = \tan t \longrightarrow t = 0$$
$$1 = \tan t \longrightarrow t = \frac{\pi}{4}$$

Integral by substitution $x = k(t)$: Example

$f(x)$ can be expressed as a *simpler* function of t as $h(t)$

Finally

$$\int_0^1 \frac{x+1}{\left(x^2+1\right)^2} dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2}\sin 2t + \frac{1+\cos 2t}{2}\right) dt$$

$$= \left[\frac{-1}{4}\cos 2t + \frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8} + \frac{1}{4} + \frac{1}{4} = \frac{\pi}{8} + \frac{1}{2}$$

Integral of $\frac{P(x)}{Q(x)}$

$Q(x) = (ax+b)(cx+d)(ex+f)$ where a,b, $\cdots$ are real, not complex

Note: the order of $P(x)$ < the order of $Q(x)$

1. **separate $\frac{P(x)}{Q(x)}$ into several small fractions such as** $\frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f}$

Integral of $\dfrac{P(x)}{Q(x)}$

$Q(x) = (ax+b)(cx+d)(ex+f)$ where a,b, $\cdots$ are real, not complex

Note: the order of $P(x) <$ the order of $Q(x)$

1. **separate $\dfrac{P(x)}{Q(x)}$ into several small fractions such as** $\dfrac{A}{ax+b} + \dfrac{B}{cx+d} + \dfrac{C}{ex+f}$

$$\int \frac{P(x)}{Q(x)} dx = \int \frac{A}{ax+b} dx + \int \frac{B}{cx+d} dx + \int \frac{C}{ex+f} dx$$

Integral of $\dfrac{P(x)}{Q(x)}$ - Summary

$Q(x) = h\{(x+a)^2 + b^2\}$ where a and b are real, not complex

Note: the order of $P(x) \geq$ the order of $Q(x)$

1. **Find a, b, h.**
2. **Find $A(x)$, C and E**
   - $A(x)$: **answer of** $P(x)/Q(x)$
   - C : **answer of** $\frac{\frac{P(x)}{Q(x)}\text{'s remainder}}{Q'(x)}$
   - E : **remainder of** $\frac{\frac{P(x)}{Q(x)}\text{'s remainder}}{Q'(x)}$
3. $$\int \frac{P(x)}{Q(x)} dx =$$

$$\int A(x)dx + C \ln|Q(x)| + \frac{E}{h \cdot b} \tan^{-1}\left(\frac{x+a}{b}\right) + c$$

Integral of $\frac{P(x)}{Q(x)}$ -Proof

$Q(x) = h\{(x+a)^2 + b^2\}$ where a and b are real, not complex

Note: the order of $P(x) \geq$ the order of $Q(x)$

1. **polynomial division**

$$
\begin{array}{r|l}
 & A(x) \\ \hline
Q(x) & P(x) \\
 & \cdots \\ \hline
 & R(x)
\end{array}
$$

Note: $P(x) = Q(x)A(x) + R(x)$

Integral of $\frac{P(x)}{Q(x)}$ -Proof

$Q(x) = h\{(x+a)^2 + b^2\}$ where a and b are real, not complex

Note: the order of $P(x) \geq$ the order of $Q(x)$

2. **polynomial division**

$$
\begin{array}{r|l}
 & C \\
\hline
Q'(x) \;) & R(x) \\
 & \cdots \\
\hline
 & E
\end{array}
$$

Note: $R(x) = Q'(x) \cdot C + E$

In the end $P(x) = Q(x)A(x) + Q'(x) \cdot C + E$

Integral of $\dfrac{P(x)}{Q(x)}$ -Proof

$Q(x) = h\{(x+a)^2 + b^2\}$ where a and b are real, not complex

Note: the order of $P(x) \geq$ the order of $Q(x)$

3. **re-write the original fraction**

$$\frac{P(x)}{Q(x)} = A(x) + C\frac{Q'(x)}{Q(x)} + \frac{E}{Q(x)}$$

$$\textbf{Note:} \frac{d\ln|Q(x)|}{dx} = \frac{Q'(x)}{Q(x)}$$

$$\downarrow$$

$$\int \frac{d\ln|Q(x)|}{dx}dx = \ln|Q(x)| = \int \frac{Q'(x)}{Q(x)}dx$$

Integral of $\frac{P(x)}{Q(x)}$ -Proof

$Q(x) = h\{(x+a)^2 + b^2\}$ where a and b are real, not complex

Note: the order of $P(x) \geq$ the order of $Q(x)$

4. **solve the re-written integral**

$$\int \frac{P(x)}{Q(x)} dx$$

$$= \int A(x)dx + \int C\frac{Q'(x)}{Q(x)}dx + \int \frac{E}{Q(x)}dx$$

$$= \int A(x)dx + C\ln|Q(x)| + \frac{E}{h \cdot b}\tan^{-1}\left(\frac{x+a}{b}\right)$$

Note: see the bottom line of the yellow card