# Engineering Maths

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## Contents

## I. PREREQUISITES

In order to successfully complete this Engineering Mathematics course you must be competent with the following material. If you are unfamiliar with the any of the following material it is recommended that you attempt some practice questions before undertaking the main course material.

1) **Logarithms**

$$\begin{aligned}
\log_a(x) = m &\equiv a^m = x \\
\log(x) &\equiv \log_{10}(x) \\
\ln(x) &\equiv \log_e(x) \\
\log_a(a) &= 1 \\
\log_a(m \cdot n) &= \log_a(m) + \log_a(n) \\
\log_a\left(\frac{m}{n}\right) &= \log_a(m) - \log_a(n) \\
\log_a(m^n) &= n \cdot \log_a(m) \\
\log_a b &= \frac{\log_c b}{\log_c a}
\end{aligned}$$

2) **Indices**

$$\begin{aligned}
a^m \cdot a^n &= a^{(m+n)} \\
\frac{a^m}{a^n} &= a^{(m-n)} \\
(a^m)^n &= a^{(m \cdot n)} \\
a^{-m} &= \frac{1}{a^m} \\
a^{(m/n)} &= \sqrt[n]{a^m} \\
a^0 &= 1 \\
a^1 &= a
\end{aligned}$$

3) **Trigonometric Identities**

$$\begin{aligned}
y = \sin^{-1} x = \arcsin x &\iff x = \sin y \\
y = \cos^{-1} x = \arccos x &\iff x = \cos y \\
y = \tan^{-1} x = \arctan x &\iff x = \tan y \\
\text{cosec } x &= \frac{1}{\sin x} \\
\sec x &= \frac{1}{\cos x} \\
\cot x &= \frac{1}{\tan x} \\
y = \text{cosec }^{-1} x \iff x = \text{ cosec } y &= \frac{1}{\sin y} \\
y = \sec^{-1} x \iff x = \sec y &= \frac{1}{\cos y} \\
y = \cot^{-1} x \iff x = \cot y &= \frac{1}{\tan y} \\
\tan(x) &= \frac{\sin(x)}{\cos(x)} \\
\sin^2(x) + \cos^2(x) &= 1 \\
\sec^2(x) &= 1 + \tan^2(x)
\end{aligned}$$

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$
$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin(2A) = 2\sin(A)\cos(A)$$
$$\begin{aligned}\cos(2A) &= \cos^2(A) - \sin^2(A)\\ &= 2\cos^2(A) - 1\\ &= 1 - 2\sin^2(A)\end{aligned}$$
$$\tan(2A) = \frac{2\tan(A)}{1-\tan^2(A)}$$
$$2\sin(A)\cos(B) = \sin(A+B) + \sin(A-B)$$
$$2\cos(A)\sin(B) = \sin(A+B) - \sin(A-B)$$
$$2\cos(A)\cos(B) = \cos(A+B) + \cos(A-B)$$
$$-2\sin(A)\sin(B) = \cos(A+B) - \cos(A-B)$$
$$\cos(x) = \frac{\mathfrak{e}^{\jmath x} + \mathfrak{e}^{-\jmath x}}{2}$$
$$\sin(x) = \frac{\mathfrak{e}^{\jmath x} - \mathfrak{e}^{-\jmath x}}{2\jmath}$$

4) **Hyperbolic Identities**

$$\cosh(x) = (\mathfrak{e}^{x} + \mathfrak{e}^{-x})/2$$
$$x = \cosh^{-1}\left(\frac{\mathfrak{e}^{x} + \mathfrak{e}^{-x}}{2}\right)$$
$$\sinh(x) = (\mathfrak{e}^{x} - \mathfrak{e}^{-x})/2$$
$$x = \sinh^{-1}\left(\frac{\mathfrak{e}^{x} - \mathfrak{e}^{-x}}{2}\right)$$
$$\tanh(x) = (\mathfrak{e}^{x} - \mathfrak{e}^{-x})/(\mathfrak{e}^{x} + \mathfrak{e}^{-x})$$
$$\cosh^2(A) - \sinh^2(A) = 1$$

When you need $x$ which satisfies $\cosh(x) = \alpha$ where $\alpha$ is a real number,

$$\frac{\mathfrak{e}^{x} + \mathfrak{e}^{-x}}{2} = \alpha$$
$$\therefore \mathfrak{e}^{x} + \mathfrak{e}^{-x} = 2\alpha$$
$$\therefore \mathfrak{e}^{2x} + 1 = 2\alpha\mathfrak{e}^{x}$$
$$\therefore \mathfrak{e}^{2x} - 2\alpha\mathfrak{e}^{x} + 1 = 0$$
$$\therefore \mathfrak{e}^{x} = \alpha \pm \sqrt{\alpha^2 - 1}$$
$$\therefore x = \ln(\alpha \pm \sqrt{\alpha^2 - 1})$$

When you need $x$ which satisfies $\sinh(x) = \alpha$ where $\alpha$ is a real number,

$$\frac{\mathfrak{e}^{x} - \mathfrak{e}^{-x}}{2} = \alpha$$
$$\therefore \mathfrak{e}^{x} - \mathfrak{e}^{-x} = 2\alpha$$
$$\therefore \mathfrak{e}^{2x} - 1 = 2\alpha\mathfrak{e}^{x}$$
$$\therefore \mathfrak{e}^{2x} - 2\alpha\mathfrak{e}^{x} - 1 = 0$$
$$\therefore \mathfrak{e}^{x} = \alpha \pm \sqrt{\alpha^2 + 1}$$

$$\therefore x = \ln(\alpha \pm \sqrt{\alpha^2 + 1})$$
$$\therefore x = \ln(\alpha + \sqrt{\alpha^2 + 1}) (\because A > 0 \text{ for } \ln A)$$

5) **Completing the Square**

$$4x^2 - 2x - 5 = 0$$

We can solve the above equation by completing the square as follows

$$\begin{aligned}
4x^2 - 2x - 5 &= 0 \\
4x^2 - 2x &= 5 \\
x^2 - \frac{1}{2}x &= \frac{5}{4} \\
\left(x - \frac{1}{4}\right)^2 - \frac{1}{16} &= \frac{5}{4} \\
\left(x - \frac{1}{4}\right)^2 &= \frac{5}{4} + \frac{1}{16} \\
\left(x - \frac{1}{4}\right)^2 &= \frac{21}{16} \\
\therefore x &= \frac{1}{4} \pm \sqrt{\frac{21}{16}}
\end{aligned}$$

6) **Quadratic Equation**
We can use completing the square to derive the quadratic equation.

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
ax^2 + bx &= -c \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} &= -\frac{c}{a} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

7) **Polynomial Long Division**
If we know one factor of a polynomial equation, in order to find out the other factor we perform a division. In this example we know that $x^2 - 9x - 10$ has a factor of $x + 1$. Therefore

$$\begin{array}{r} x-10 \\ x+1\;\overline{)\,x^2\;\; -9x-10} \\ -)\,x^2\quad +x \qquad \\ \hline -10x-10 \\ -)\quad -10x-10 \\ \hline \\ 0 \quad 0 \end{array}$$

Thus, we find the other factor to be

$$x - 10$$

In order to confirm this is correct we can multiply this factor by the known factor to find the original polynomial.

$$\begin{aligned}(x - 10)(x + 1) &= x^2 + x - 10x - 10 \\ &= x^2 - 9x - 10\end{aligned}$$

8) **Area of a Triangle in Vector Form**
When a triangle is defined with two sides $|\boldsymbol{p}|$ and $|\boldsymbol{q}|$ and the angle between these two sides is $\theta$, the area of triangle is

$$\frac{1}{2}|\boldsymbol{p}| \cdot |\boldsymbol{q}| \cdot \sin\theta$$

9) **Inequalities**

| Symbol | Meaning |
|---|---|
| $<$ | is less than |
| $>$ | is greater than |
| $\leq$ | is less than or equal to |
| $\geq$ | is greater than or equal to |

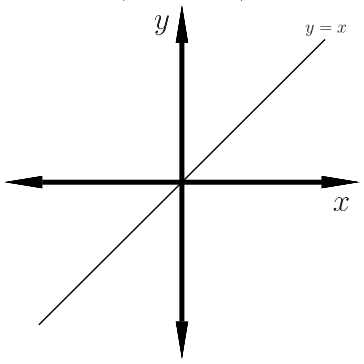
The one rule for inequalities is if you multiply or divide by a negative number the inequality sign is reversed as follows

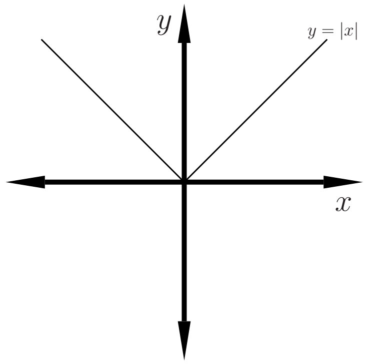
$$\begin{aligned}-ax + c &\leq d \\ -ax &\leq d - c \\ x &\geq -\frac{(d - c)}{a}\end{aligned}$$

$$\begin{aligned}\frac{x}{-e} - f &> g \\ \frac{x}{-e} &> g + f \\ x &< -e(g + f)\end{aligned}$$

10) **Modulus**
The modulus symbol is $||$. Anything that is enclosed within this can not evaluate to a negative number. For

example $|-4+2| = 2$.



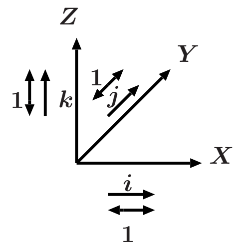



## II. Key points on vectors

**Key Points**

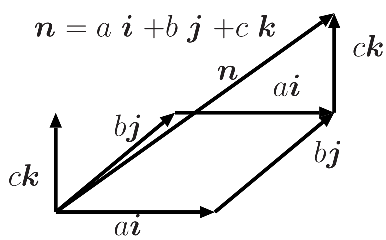
$\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are unit vector in $x$, $y$, and $z$ directions respectively. $\jmath$ is $\sqrt{-1}$.

1) A vector has a $x$ component, $y$ component, and $z$ component
   - A vector is expressed as $\boldsymbol{i}$ when it has only a $x$ component and its modulus is 1.
   - A vector is expressed as $\boldsymbol{j}$ when it has only a $y$ component and its modulus is 1.
   - A vector is expressed as $\boldsymbol{k}$ when it has only a $z$ component and its modulus is 1.




2) When a vector has an amount of $a$ in $x$ component, an amount of $b$ in $y$ component, and an amount of $c$ in $z$ component, the vector can be expressed as

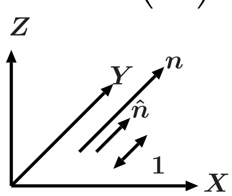
$$\begin{aligned} \boldsymbol{n} &= a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} \\ &\equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix} \end{aligned} \tag{1}$$



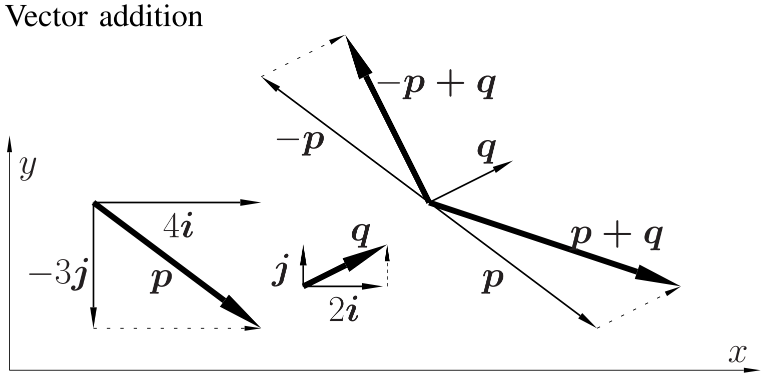

3) A unit vector can be found by dividing a vector by its modulus.

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{|\boldsymbol{n}|} \tag{2}$$

where $|\boldsymbol{n}|$ is $\sqrt{a^2+b^2+c^2}$ when $\boldsymbol{n} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} \equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.




4) Vector addition



When there are two vectors

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and

$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

the addition of the vectors is

$$\begin{aligned} \boldsymbol{a} + \boldsymbol{b} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \end{aligned} \tag{3}$$

5) The position vector of $P$ with coordinates $(a, b, c)$ is

$$\overrightarrow{OP} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} \tag{4}$$

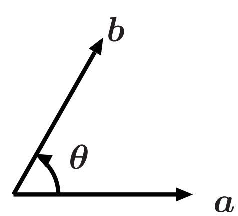
6) When there are two vectors

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and

$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

and these two vectors subtend an angle $\theta$,




the scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3 = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta \tag{5}$$

7) When there are two vectors

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and

$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

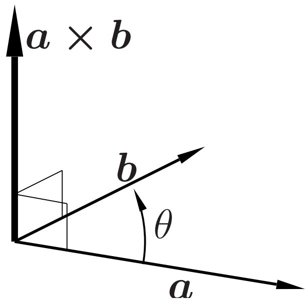



Fig. 1. $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$

and these two vectors subtend an angle $\theta$, the vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} = & \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \boldsymbol{i} \\ & + \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} \boldsymbol{j} \\ & + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \boldsymbol{k} \\ = & (a_2 b_3 - a_3 b_2)\boldsymbol{i} \\ & + (a_3 b_1 - a_1 b_3)\boldsymbol{j} \\ & + (a_1 b_2 - a_2 b_1)\boldsymbol{k} \\ = & |\boldsymbol{a}||\boldsymbol{b}| \sin\theta \hat{\boldsymbol{n}} \end{aligned} \tag{6}$$

where $\hat{\boldsymbol{n}}$ is a unit vector and the direction of $\hat{\boldsymbol{n}}$ is the same as $\boldsymbol{a} \times \boldsymbol{b}$ in Fig. 1.

8) The vector equation of the line which goes through a point $A$ and is parallel to a vector $c$ is

$$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{c} \tag{7}$$

where $t$ is the real number. Please note that '$x$','$y$','$z$' are not involved in the vector equation. The cartesian form of Equation (7) is obtained as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = t \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x - a_1 \\ y - a_2 \\ z - a_3 \end{pmatrix} = t \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

This can be expressed in the scalar manner as

$$x - a_1 = tc_1$$
$$\therefore \frac{x - a_1}{c_1} = t$$
$$y - a_2 = tc_2$$
$$\therefore \frac{y - a_2}{c_2} = t$$
$$z - a_3 = tc_3$$
$$\therefore \frac{z - a_3}{c_3} = t$$

By getting rid of $t$ in these three equations, we get the cartesian equation:

$$\frac{x - a_1}{c_1} = \frac{y - a_2}{c_2} = \frac{z - a_3}{c_3} \tag{8}$$

9) The vector equation of the line through points $A$ and $B$ with position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ is

$$\boldsymbol{r} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}) \tag{9}$$

where $t$ is the real number. Please note that '$x$','$y$','$z$' are not involved in the vector equation. When $0 \leq t \leq 1$, then $\boldsymbol{r}$ is in-between $A$ and $B$. The cartesian form of Equation (9) is obtained as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \left( \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right)$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = t \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x - a_1 \\ y - a_2 \\ z - a_3 \end{pmatrix} = t \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

This can be expressed in the scalar manner as

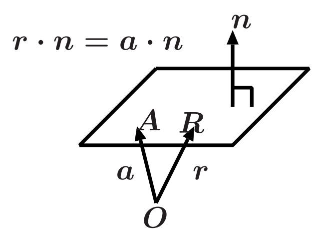
$$x - a_1 = t(b_1 - a_1)$$
$$\therefore \frac{x - a_1}{b_1 - a_1} = t$$
$$y - a_2 = t(b_2 - a_2)$$
$$\therefore \frac{y - a_2}{b_2 - a_2} = t$$
$$z - a_3 = t(b_3 - a_3)$$
$$\therefore \frac{z - a_3}{b_3 - a_3} = t$$

By getting rid of $t$ in these three equations, we get the cartesian equation:

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3} \tag{10}$$

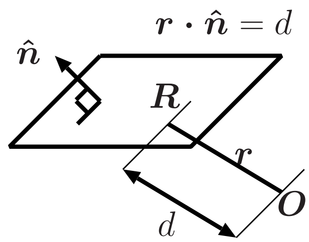
10) A plane perpendicular to the vector $\boldsymbol{n}$ and passing through the point with position vector $\boldsymbol{a}$, has equation

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n} \tag{11}$$




11) A plane with unit normal $\hat{\boldsymbol{n}}$, which has a perpendicular distance $d$ from the origin is given by

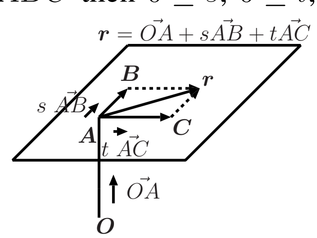
$$\boldsymbol{r} \cdot \hat{\boldsymbol{n}} = d \tag{12}$$




12) A plane which goes through $A(\boldsymbol{a})$, $B(\boldsymbol{b})$ and $C(\boldsymbol{c})$ is given by

$$\boldsymbol{r} = \overrightarrow{OA} + s\overrightarrow{AB} + t\overrightarrow{AC} \tag{13}$$

If the point $R(\boldsymbol{r})$ is inside of the triangle $ABC$ then $0 \leq s$, $0 \leq t$, and $s + t \leq 1$.

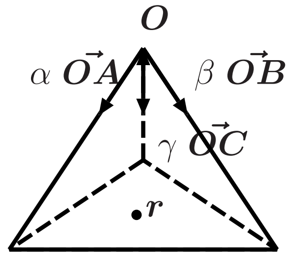



13) A point $R(\boldsymbol{r})$ which is inside the tetrahedron $O$, $A(\boldsymbol{a})$, $B(\boldsymbol{b})$ and $C(\boldsymbol{c})$ is given by

$$\boldsymbol{r} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \tag{14}$$

where $\alpha$, $\beta$, $\gamma$ are real numbers and satisfy

$$\alpha + \beta + \gamma < 1,\ 0 < \alpha,\ 0 < \beta,\ 0 < \gamma \tag{15}$$




$$\boldsymbol{r} = \alpha \boldsymbol{a} + \beta \boldsymbol{b} + \gamma \boldsymbol{c}$$

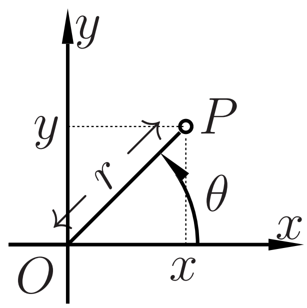



Fig. 2. The relashionship between polar and Cartesian coordinates

## III. Key points on coordinates

### Key Points

1) If the Cartesian coordinates of a point $P$ are $(x, y)$ then $P$ can be located on a Cartesian plane as indicated in Fig. 2. $r$ is the distance of $P$ from the origin and $\theta$ is the angle, measured anti-clockwise, which the line $OP$ makes when measured from the positive $x-$direction. If $(x, y)$ are the Cartesian coordinates and $[r, \theta]$ the polar coordinates of a point $P$, then

$$x = r\cos\theta, \quad y = r\sin\theta \tag{16}$$

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = y/x \tag{17}$$

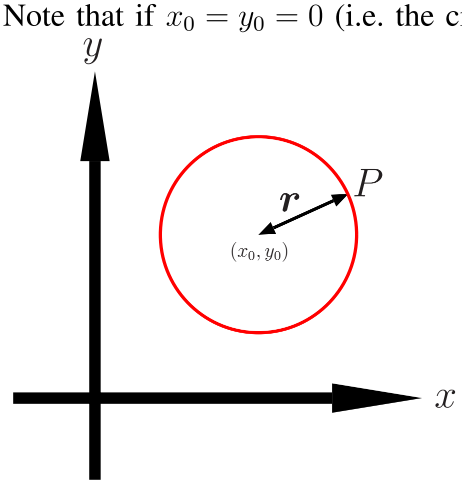
2) If the Cartesian coordinates $(x, y)$ are any point $P$ on a circle of radius $r$ whose centre is at the origin. Then since $\sqrt{x^2 + y^2}$ is the distance of $P$ from the origin, the equation of the circle is,

$$r = \sqrt{x^2 + y^2}, \quad x^2 + y^2 = r^2 \tag{18}$$

3) If the Cartesian coordinates $(x, y)$ are any point $P$ on a circle of radius $r$ whose centre is $(x_0, y_0)$. Then since $\sqrt{(x - x_o)^2 + (y - y_0)^2}$ is the distance of $P$ from the origin, the equation of the circle is,

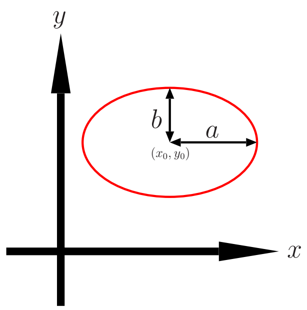
$$r = \sqrt{(x - x_o)^2 + (y - y_0)^2}, \quad (x - x_o)^2 + (y - y_0)^2 = r^2 \tag{19}$$

Note that if $x_0 = y_0 = 0$ (i.e. the circle is at the origin) then Equation (19) reduces to Equation (18).



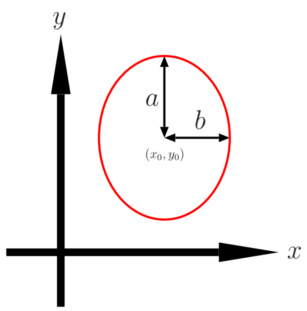
4) An ellipse with centre $(x_0, y_0)$ satifies the equation

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \tag{20}$$




or

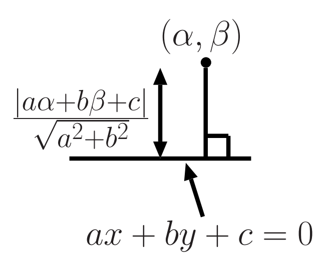
$$\frac{(x-x_0)^2}{b^2} + \frac{(y-y_0)^2}{a^2} = 1 \tag{21}$$




The parameter $b$ is called the semiminor axis by analogy with the parameter $a$, which is called the semimajor axis (assuming $a > b$). When the major axis is horizontal use Equation (20). If on the other hand the major axis is vertical use Equation (21).

5) The minimum distance between a point $Q(\alpha, \beta)$ and a line $ax + by + c = 0$ is expressed as

$$\frac{|a\alpha + b\beta + c|}{\sqrt{a^2 + b^2}} \tag{22}$$




Proof: The line $ax + by + c = 0$ goes through the point $R(\boldsymbol{r})$ where

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -\frac{c}{b} \end{pmatrix}$$

and it is parallel to

$$\boldsymbol{l} = \begin{pmatrix} b \\ -a \end{pmatrix}$$

A point $P(\boldsymbol{p})$ on the line can be written as

$$\boldsymbol{p} = \boldsymbol{r} + t\boldsymbol{l}$$

where $t$ is a real value. Since

$$\overrightarrow{QP} \perp \boldsymbol{l}$$

we can express this as the following equation:

$$\begin{aligned}
&\overrightarrow{QP} \cdot \boldsymbol{l} \\
&= (\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{l} \\
&= (\boldsymbol{r} + t\boldsymbol{l} - qv) \cdot \boldsymbol{l} \\
&= (\boldsymbol{r} - \boldsymbol{q}) \cdot \boldsymbol{l} + t|\boldsymbol{l}|^2 = 0 \\
&\therefore t = \frac{(\boldsymbol{q} - \boldsymbol{r}) \cdot \boldsymbol{l}}{|\boldsymbol{l}|^2}
\end{aligned}$$

Now we need to get $\overrightarrow{QP}$ as follows:

$$\begin{aligned}
|\overrightarrow{QP}|^2 &= |\boldsymbol{p} - \boldsymbol{q}|^2 \\
&= |\boldsymbol{r} + t\boldsymbol{l} - \boldsymbol{q}|^2 \\
&= |\boldsymbol{r}|^2 + |\boldsymbol{q}|^2 + t^2|\boldsymbol{l}|^2 + 2t\boldsymbol{r}\boldsymbol{l} - 2t\boldsymbol{l}\boldsymbol{q} - 2\boldsymbol{r}\boldsymbol{q} \\
&= |\boldsymbol{r}|^2 + |\boldsymbol{q}|^2 + \frac{((\boldsymbol{q} - \boldsymbol{r}) \cdot \boldsymbol{l})^2}{|\boldsymbol{l}|^4} \cdot |\boldsymbol{l}|^2 + 2\frac{(\boldsymbol{q} - \boldsymbol{r}) \cdot \boldsymbol{l}}{|\boldsymbol{l}|^2}(\boldsymbol{r}\boldsymbol{l} - \boldsymbol{l}\boldsymbol{q}) - 2\boldsymbol{r}\boldsymbol{q} \\
&= |\boldsymbol{r}|^2 + |\boldsymbol{q}|^2 + \frac{((\boldsymbol{q} - \boldsymbol{r}) \cdot \boldsymbol{l})^2}{|\boldsymbol{l}|^2} - 2\frac{(\boldsymbol{q} - \boldsymbol{r}) \cdot \boldsymbol{l}}{|\boldsymbol{l}|^2}(\boldsymbol{q} - \boldsymbol{r})\boldsymbol{l} - 2\boldsymbol{r}\boldsymbol{q} \\
&= |\boldsymbol{r}|^2 + |\boldsymbol{q}|^2 + \frac{((\boldsymbol{q} - \boldsymbol{r}) \cdot \boldsymbol{l})^2}{|\boldsymbol{l}|^2} - 2\frac{((\boldsymbol{q} - \boldsymbol{r})\boldsymbol{l})^2}{|\boldsymbol{l}|^2} - 2\boldsymbol{r}\boldsymbol{q} \\
&= |\boldsymbol{r}|^2 + |\boldsymbol{q}|^2 - \frac{((\boldsymbol{q} - \boldsymbol{r}) \cdot \boldsymbol{l})^2}{|\boldsymbol{l}|^2} - 2\boldsymbol{r}\boldsymbol{q} \\
&= \frac{|a\alpha + b\beta + c|^2}{a^2 + b^2} \\
\therefore |\overrightarrow{QP}| &= \frac{|a\alpha + b\beta + c|}{\sqrt{a^2 + b^2}}
\end{aligned}$$

6) 3D Cylindrical polar coordinate $(\rho, \phi, z)$ in Fig. 3 can be obtained from

$$\rho = \sqrt{x^2 + y^2}; \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{23}$$

$$\because x = \rho\cos\phi, y = \rho\sin\phi \tag{24}$$

You need to draw a diagram to determine the correct $\phi$

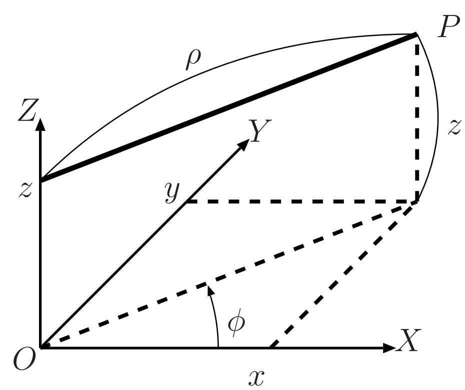



Fig. 3. The relashionship between Cylindrical and Cartesian coordinates

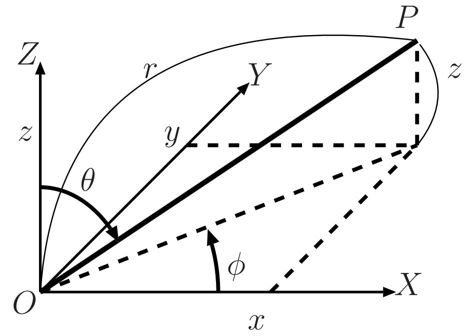



Fig. 4. The relashionship between Spherical and Cartesian coordinates

7) 3D Spherical polar coordinate $(r, \theta, \phi)$ in Fig. 4 can be obtained from

$$r = \sqrt{x^2 + y^2 + z^2}; \theta = \cos^{-1}\left(\frac{z}{r}\right); \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{25}$$

$$\because x = r\sin\theta\cos\phi; y = r\sin\theta\sin\phi; z = r\cos\theta \tag{26}$$

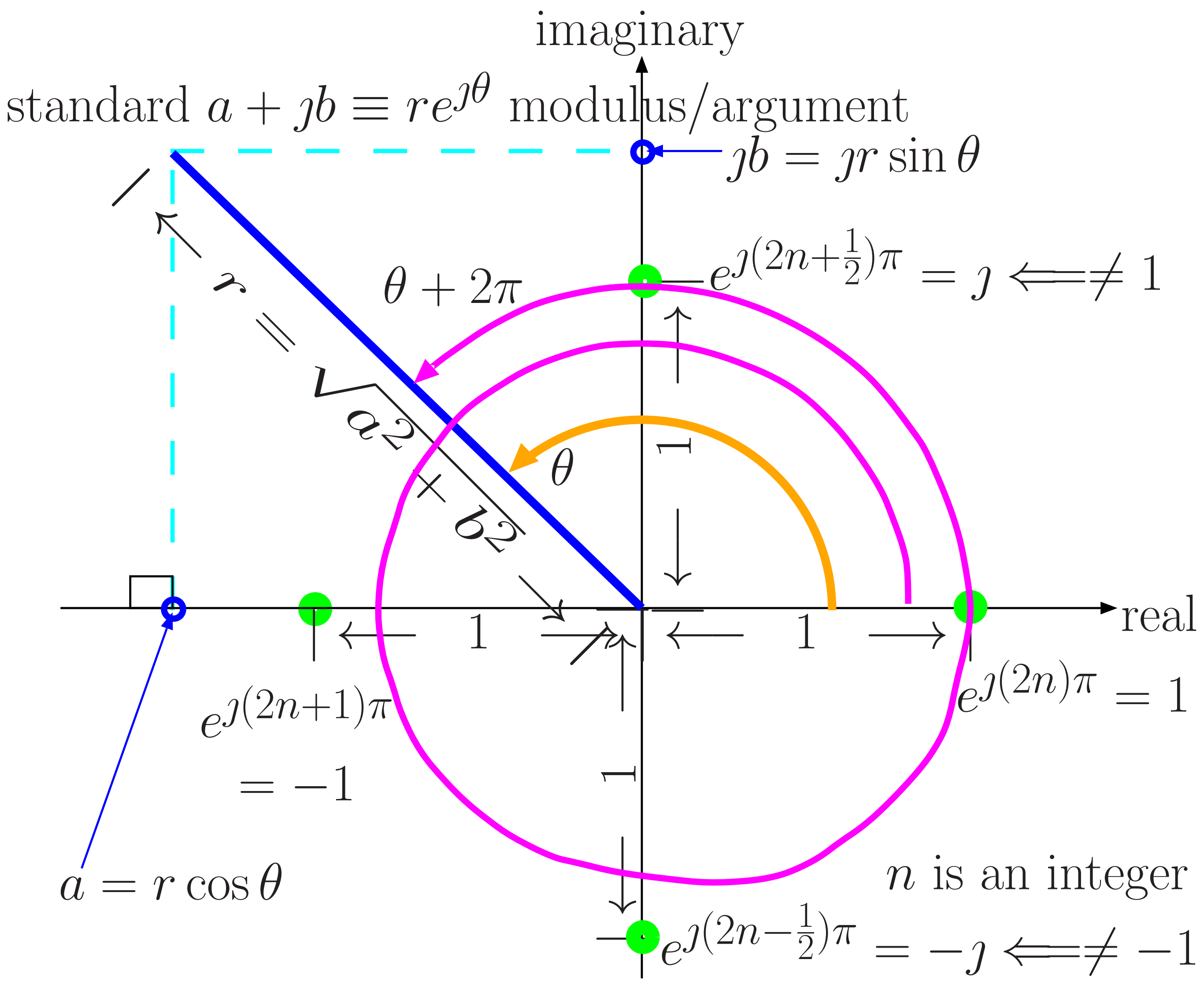
You need to draw a diagram to determine the correct $\phi$. $\theta$ should satisfy $0 \leq \theta \leq \pi$ without a diagram

## IV. Key points on complex numbers

**Key Points**

1) The symbol $\jmath$ is such that

$$\jmath^2 = -1 \qquad \jmath = \sqrt{-1} \tag{27}$$




2) In Argand diagram, the complex number $a + \jmath b$ (the standard form) can be expressed as

$$a + \jmath b = r\mathfrak{e}^{\jmath\theta} = r(\cos\theta + \jmath\sin\theta) \tag{28}$$

,which is the modulus/argument form,where

$$r = |a + \jmath b| = \sqrt{a^2 + b^2} \qquad \tan\theta = \frac{b}{a} \tag{29}$$

$$a = r\cos\theta \qquad b = r\sin\theta \tag{30}$$

Be careful: $a^2 - b^2 + 2ab\jmath = (a + \jmath b)^2 \neq |a + \jmath b|^2 = a^2 + b^2$.
$\frac{a\jmath}{b\jmath} = \frac{a}{b}$, *i.e.*, $\frac{a\jmath}{b\jmath} \neq a - b$.
$\frac{\mathfrak{e}^{a\jmath}}{\mathfrak{e}^{b\jmath}} = \mathfrak{e}^{a\jmath - b\jmath} = \mathfrak{e}^{(a-b)\jmath}$

3) From the figure, $\pm \jmath$ can be expressed as

$$\jmath = \mathfrak{e}^{\frac{\pi}{2}\jmath}, -\jmath = \mathfrak{e}^{-\frac{\pi}{2}\jmath} \tag{31}$$

4) If $a + \jmath b$ is any complex number then its complex conjugate is

$$a - \jmath b \tag{32}$$

5) In the Argand diagram, the argument can be $2\pi n$ rotated to have an identical value:

$$\mathfrak{e}^{\jmath\theta} = \mathfrak{e}^{\jmath(\theta+2\pi n)} \tag{33}$$

where $n$ is an integer.

6) De Moivre's theorem

$$(r\mathfrak{e}^{\jmath\theta})^n = [r(\cos\theta + \jmath\sin\theta)]^n = r^n(\cos n\theta + \jmath\sin n\theta) = r^n\mathfrak{e}^{\jmath n\theta} \tag{34}$$

7) n$^{th}$ roots of complex numbers

If

$$z^n = r\mathfrak{e}^{\jmath\theta} = r(\cos\theta + \jmath\sin\theta)$$

then

$$z = \sqrt[n]{r}\mathfrak{e}^{\jmath(\theta+2k\pi)/n} \qquad k = 0, \pm 1, \pm 2, \ldots \tag{35}$$

In other words, if

$$a\mathfrak{e}^{\jmath b} = c\mathfrak{e}^{\jmath d}$$

then

$$a = c$$
$$b = d + 2n\pi$$

8) If $a + \jmath b = c + \jmath d$ , where $a$, $b$, $c$, and $d$, are real, then we can say

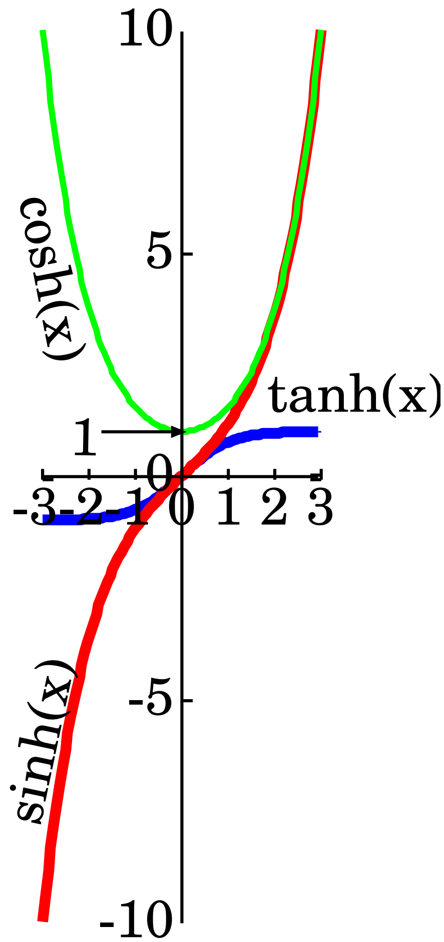
$$a = c, b = d \tag{36}$$

If $a + \jmath b = 0$, then $a = b = 0$

9) $\cosh x$ and $\sinh x$ are defined as

$$\cosh x = \frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}, \sinh x = \frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2} \tag{37}$$
$$x = \cosh^{-1}\left(\frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}\right), x = \sinh^{-1}\left(\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2}\right)$$

$$\tanh(x) = (\mathfrak{e}^x - \mathfrak{e}^{-x})/(\mathfrak{e}^x + \mathfrak{e}^{-x})$$
$$\cosh^2(A) - \sinh^2(A) = 1$$




When you need $x$ which satisfies $\cosh(x) = \alpha$ where $\alpha$ is a real number, using $x = \cosh^{-1}\left(\frac{e^x + e^{-x}}{2}\right)$ we get

$$\begin{aligned} \frac{e^x + e^{-x}}{2} &= \alpha \\ \therefore e^x + e^{-x} &= 2\alpha \\ \therefore e^{2x} + 1 &= 2\alpha e^x \\ \therefore e^{2x} - 2\alpha e^x + 1 &= 0 \\ \therefore e^x &= \alpha \pm \sqrt{\alpha^2 - 1} \\ \therefore x = \cosh^{-1}(\alpha) &= \ln(\alpha \pm \sqrt{\alpha^2 - 1}) \end{aligned}$$

When you need $x$ which satisfies $\sinh(x) = \alpha$ where $\alpha$ is a real number, using $x = \sinh^{-1}\left(\frac{e^x - e^{-x}}{2}\right)$ we get

$$\begin{aligned} \frac{e^x - e^{-x}}{2} &= \alpha \\ \therefore e^x - e^{-x} &= 2\alpha \\ \therefore e^{2x} - 1 &= 2\alpha e^x \\ \therefore e^{2x} - 2\alpha e^x - 1 &= 0 \\ \therefore e^x &= \alpha \pm \sqrt{\alpha^2 + 1} \\ \therefore x &= \ln(\alpha \pm \sqrt{\alpha^2 + 1}) \\ \therefore x = \sinh^{-1}(\alpha) &= \ln(\alpha + \sqrt{\alpha^2 + 1})(\because A > 0 \text{ for } \ln A) \end{aligned}$$

10) $\cos\theta$ and $\sin\theta$ are defined as

$$\cos\theta = \frac{\mathfrak{e}^{\jmath\theta} + \mathfrak{e}^{-\jmath\theta}}{2}, \sin\theta = \frac{\mathfrak{e}^{\jmath\theta} - \mathfrak{e}^{-\jmath\theta}}{2\jmath} \tag{38}$$

Proof: We know that

$$\mathfrak{e}^{\jmath\theta} = \cos\theta + \jmath\sin\theta \qquad ①$$

By replacing $\jmath$ in ① with $-\jmath$ we get

$$\mathfrak{e}^{-\jmath\theta} = \cos\theta - \jmath\sin\theta \qquad ②$$

① + ② gives us

$$\mathfrak{e}^{\jmath\theta} + \mathfrak{e}^{-\jmath\theta} = 2\cos\theta$$
$$\therefore \frac{\mathfrak{e}^{\jmath\theta} + \mathfrak{e}^{-\jmath\theta}}{2} = \cos\theta$$

① - ② gives us

$$\mathfrak{e}^{\jmath\theta} - \mathfrak{e}^{-\jmath\theta} = 2\jmath\sin\theta$$
$$\therefore \frac{\mathfrak{e}^{\jmath\theta} - \mathfrak{e}^{-\jmath\theta}}{2\jmath} = \sin\theta$$

11) The expression of $\sinh(z), \cosh(z), \sin(z), \cos(z)$ in in standard form.

$$\begin{aligned}
\sinh(a+\jmath b) &= \sinh(a)\cos(b) + \jmath\cosh(a)\sin(b) \qquad ① \\
\cosh(a+\jmath b) &= \cosh(a)\cos(b) + \jmath\sinh(a)\sin(b) \qquad ② \\
\sin(a\pm\jmath b) &= \sin(a)\cosh(b) \pm \jmath\cos(a)\sinh(b) \qquad ③ \\
\cos(a\pm\jmath b) &= \cos(a)\cosh(b) \mp \jmath\sin(a)\sinh(b) \qquad ④
\end{aligned}$$

They are useful but in most cases you need to prove these before you can use them. Therefore you should go through the following proof and practice the proof.

- The proof of ①

$$\sinh(a+\jmath b) = \frac{\mathfrak{e}^{a+\jmath b} - \mathfrak{e}^{-(a+\jmath b)}}{2} = \frac{\mathfrak{e}^{a+\jmath b} - \mathfrak{e}^{-a-\jmath b}}{2} = \frac{\mathfrak{e}^{a}\mathfrak{e}^{\jmath b} - \mathfrak{e}^{-a}\mathfrak{e}^{-\jmath b}}{2}$$
$$= \frac{\mathfrak{e}^{a}(\cos b + \jmath\sin b) - \mathfrak{e}^{-a}(\cos(-b) + \jmath\sin(-b))}{2} = \frac{\mathfrak{e}^{a}(\cos b + \jmath\sin b) - \mathfrak{e}^{-a}(\cos b - \jmath\sin b)}{2}$$
$$= \frac{\cos b\mathfrak{e}^{a} + \jmath\sin b\mathfrak{e}^{a}}{2} - \frac{\cos b\mathfrak{e}^{-a} - \jmath\sin b\mathfrak{e}^{-a}}{2} = \frac{\cos b\mathfrak{e}^{a} + \jmath\sin b\mathfrak{e}^{a}}{2} + \frac{-\cos b\mathfrak{e}^{-a} + \jmath\sin b\mathfrak{e}^{-a}}{2}$$
$$= \frac{\cos b\mathfrak{e}^{a} - \cos b\mathfrak{e}^{-a}}{2} + \jmath\frac{\sin b\mathfrak{e}^{a} + \sin b\mathfrak{e}^{-a}}{2} = \cos(b)\frac{\mathfrak{e}^{a} - \mathfrak{e}^{-a}}{2} + \jmath\sin(b)\frac{\mathfrak{e}^{a} + \mathfrak{e}^{-a}}{2}$$
$$= \sinh(a)\cos(b) + \jmath\cosh(a)\sin(b)$$

- The proof of ②

$$\cosh(a+\jmath b) = \frac{\mathfrak{e}^{a+\jmath b} + \mathfrak{e}^{-(a+\jmath b)}}{2} = \frac{\mathfrak{e}^{a+\jmath b} + \mathfrak{e}^{-a-\jmath b}}{2} = \frac{\mathfrak{e}^{a}\mathfrak{e}^{\jmath b} + \mathfrak{e}^{-a}\mathfrak{e}^{-\jmath b}}{2}$$
$$= \frac{\mathfrak{e}^{a}(\cos(b) + \jmath\sin(b)) + \mathfrak{e}^{-a}(\cos(-b) + \jmath\sin(-b))}{2} = \frac{\mathfrak{e}^{a}(\cos(b) + \jmath\sin(b)) + \mathfrak{e}^{-a}(\cos(b) - \jmath\sin(b))}{2}$$
$$= \frac{\mathfrak{e}^{a}\cos(b) + \jmath\mathfrak{e}^{a}\sin(b) + \mathfrak{e}^{-a}\cos(b) - \jmath\mathfrak{e}^{-a}\sin(b)}{2} = \frac{\mathfrak{e}^{a}\cos(b) + \mathfrak{e}^{-a}\cos(b) - \jmath\mathfrak{e}^{-a}\sin(b) + \jmath\mathfrak{e}^{a}\sin(b)}{2}$$
$$= \frac{\mathfrak{e}^{a}\cos(b) + \mathfrak{e}^{-a}\cos(b) + \jmath(-\mathfrak{e}^{-a}\sin(b) + \mathfrak{e}^{a}\sin(b))}{2} = \frac{\mathfrak{e}^{a}\cos(b) + \mathfrak{e}^{-a}\cos(b)}{2} + \frac{\jmath(-\mathfrak{e}^{-a}\sin(b)) + \mathfrak{e}^{a}\sin(b))}{2}$$
$$= \cos(b)\frac{\mathfrak{e}^{a} + \mathfrak{e}^{-a}}{2} + \jmath\sin(b)\frac{-\mathfrak{e}^{-a} + \mathfrak{e}^{a}}{2} = \cosh(a)\cos(b) + \jmath\sinh(a)\sin(b)$$

- The proof of ③

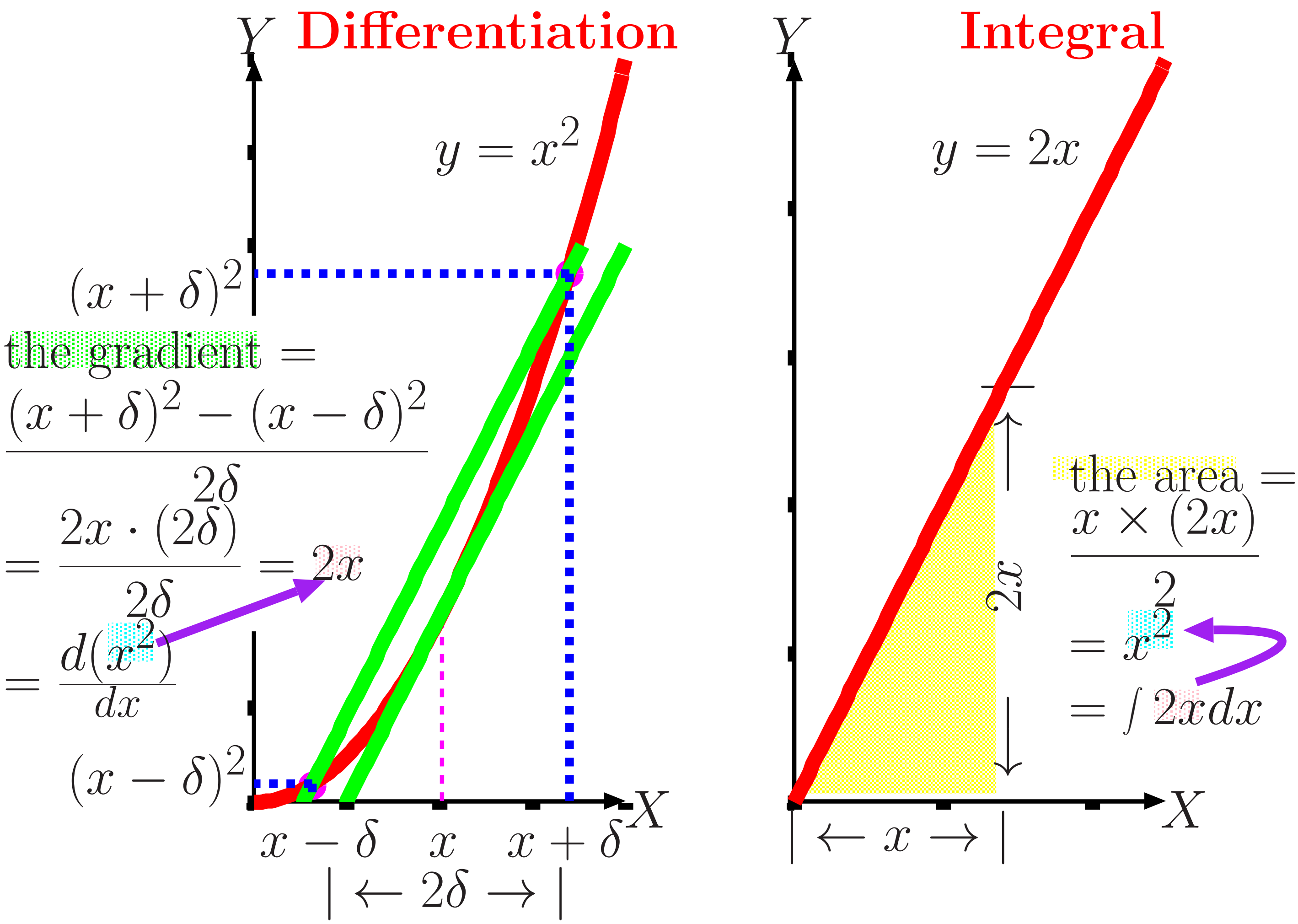
$$\sin(a \pm \jmath b) = \frac{\mathfrak{e}^{\jmath(a \pm \jmath b)} - \mathfrak{e}^{-\jmath(a \pm \jmath b)}}{2\jmath} = \frac{\mathfrak{e}^{a\jmath \pm \jmath^2 b} - \mathfrak{e}^{-a\jmath \mp \jmath^2 b}}{2\jmath} = \frac{\mathfrak{e}^{a\jmath \mp b} - \mathfrak{e}^{-a\jmath \pm b}}{2\jmath} = \frac{\mathfrak{e}^{\jmath a}\mathfrak{e}^{\mp b} - \mathfrak{e}^{-\jmath a}\mathfrak{e}^{\pm b}}{2\jmath}$$
$$= \frac{(\cos(a) + \jmath \sin(a))\mathfrak{e}^{\mp b} - (\cos(a) - \jmath \sin(a))\mathfrak{e}^{\pm b}}{2\jmath} = \frac{\cos(a)\mathfrak{e}^{\mp b} + \jmath \sin(a)\mathfrak{e}^{\mp b} - \cos(a)\mathfrak{e}^{\pm b} + \jmath \sin(a)\mathfrak{e}^{\pm b}}{2\jmath}$$
$$= \frac{\cos(a)(\mathfrak{e}^{\mp b} - \mathfrak{e}^{\pm b}) + \jmath \sin(a)(\mathfrak{e}^{\mp b} + \mathfrak{e}^{\pm b})}{2\jmath} = -\jmath \frac{\cos(a)(\mathfrak{e}^{\mp b} - \mathfrak{e}^{\pm b}) + \jmath \sin(a)(\mathfrak{e}^{\mp b} + \mathfrak{e}^{\pm b})}{2}$$
$$= \frac{-\jmath \cos(a)(\mathfrak{e}^{\mp b} - \mathfrak{e}^{\pm b}) - \jmath^2 \sin(a)(\mathfrak{e}^{\mp b} + \mathfrak{e}^{\pm b})}{2} = \frac{-\jmath \cos(a)(\mathfrak{e}^{\mp b} - \mathfrak{e}^{\pm b}) + \sin(a)(\mathfrak{e}^{\mp b} + \mathfrak{e}^{\pm b})}{2}$$
$$= \frac{-\jmath \cos(a)(\mathfrak{e}^{\mp b} - \mathfrak{e}^{\pm b})}{2} + \frac{\sin(a)(\mathfrak{e}^{\mp b} + \mathfrak{e}^{\pm b})}{2} = \frac{\jmath \cos(a)(-\mathfrak{e}^{\mp b} + \mathfrak{e}^{\pm b})}{2} + \frac{\sin(a)(\mathfrak{e}^{\mp b} + \mathfrak{e}^{\pm b})}{2}$$
$$= \jmath \cos(a)\frac{-\mathfrak{e}^{\mp b} + \mathfrak{e}^{\pm b}}{2} + \sin(a)\frac{\mathfrak{e}^{\mp b} + \mathfrak{e}^{\pm b}}{2} = \pm\jmath \cos(a)\sinh(b) + \sin(a)\cosh(b)$$

- The proof of ④

$$\cos(a \pm \jmath b) = \frac{\mathfrak{e}^{\jmath(a \pm \jmath b)} + \mathfrak{e}^{-\jmath(a \pm \jmath b)}}{2} = \frac{\mathfrak{e}^{a\jmath \pm \jmath^2 b} + \mathfrak{e}^{-a\jmath \mp \jmath^2 b}}{2} = \frac{\mathfrak{e}^{a\jmath \mp b} + \mathfrak{e}^{-a\jmath \pm b}}{2\jmath} = \frac{\mathfrak{e}^{\jmath a}\mathfrak{e}^{\mp b} + \mathfrak{e}^{-\jmath a}\mathfrak{e}^{\pm b}}{2}$$
$$= \frac{(\cos(a) + \jmath \sin(a))\mathfrak{e}^{\mp b} + (\cos(a) - \jmath \sin(a))\mathfrak{e}^{\pm b}}{2} = \frac{\cos(a)\mathfrak{e}^{\mp b} + \jmath \sin(a)\mathfrak{e}^{\mp b} + \cos(a)\mathfrak{e}^{\pm b} - \jmath \sin(a)\mathfrak{e}^{\pm b}}{2}$$
$$= \frac{\cos(a)(\mathfrak{e}^{\mp b} + \mathfrak{e}^{\pm b}) + \jmath \sin(a)(\mathfrak{e}^{\mp b} - \mathfrak{e}^{\pm b})}{2} = \frac{\cos(a)(\mathfrak{e}^{\mp b} + \mathfrak{e}^{\pm b})}{2} + \frac{\jmath \sin(a)(\mathfrak{e}^{\mp b} - \mathfrak{e}^{\pm b})}{2}$$
$$= \cos(a)\frac{\mathfrak{e}^{\mp b} + \mathfrak{e}^{\pm b}}{2} + \jmath \sin(a)\frac{\mathfrak{e}^{\mp b} - \mathfrak{e}^{\pm b}}{2} = \cos(a)\cosh(b) - \jmath \sin(a)\frac{-\mathfrak{e}^{\mp b} + \mathfrak{e}^{\pm b}}{2}$$
$$= \cos(a)\cosh(b) - \jmath \sin(a)(\pm \sinh(b)) = \cos(a)\cosh(b) \mp \jmath \sin(a)\sinh(b)$$

# V. Key points on differentiation

**Key points**




1) Product rule

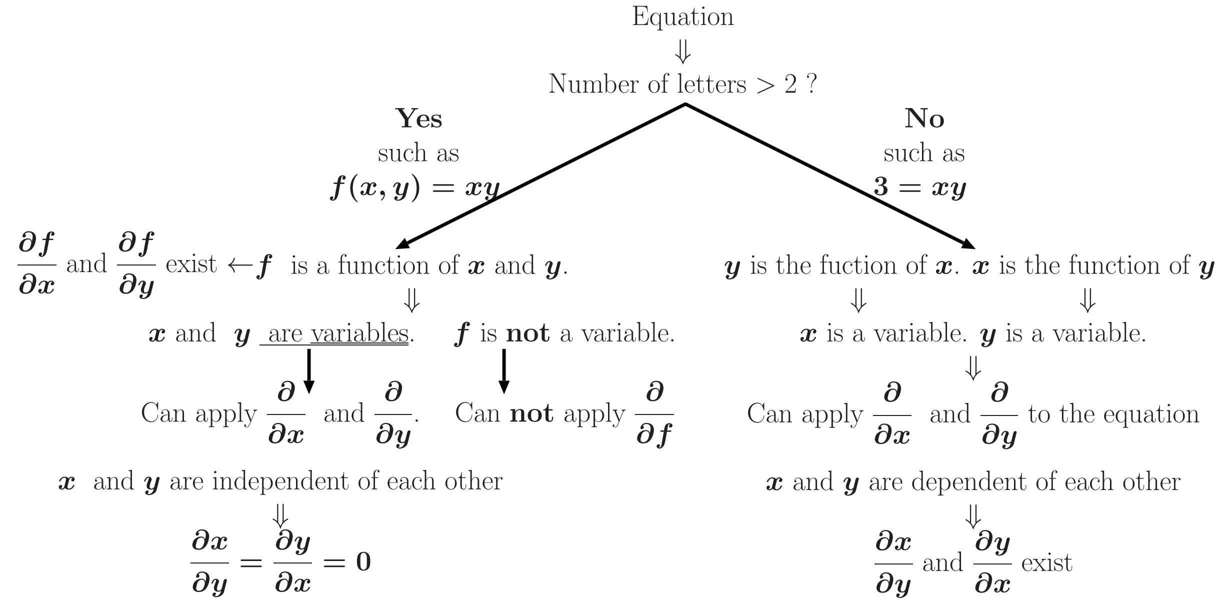
$$\frac{d\{f(x)g(x)\}}{dx} = f(x)\frac{d\{g(x)\}}{dx} + \frac{d\{f(x)\}}{dx}g(x) \tag{39}$$

2) Chain rule
   a) When $y = f(u)$ and $u = g(x)$,

$$\frac{d\{y\}}{dx} = \frac{d\{u\}}{dx} \cdot \frac{\partial\{y\}}{\partial u} \tag{40}$$

   It is important that you know the fundamental differentiable functions of Equation (46) $\sim$ Equation (54) so that a complicated function can be simplified to one of the fundamental functions of Equation (46) $\sim$ Equation (54). For example, if you know that $5^x$ can be differentiable, you can change $\frac{d\left\{5^{x^4-2}\right\}}{dx}$ to $\frac{d\{5^X\}}{dx}$ where $X = x^4 - 2$.

   b) Function and variables




c) When $W$ is a function of $x$, $y$ and $z$ and $x$, $y$, $z$ are the function of $s$ and $t$, $\frac{d\{W\}}{dt}$ and $\frac{d\{W\}}{ds}$ can not be directly calculated but can be calculated as follows:

$$\frac{d\{W\}}{dt} = \frac{d\{W\}}{dx} \cdot \frac{d\{x\}}{dt} + \frac{d\{W\}}{dy} \cdot \frac{d\{y\}}{dt} + \frac{d\{W\}}{dz} \cdot \frac{d\{z\}}{dt}$$

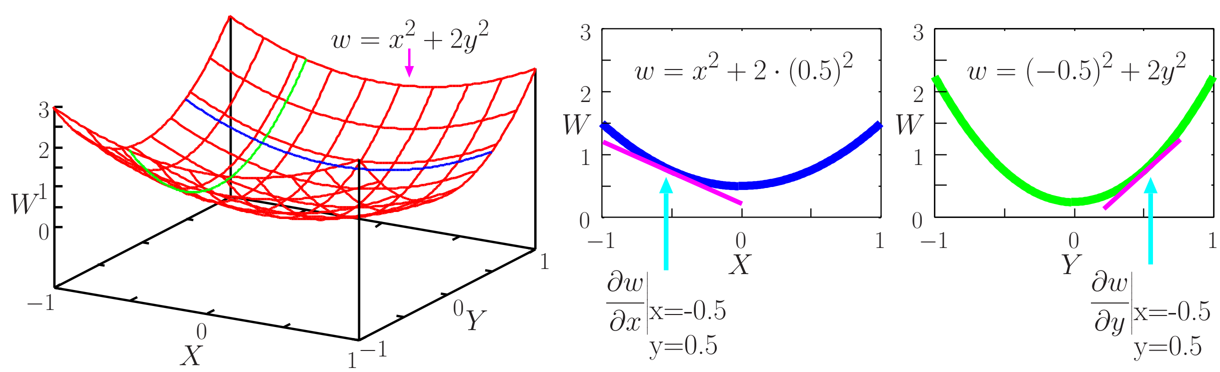
$$\frac{d\{W\}}{ds} = \frac{d\{W\}}{dx} \cdot \frac{d\{x\}}{ds} + \frac{d\{W\}}{dy} \cdot \frac{d\{y\}}{ds} + \frac{d\{W\}}{dz} \cdot \frac{d\{z\}}{ds}$$

d) When $W$ is a function of $x$, $y$ and $z$, the total differential $dW$ can be obtained by

$$dW = \frac{d\{W\}}{dx}dx + \frac{d\{W\}}{dy}dy + \frac{d\{W\}}{dz}dz$$

e) When $W$ is a function of $x$, $y$ and $z$, the gradient $\nabla W$ is defined as

$$\nabla W = \frac{d\{W\}}{dx}\boldsymbol{i} + \frac{d\{W\}}{dy}\boldsymbol{j} + \frac{d\{W\}}{dy}\boldsymbol{k} = \begin{pmatrix} \frac{d\{W\}}{dx} \\ \frac{d\{W\}}{dy} \\ \frac{d\{W\}}{dz} \end{pmatrix}$$



3) Quotient rule

$$\frac{d\left\{\dfrac{f(x)}{g(x)}\right\}}{dx}=\frac{\dfrac{d\{f(x)\}}{dx}g(x)-f(x)\dfrac{d\{g(x)\}}{dx}}{(g(x))^2} \tag{41}$$

Check if $g(x)$ is really a function. If $g(x)$ is a constant, you do not have to use the quatient rule. If $f(x)$ and $g(x)$ are polynomial, check the order of $f(x)$ and $g(x)$. If the order of $f(x)$ is higher than that of $g(x)$ then modify $\dfrac{f(x)}{g(x)}$ so that the order of the numerator of the resultant function is always lower than the order of denominator.

4) When $x$ and $y$ are the function of $t$,

$$\frac{d\{y\}}{dx}=\frac{d\{y\}}{dt}\cdot\frac{d\{t\}}{dx}=\frac{d\{y\}}{dt}\cdot\left(\frac{d\{x\}}{dt}\right)^{-1}$$

and

$$\frac{d^2y}{dx^2}=\frac{d\left\{\dfrac{d\{y\}}{dx}\right\}}{dx}=\frac{d\{t\}}{dx}\frac{d\left\{\dfrac{d\{y\}}{dx}\right\}}{dt}=\left(\frac{d\{x\}}{dt}\right)^{-1}\frac{d\left\{\dfrac{d\{y\}}{dx}\right\}}{dt}$$

5) Let $F(x)$ and $G(y)$ the function of $x$ and $y$, respectively.

a) $\dfrac{d\{y\}}{dx}$ for $F(x)+G(y)=0$ is obtained as

$$\begin{aligned}F(x)+G(y)&=0\\ \therefore\frac{d\{F(x)+G(y)\}}{dx}&=\frac{d\{0\}}{dx}\\ \therefore\frac{d\{F(x)\}}{dx}+\frac{d\{G(y)\}}{dx}&=0\\ \therefore\frac{d\{F(x)\}}{dx}+\frac{d\{y\}}{dx}\frac{d\{G(y)\}}{dy}&=0\\ \therefore\frac{d\{y\}}{dx}&=-\frac{\dfrac{d\{F(x)\}}{dx}}{\dfrac{d\{G(y)\}}{dy}}\end{aligned}$$

b) $\dfrac{d\{y\}}{dx}$ for $F(x)\cdot G(y)=0$ is obtained as

$$\begin{aligned}F(x)\cdot G(y)&=0\\ \therefore\frac{d\{F(x)\cdot G(y)\}}{dx}&=\frac{d\{0\}}{dx}\\ \therefore\frac{d\{F(x)\}}{dx}G(y)+F(x)\frac{d\{G(y)\}}{dx}&=0\\ \therefore\frac{d\{F(x)\}}{dx}G(y)+F(x)\cdot\frac{d\{y\}}{dx}\frac{d\{G(y)\}}{dy}&=0\\ \therefore\frac{d\{y\}}{dx}&=-\frac{\dfrac{d\{F(x)\}}{dx}G(y)}{F(x)\dfrac{d\{G(y)\}}{dy}}\end{aligned}$$

6) When a graph has a local minimum and local maximum at $(x_m, y_m)$, $\frac{d\{y\}}{dx}|_{(x,y)=(x_m,y_m)} = 0$. Furthermore, if $\frac{d^2y}{dx^2}|_{(x,y)=(x_m,y_m)} > 0$, then $(x_m, y_m)$ is the local minimum point. If $\frac{d^2y}{dx^2}|_{(x,y)=(x_m,y_m)} < 0$, then $(x_m, y_m)$ is the local maximum point.
7) L'Hôpital's Rule
Let's assume we have a function of

$$y = f(x) = \frac{P(x)}{Q(x)}.$$

If we want $\lim_{x\to a} f(x)$ but we find out $P(a) = Q(a) = 0$ then we can still find $f(a)$ by

$$\lim_{x\to a} f(x) = \frac{P'(a)}{Q'(a)}.$$

Please do not mix up with $\frac{df(x)}{dx}$ here. $\dfrac{\left.\dfrac{d\frac{P(x)}{Q(x)}}{dx}\right|_{x=a} = \left.\dfrac{P'(x)Q(x) - P(x)'Q(x)}{Q^2(x)}\right|_{x=a} \neq \dfrac{P'(a)}{Q'(a)}}{\text{You are NOT finding a gradient but you are trying to obtain the value of } f(a) = \frac{P(a)}{Q(a)}}$.

**Proof:** When we use Equation (82) we can write

$$P(a+h) = P(a) + h\left.\frac{d\{P\}}{dx}\right|_{x=a} + \cdots$$

and

$$Q(a+h) = Q(a) + h\left.\frac{d\{Q\}}{dx}\right|_{x=a} + \cdots$$

Then we can get the limit as

$$\lim_{x\to a}\frac{P(x)}{Q(x)} = \lim_{h\to 0}\frac{P(a+h)}{Q(a+h)} = \lim_{h\to 0}\frac{P(a) + h\left.\frac{d\{P\}}{dx}\right|_{x=a}}{Q(a) + h\left.\frac{d\{Q\}}{dx}\right|_{x=a}}$$

$$= \lim_{h\to 0}\frac{0 + h\left.\frac{d\{P\}}{dx}\right|_{x=a}}{0 + h\left.\frac{d\{Q\}}{dx}\right|_{x=a}} = \lim_{h\to 0}\frac{h\left.\frac{d\{P\}}{dx}\right|_{x=a}}{h\left.\frac{d\{Q\}}{dx}\right|_{x=a}} = \frac{\left.\frac{d\{P\}}{dx}\right|_{x=a}}{\left.\frac{d\{Q\}}{dx}\right|_{x=a}}$$

8) Newton-Raphson method The crossing point between $y = f(x)$ and $X$ axis can be estimated in an iterative manner as is shown in Fig. 5. The $(n+1)$th guess of the crossing point is obtained using $n$th guess as in Equation (42).

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{42}$$

9) Multivariable higher order differentiation

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx} \tag{43}$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dy} \tag{44}$$

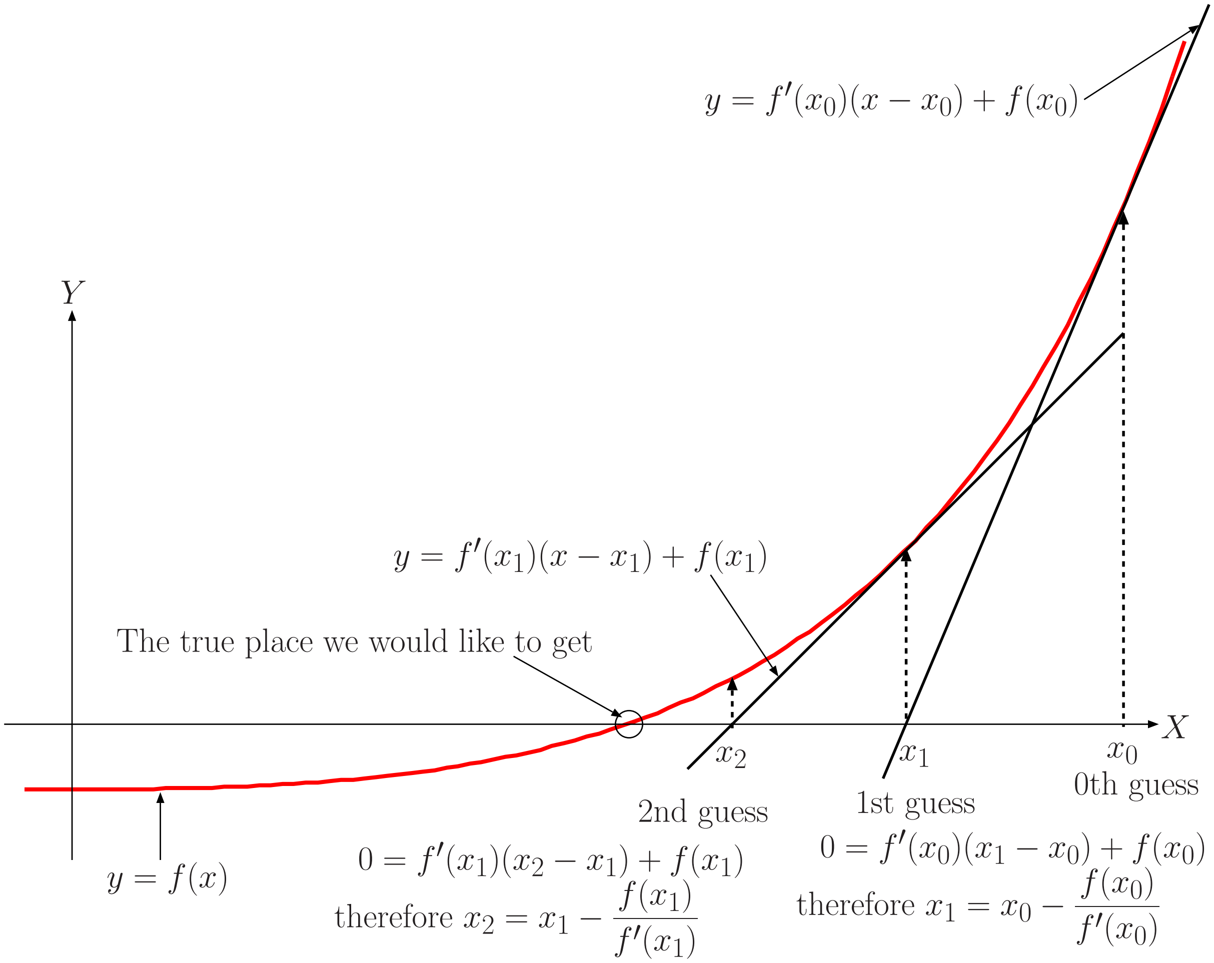



Fig. 5. Estimation of the crossing point between $y = f(x)$ and $X$ axis.

Please pay attention

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} \neq \frac{d\{f(x,y\}}{dy} \cdot \frac{d\{f(x,y)\}}{dx}.$$

Please also be aware the following difference: Let

$$f(x,y) = axy + bx + cy.$$

When we need $\frac{d\{f(x,y\}}{dx}$, then you assume $x$ and $y$ are independent and we obtain

$$\frac{d\{f(x,y)\}}{dx} = ay + b$$

but if we need $\frac{d\{y\}}{dx}$ for $f(x,y) = 0$, then $f(x,y) = 0$ tells you that $x$ and $y$ are dependent of each other and

$xy$ can be regarded as the multiplication of two function $x$ and $y$ and then we obtain

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{0\}}{dx}$$
$$\therefore \frac{d\{axy+bx+cy\}}{dx} = 0$$
$$\therefore a\frac{d\{x\}}{dx}y + ax\frac{d\{y\}}{dx} + b\frac{d\{x\}}{dx} + c\frac{d\{y\}}{dx} = 0$$
$$\therefore ay + ax\frac{d\{y\}}{dx} + b + c\frac{d\{y\}}{dx} = 0$$
$$\therefore (ax+c)\frac{d\{y\}}{dx} = -ay-b$$
$$\therefore \frac{d\{y\}}{dx} = \frac{-ay-b}{ax+c}$$

10) Local minimum and local maximum
When $f(x,y)$ has a local minimum or a local maximum at $x=a$ and $y=b$, then $f(x,y)$ satisfies:

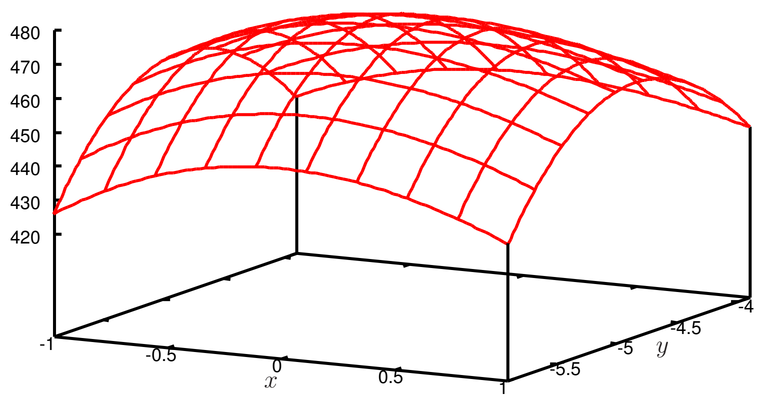
$$\left.\frac{d\{f(x,y)\}}{dx}\right|_{x=a,y=b} = 0, \qquad \left.\frac{d\{f(x,y)\}}{dy}\right|_{x=a,y=b} = 0 \tag{45}$$

This does NOT mean that if $\frac{d\{f(a,b)\}}{dx} = 0$ ,$\frac{d\{f(a,b)\}}{dy} = 0$, then $f(a,b)$ is a local minimum or a local maximum.
When $\frac{d\{f(a,b)\}}{dx} = 0$ ,$\frac{d\{f(a,b)\}}{dy} = 0$ is satisfied;

a) $f(a,b)$ is the local maximum when

$$\frac{d^2f(a,b)}{dx^2}\frac{\partial^2 f(a,b)}{\partial y^2} - \left(\frac{\partial^2 f(a,b)}{\partial y \partial x}\right)^2 > 0$$

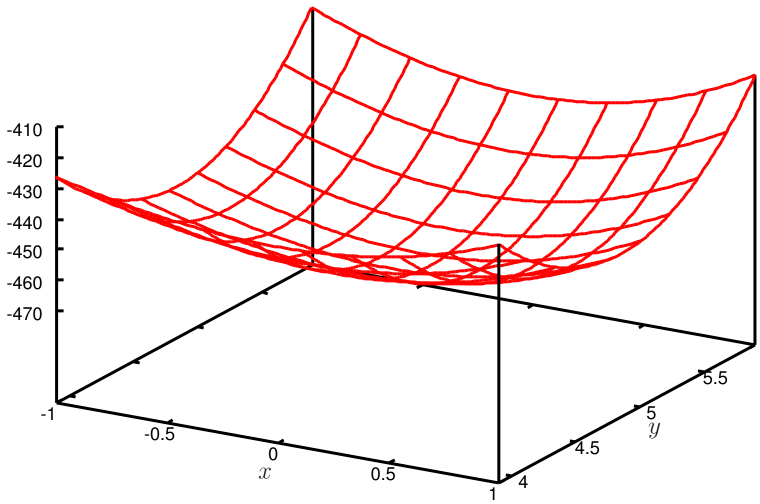
and $\frac{d^2f(a,b)}{dx^2} < 0$




b) $f(a,b)$ is the local minimum when

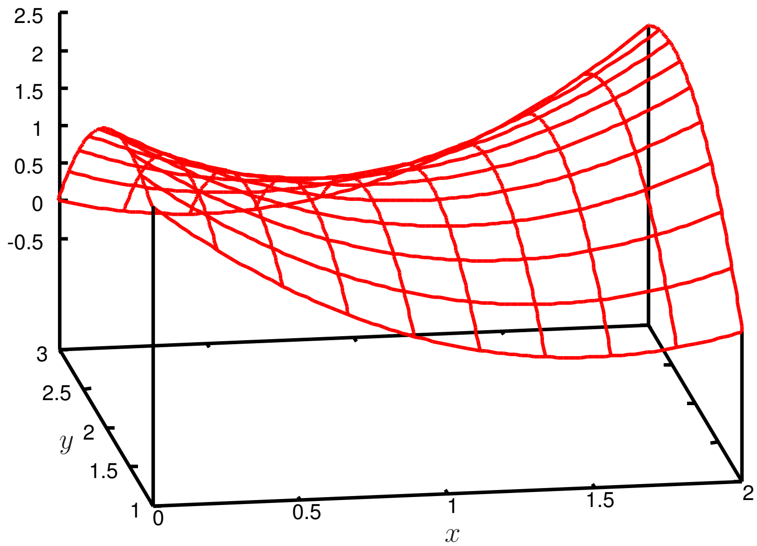
$$\frac{d^2f(a,b)}{dx^2}\frac{\partial^2 f(a,b)}{\partial y^2} - \left(\frac{\partial^2 f(a,b)}{\partial y \partial x}\right)^2 > 0$$

and $\frac{d^2f(a,b)}{dx^2} > 0$




c) $f(a,b)$ is a saddle point when

$$\frac{d^2 f(a,b)}{dx^2}\frac{\partial^2 f(a,b)}{\partial y^2} - \left(\frac{\partial^2 f(a,b)}{\partial y \partial x}\right)^2 < 0$$




d) We do not know whether or not $f(a,b)$ is a local maximum or minimum when

$$\frac{d^2 f(a,b)}{dx^2}\frac{\partial^2 f(a,b)}{\partial y^2} - \left(\frac{\partial^2 f(a,b)}{\partial y \partial x}\right)^2 = 0$$

Attention: $\frac{\partial^2 f}{\partial y \partial x}$ is different from $\frac{d\{f\}}{dx} \cdot \frac{d\{f\}}{dy}$.

Basic derivative:

$$\frac{d\{x^\alpha\}}{dx} = \alpha x^{\alpha-1} \tag{46}$$

Attention: When you see a fraction, get rid of a fraction such as $\frac{1}{x^a}$ immediately by changing it to $x^{-a}$.

$$\frac{d\left\{x^{a}\right\}}{dx} = a \cdot x^{a-1} \tag{47}$$

$$\frac{d\left\{\mathfrak{e}^{kx}\right\}}{dx} = k\mathfrak{e}^{kx} \tag{48}$$

$$\frac{d\left\{\ln(kx)\right\}}{dx} = \frac{1}{x} \tag{49}$$

$$\frac{d\left\{\log_a(kx)\right\}}{dx} = \frac{1}{x\ln a} \tag{50}$$

$$\frac{d\left\{a^{x}\right\}}{dx} = a^{x}\ln a \tag{51}$$

$$\frac{d\left\{\sin kx\right\}}{dx} = k\cos kx \tag{52}$$

$$\frac{d\left\{\cos kx\right\}}{dx} = -k\sin kx \tag{53}$$

$$\frac{d\left\{\tan kx\right\}}{dx} = \frac{k}{\cos^{2} kx} \tag{54}$$

## VI. Key points on integration

### Key points

1) Integral by Parts

$$\int_a^b f(x)\cdot g(x)dx \tag{55}$$
$$= \left[f(x)\cdot\int g(x)dx\right]_a^b - \int_a^b \left(\frac{d\{f(x)\}}{dx}\cdot\int g(x)dx\right)dx$$

<u>Hint: Let $f(x)$ equate the polynomial part or logarithmic part of the intergral.</u>

$\int \sin^n x dx$ and $\int \cos^n x dx$ can be obtained using "Integral by Parts" in order to reduce the power as follows

$$\begin{aligned}
\int \cos^n x dx &= \int \cos^{n-1} x\cdot\cos x dx \\
&= f(x)\cdot\int g(x)dx - \int\left(\frac{d\{f(x)\}}{dx}\cdot\int g(x)dx\right)dx \\
&= \cos^{n-1}x\cdot\int\cos x dx - \int\left(\frac{d\{\cos^{n-1}x\}}{dx}\cdot\int\cos x dx\right)dx \\
&= \cos^{n-1}x\cdot\sin x - \int\left((n-1)\cos^{n-2}x(-\sin x)\cdot\sin x\right)dx \\
&= \cos^{n-1}x\cdot\sin x + (n-1)\int\left(\cos^{n-2}x\cdot\sin^2 x\right)dx \\
&= \cos^{n-1}x\cdot\sin x + (n-1)\int\left(\cos^{n-2}x\cdot(1-\cos^2 x)\right)dx \\
&= \cos^{n-1}x\cdot\sin x + (n-1)\int\left(\cos^{n-2}x - \cos^n x\right)dx \\
&= \cos^{n-1}x\cdot\sin x + (n-1)\int\cos^{n-2}x dx - (n-1)\int\cos^n x dx \\
\therefore \int\cos^n x dx + (n-1)\int\cos^n x dx &= \cos^{n-1}x\cdot\sin x + (n-1)\int\cos^{n-2}x dx \\
\therefore n\int\cos^n x dx &= \cos^{n-1}x\cdot\sin x + (n-1)\int\cos^{n-2}x dx
\end{aligned}$$

$$\begin{aligned}
\int\sin^n x dx = \int\sin^{n-1}x\cdot\sin x dx &= f(x)\cdot\int g(x)dx - \int\left(\frac{d\{f(x)\}}{dx}\cdot\int g(x)dx\right)dx \\
&= \sin^{n-1}x\cdot\int\sin x dx - \int\left(\frac{d\left\{\sin^{n-1}x\right\}}{dx}\cdot\int\sin x dx\right)dx \\
&= \sin^{n-1}x\cdot(-\cos x) - \int\left((n-1)\sin^{n-2}x(\cos x)\cdot(-\cos x)\right)dx \\
&= -\sin^{n-1}x\cdot\cos x + \int\left((n-1)\sin^{n-2}x(\cos^2 x)\right)dx \\
&= -\sin^{n-1}x\cdot\cos x + (n-1)\int\left(\sin^{n-2}x(1-\sin^2 x)\right)dx \\
&= -\sin^{n-1}x\cdot\cos x + (n-1)\int\left(\sin^{n-2}x - \sin^n x\right)dx
\end{aligned}$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \left(\sin^{n-2} x\right) dx - (n-1) \int \left(\sin^{n} x\right) dx$$

$$\therefore \int \sin^n x dx + (n-1) \int \left(\sin^n x\right) dx = -\sin^{n-1} x \cdot \cos x + (n-1) \int \left(\sin^{n-2} x\right) dx$$

$$\therefore n \int \left(\sin^n x\right) dx = -\sin^{n-1} x \cdot \cos x + (n-1) \int \left(\sin^{n-2} x\right) dx$$

2) Integral by substitution

When a function $f(x)$ can be written as $h(g(x))\dfrac{d\left\{g(x)\right\}}{dx}$, you can let $t = g(x)$ therefore, $\dfrac{d\left\{t\right\}}{dx} = \dfrac{d\left\{g(x)\right\}}{dx}$.

$$\int f(x)dx = \int h(g(x))\frac{d\left\{g(x)\right\}}{dx}dx \tag{56}$$

$$= \int h(t)\frac{d\left\{t\right\}}{dx}dx = \int h(t)dt$$

- For $\int \sin^{2m+1} x dx$ , set $t = \cos x$.
- For $\int \sin^{2m} x dx$ , set $t = \sin x$.
- For $\int \cos^{2m+1} x dx$ , set $t = \sin x$.
- For $\int \cos^{2m} x dx$ , set $t = \cos x$.

where $m$ is an integer. But in case of even power such as $2m$, it is better to decrease the power such as

$$\sin^4 x = \left(\sin^2 x\right)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1 - 2\cos 2x + \cos^2 2x}{4} = \frac{1-2\cos 2x}{4} + \frac{1}{4}\cos^2 2x$$

$$= \frac{1-2\cos 2x}{4} + \frac{1}{4}\frac{1+\cos 4x}{2} = \frac{2-4\cos 2x}{8} + \frac{1+\cos 4x}{8} = \frac{3 - 4\cos 2x + \cos 4x}{8}$$

If the power is higher than 4, then use "Integral by Parts" as shown above.

When we carry out $\int_{x_{\mathrm{L}}}^{x_{\mathrm{H}}} f(x)dx$, the procedure of 'integral by substitution' is as follows

a) set the new variable $\theta$ for substitution such as $x = \dfrac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2}$

b) find the relationship between $dx$ and $d\theta$ such as $dx = \dfrac{\mathfrak{e}^{\theta} + \mathfrak{e}^{-\theta}}{2}d\theta$

c) find the range for the new variable $\theta$

$$x_{\mathrm{L}} = \frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2} \rightarrow \theta_{\mathrm{L}} = \ln(x_{\mathrm{L}} + \sqrt{x_{\mathrm{L}}^2 + 1})$$

$$x_{\mathrm{H}} = \frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2} \rightarrow \theta_{\mathrm{H}} = \ln(x_{\mathrm{H}} + \sqrt{x_{\mathrm{H}}^2 + 1})$$

d) manipulate the original function $f(x)$ to remove $x$ . $f(x) \Longrightarrow g(\theta)$

e) calculate the final modified integral such as $\int_{\theta_{\mathrm{L}}}^{\theta_{\mathrm{H}}} g(\theta)\dfrac{\mathfrak{e}^{\theta} + \mathfrak{e}^{-\theta}}{2}d\theta$

3) Integral of $f(x)^k\dfrac{d\left\{f(x)\right\}}{dx}$ for $k = -1$, *i.e.*, $\int \dfrac{f'(x)}{f(x)}dx$

$$\int \frac{1}{f(x)}\frac{d\left\{f(x)\right\}}{dx}dx = \ln|f(x)| + c \tag{57}$$

Proof:

$$\frac{d\{\ln|f(x)|\}}{dx} = \frac{d\{\ln|A|\}}{dx}(\because A \triangleq f(x)) = \frac{d\{A\}}{dx}\frac{d\{\ln|A|\}}{dA} = \frac{d\{f(x)\}}{dx}\frac{1}{A} = \frac{f'(x)}{f(x)}$$

$$\therefore \frac{f'(x)}{f(x)} = \frac{d\{\ln|f(x)|\}}{dx}$$

$$\therefore \int \frac{f'(x)}{f(x)}dx = \int \frac{d\{\ln|f(x)|\}}{dx}dx = \int \partial(\ln|f(x)|) = \ln|f(x)|$$

4) Integral of $f(x)^k \frac{d\{f(x)\}}{dx}$ for $k \neq -1$

$$\int f(x)^k \cdot \frac{d\{f(x)\}}{dx}dx = \frac{1}{k+1}f(x)^{k+1} + c \tag{58}$$

5) $P(x)$ and $Q(x)$ are the $m$th and $n$th order polynomials, respectively.
   - When $m > n$, $\int \frac{P(x)}{Q(x)}dx$ can be obtained as follows:
     a) Find the answer of $A(x)$ and the remainer $R(x)$ of $\frac{P(x)}{Q(x)}$ which satisfy $P(x) = Q(x)A(x) + R(x)$
     b) Find the answer of $C$ and the remainder of $E$ of $\frac{R(x)}{Q'(x)}$ which satisfy $R(x) = C \cdot Q'(x) + E$
     c) $\int \frac{P(x)}{Q(x)}dx = \int \left(A(x) + C\frac{Q'(x)}{Q(x)} + \frac{E}{Q(x)}\right)dx = \int A(x)dx + C\ln|Q(x)| + \int \frac{E}{Q(x)}dx$
   - When $m < n$, $\int \frac{P(x)}{Q(x)}dx$ can be obtained as follows:
     a) Find the answer of $C$ and the remainder of $E$ of $\frac{P(x)}{Q'(x)}$ which satisfy $P(x) = C \cdot Q'(x) + E$
     b) $\int \frac{P(x)}{Q(x)}dx = \int \left(C\frac{Q'(x)}{Q(x)} + \frac{E}{Q(x)}\right)dx = C\ln|Q(x)| + \int \frac{E}{Q(x)}dx$

6) Calculation of Area($A$), Arc-length ($L$), Surface area($S$), Volume($V$)
   Example:

$$y = (x-1)^3 + 1 \iff \quad x = (y-1)^{\frac{1}{3}} + 1$$
$$\frac{d\{y\}}{dx} = 3(x-1)^2; \quad \frac{d\{x\}}{dy} = \frac{1}{3}(y-1)^{-\frac{2}{3}}$$

Using a parameter $t$, $y = (x-1)^3 + 1$ can be expressed as

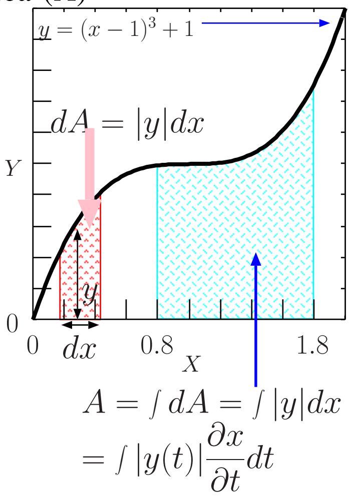
$$x = t + 1$$
$$y = t^3 + 1$$

| $x$ | 0.8 | 1.8 | $y$ | 1 | 2 |
|---|---|---|---|---|---|
| $t$ | -0.2 | 0.8 | $t$ | 0 | 1 |

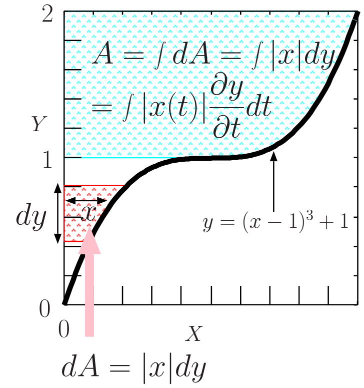
In this case

$$\frac{d\{x\}}{dt} = 1$$
$$\frac{d\{y\}}{dt} = 3t^2$$

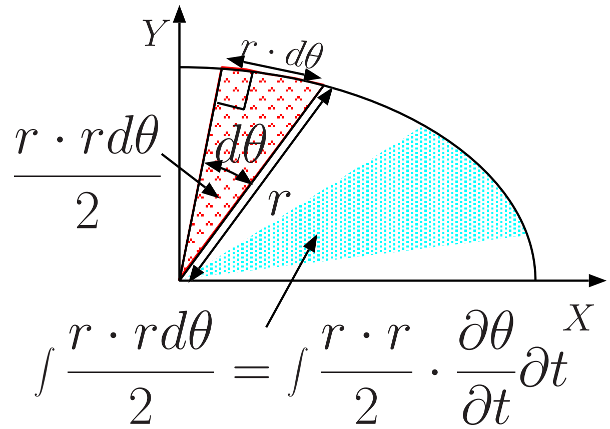
- Area ($A$)




Area bounded by the $X$-axis




Area bounded by the $Y$-axis




Area in polar coordinates

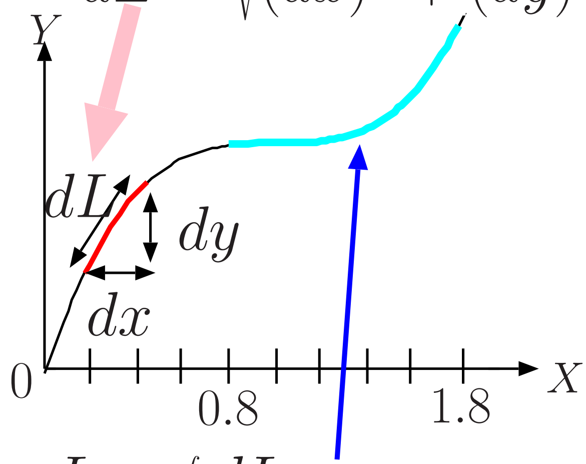
$$A=\int dA=\int_{0.8}^{1.8} ydx$$
$$=\int_{0.8}^{1.8}\left\{(x-1)^3+1\right\}dx$$

$$A=\int dA=\int_{1}^{2} xdy$$
$$=\int_{1}^{2}\left\{(y-1)^{\frac{1}{3}}+1\right\}dy$$

$$A=\int dA$$
$$=\int_{-0.2}^{0.8} y(t)\frac{d\left\{x\right\}}{dt}dt$$
$$=\int_{-0.2}^{0.8}\left\{t^3+1\right\}\cdot 1\cdot dt$$
$$=\int_{-0.2}^{0.8}\left\{t^3+1\right\}dt$$

$$A=\int dA$$
$$=\int_{0}^{1} x(t)\frac{d\left\{y\right\}}{dt}dt$$
$$=\int_{0}^{1}\left\{t+1\right\}\cdot 3t^2\cdot dt$$
$$=\int_{0}^{1}\left\{3t^3+3t^2\right\}dt$$

$$dL = \sqrt{(dx)^2 + (dy)^2}$$



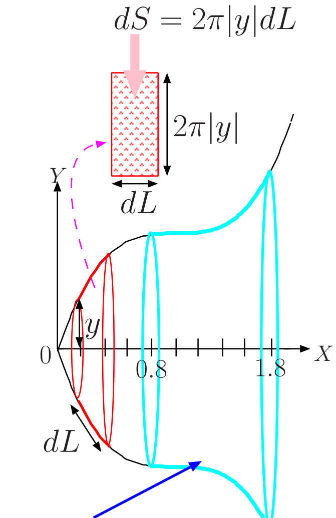

- Arc-length $(L)$

$$L = \int dL$$
$$= \int \sqrt{(dx)^2 + (dy)^2}$$
$$= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$
$$= \int \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy$$
$$= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$L = \int dL$$
$$= \int_{0.8}^{1.8} \sqrt{1 + \left(\frac{d\{y\}}{dx}\right)^2} dx$$
$$= \int_{0.8}^{1.8} \sqrt{1 + (3(x-1)^2)^2} dx$$
$$= \int_{0.8}^{1.8} \sqrt{1 + 9(x-1)^4} dx$$

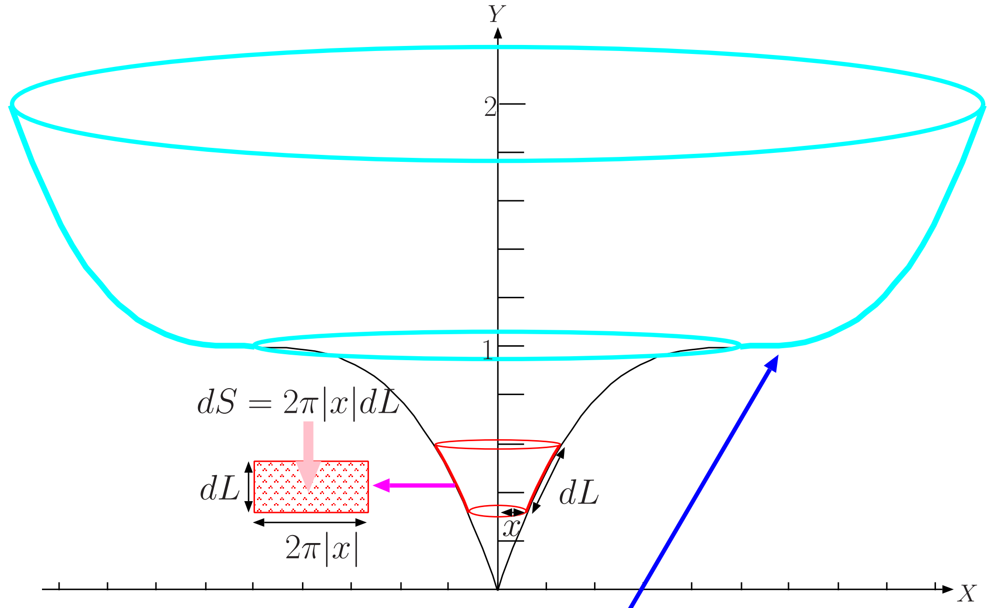
$$L = \int dL$$
$$= \int_{-0.2}^{0.8} \sqrt{\left(\frac{d\{x\}}{dt}\right)^2 + \left(\frac{d\{y\}}{dt}\right)^2} dt$$
$$= \int_{-0.2}^{0.8} \sqrt{(1)^2 + (3t^2)^2} dt$$
$$= \int_{-0.2}^{0.8} \sqrt{1 + 9t^4} dt$$

- Surface area ($S$) of solid of revolution




$$
\begin{aligned}
S &= \int dS \\
&= \int 2\pi|y|dL \\
&= \int 2\pi|y|\sqrt{(dx)^2+(dy)^2} \\
&= \int 2\pi|y|\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \\
&= \int 2\pi|y(t)|\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt
\end{aligned}
$$

Rotation about the $X$-axis




$$
\begin{aligned}
S &= \int dS \\
&= \int 2\pi|x|dL \\
&= \int 2\pi|x|\sqrt{(dx)^2+(dy)^2} \\
&= \int 2\pi|x|\sqrt{\left(\frac{dx}{dy}\right)^2+1}dy \\
&= \int 2\pi|x(t)|\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt
\end{aligned}
$$

Rotation about the $Y$-axis

$$S = \int dS = \int_{0.8}^{1.8} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)}dx$$

$$= \int_{0.8}^{1.8} 2\pi\left\{(x-1)^3+1\right\}\sqrt{1+3(x-1)^2}dx$$

$$S = \int dS = \int_{1}^{2} 2\pi x\sqrt{\left(\frac{dx}{dy}\right)+1}dy$$

$$= \int_{1}^{2} 2\pi\left\{(y-1)^{\frac{1}{3}}+1\right\}\sqrt{\frac{1}{3}(y-1)^{-\frac{2}{3}}+1}dy$$

$$S = \int dS =$$

$$\int_{-0.2}^{0.8} 2\pi y(t)\sqrt{\left(\frac{d\{x\}}{dt}\right)^2+\left(\frac{d\{y\}}{dt}\right)^2}dt$$

$$= \int_{-0.2}^{0.8} 2\pi(t^3+1)\sqrt{(1)^2+(3t^2)^2}dt$$

$$= \int_{-0.2}^{0.8} 2\pi(t^3+1)\sqrt{1+9t^4}dt$$

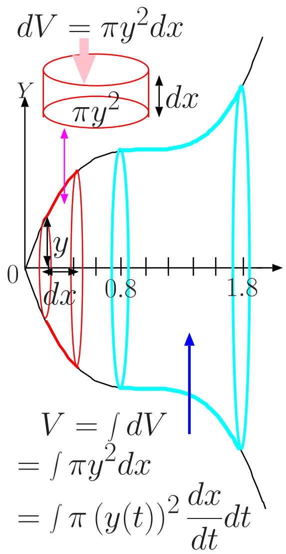
$$S = \int dS =$$

$$\int_{0}^{1} 2\pi x(t)\sqrt{\left(\frac{d\{x\}}{dt}\right)^2+\left(\frac{d\{y\}}{dt}\right)^2}dt$$

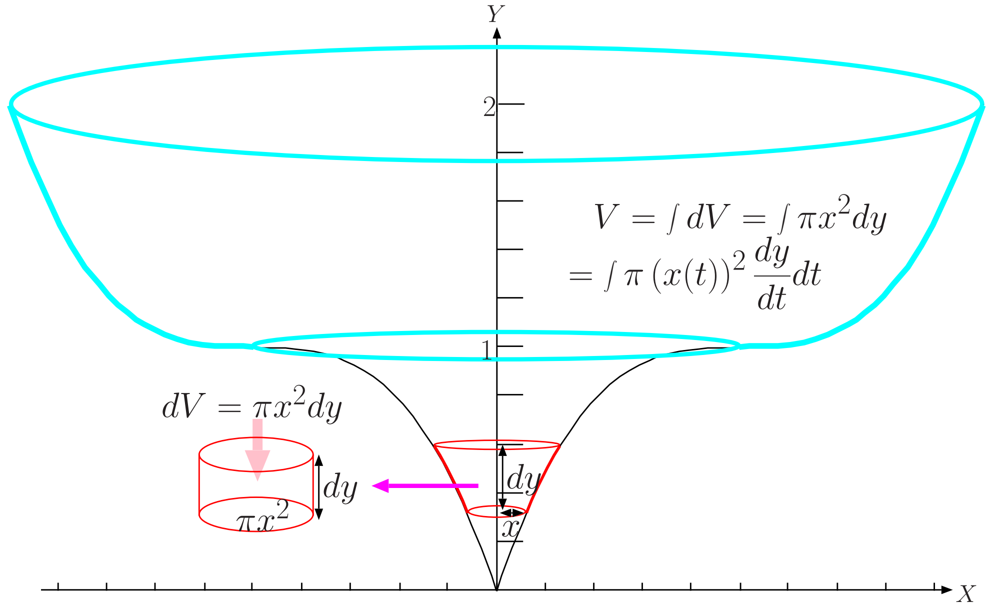
$$= \int_{0}^{1} 2\pi(t+1)\sqrt{(1)^2+(3t^2)^2}dt$$

$$= \int_{0}^{1} 2\pi(t+1)\sqrt{1+9t^4}dt$$

- Volume ($V$) of solid of revolution




Rotation about the $X$-axis




Rotation about the $Y$-axis

$$V = \int dV = \int_{0.8}^{1.8} \pi y^2 dx$$

$$= \int_{0.8}^{1.8} \pi \left\{(x-1)^3 + 1\right\}^2 dx$$

$$V = \int dV = \int_{1}^{2} \pi x^2 dy$$

$$= \int_{1}^{2} \pi \left\{(y-1)^{\frac{1}{3}} + 1\right\}^2 dy$$

$$V = \int dV =$$

$$\int_{-0.2}^{0.8} \pi (y(t))^2 \frac{d\{x\}}{dt} dt$$

$$= \int_{-0.2}^{0.8} \pi \left(t^3+1\right)^2 \cdot 1 \cdot dt$$

$$= \int_{-0.2}^{0.8} \pi \left(t^3+1\right)^2 dt$$

$$V = \int dV =$$

$$\int_{0}^{1} \pi (x(t))^2 \frac{d\{y\}}{dt} dt$$

$$= \int_{0}^{1} \pi (t+1)^2 \cdot 3t^2 \cdot dt$$

$$= \int_{0}^{1} 3\pi \left(t^2+t\right)^2 \cdot dt$$

7) Line integrals of a function which has $dx$,$dy$, and $dz$ such as $I = \int_C (F_x dx + F_y dy + F_z dz)$ .
Consider a curve $C$. The position vector of a point on the curve $C$ is written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$
$$a \le t \le b$$

Denote

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and its derivative with respect to $t$ as

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \\ \frac{d\{z\}}{dt} \end{pmatrix}.$$

When a vector function is expressed as

$$\mathbf{F}(\boldsymbol{r}) = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

a line integral of $\mathbf{F}(\boldsymbol{r})$ over a curve $C$ is defined by

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} = \int_{t=a}^{t=b} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \cdot \frac{d\{\boldsymbol{r}\}}{dt} dt$$

$$= \int_{t=a}^{t=b} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \cdot \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \\ \frac{d\{z\}}{dt} \end{pmatrix} dt$$

$$= \int_{t=a}^{t=b} (F_x \frac{d\{x\}}{dt} + F_y \frac{d\{y\}}{dt} + F_z \frac{d\{z\}}{dt}) dt \tag{59}$$

$$= \int (F_x dx + F_y dy + F_z dz) \tag{60}$$

$$= \int_{x=\hat{a}}^{x=\hat{b}} (F_x + F_y \frac{dy}{dx} + F_z \frac{dz}{dx}) dx \tag{61}$$

Thus the procedure to solve the line integral is

a) Express $x, y, z$ on the curve $C$ using $t$ and set the range of $t$
b) Express $\mathbf{F}$ as the function of $t$
c) Express $\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \\ \frac{d\{z\}}{dt} \end{pmatrix}$ using $t$
d) Put all of them into $\int \mathbf{F} \cdot \frac{d\{\boldsymbol{r}\}}{dt} dt$

8) Line integrals of a function (which does not have $dx, dy$ or $dz$ explicitly) with respect to arc length such as $\int_C f(x, y, z) ds$.

Consider a curve $C$. The position vector of a point on the curve $C$ is written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$
$$a \leq t \leq b$$

Denoting

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and its derivative with respect to t as

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \\ \frac{d\{z\}}{dt} \end{pmatrix}$$

the line integral of a function with respect to arc length is defined by

$$\int_C f(x,y,z)ds$$

$$= \int_{t=a}^{t=b} f(x,y,z)\sqrt{\left(\frac{d\{x\}}{dt}\right)^2 + \left(\frac{d\{y\}}{dt}\right)^2 + \left(\frac{d\{z\}}{dt}\right)^2}dt \tag{62}$$

where

$$ds = \sqrt{\left(\frac{d\{x\}}{dt}\right)^2 + \left(\frac{d\{y\}}{dt}\right)^2 + \left(\frac{d\{z\}}{dt}\right)^2}dt$$

The procedure to solve this type of the line integral is

a) Express $x, y, z$ on the curve $C$ using $t$ and set the range of $t$
b) Express $f(x,y,z)$ as the function of $t$
c) Express $\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \\ \frac{d\{z\}}{dt} \end{pmatrix}$ using $t$
d) Put all of them into

$$\int_{t=a}^{t=b} f(x,y,z)\sqrt{\left(\frac{d\{x\}}{dt}\right)^2 + \left(\frac{d\{y\}}{dt}\right)^2 + \left(\frac{d\{z\}}{dt}\right)^2}dt$$

9) Multiple integration

$$I = \int_a^b \int_c^d \int_e^f f(x,y,z)dxdydz$$

has the following range:

$$e \le x \le f$$
$$c \le y \le d$$
$$a \le z \le b$$

The procedure for the calculation is

a)

$$A = \int_e^f f(x,y,z)dx$$

b)

$$B = \int_a^b \int_c^d A dy$$

c)

$$I = \int_a^b B dz$$

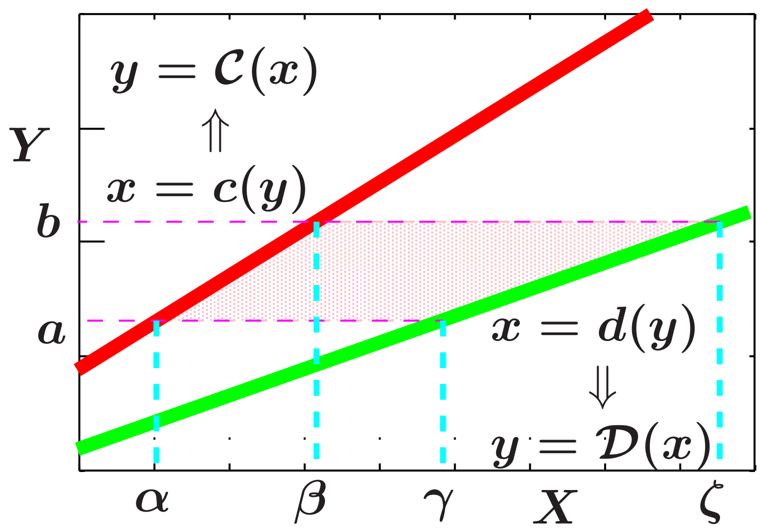
Please be aware that

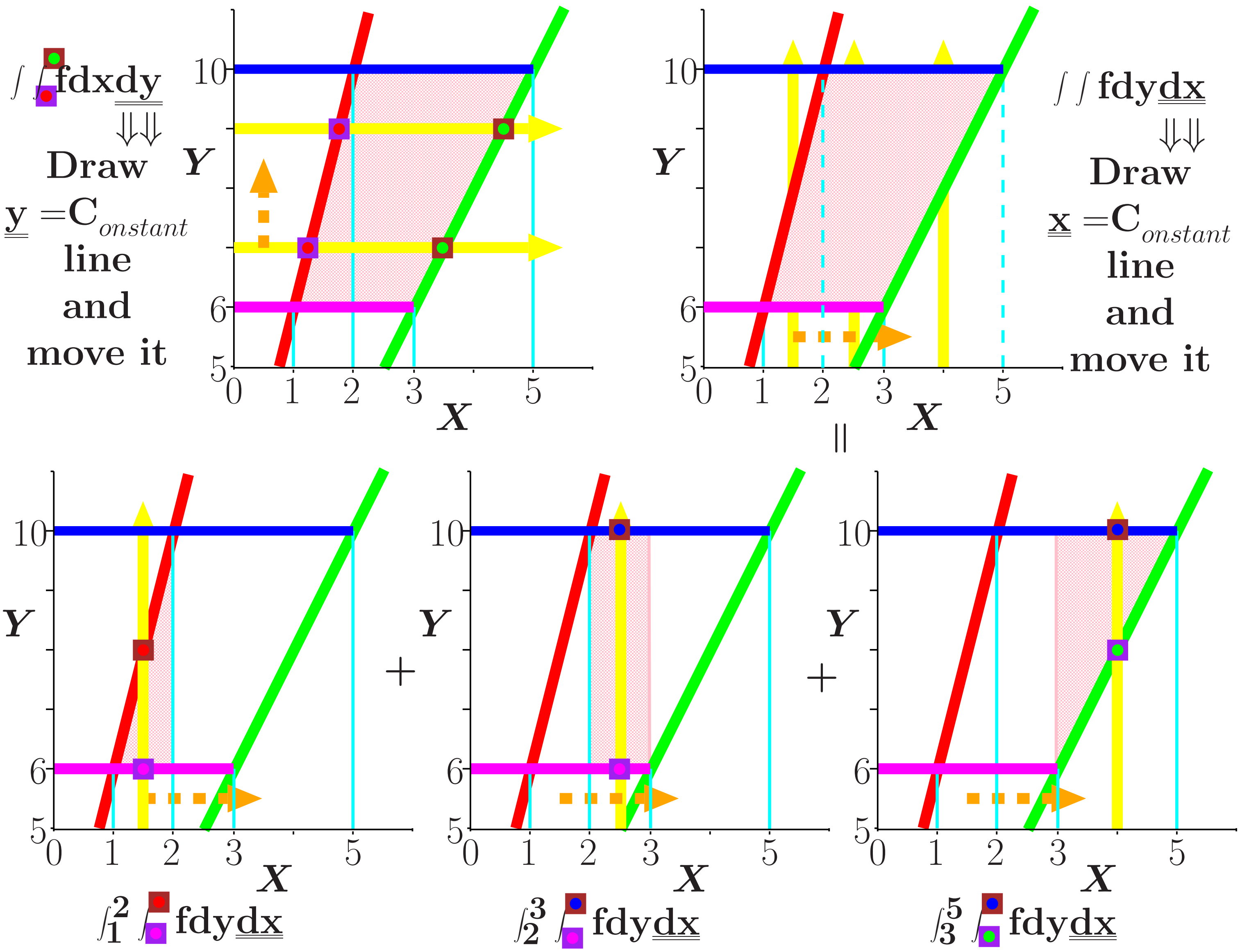
$$\int_a^b \int_c^d \int_e^f f dx dy dz \neq \int_a^b f dx \times \int_c^d f dy \times \int_e^f f dz$$

10) Reversing the order of multiple integration

-

$$I = \int_a^b \int_{c(y)}^{d(y)} f(x,y) dx dy \qquad I = \int_6^{10} \int_{\frac{y}{4}-\frac{1}{2}}^{\frac{y}{2}} f(x,y) dx dy \text{ where } c(y) = \frac{y}{4} - \frac{1}{2}, \qquad d(y) = \frac{y}{2}$$






a) Find the original range of integration

$$\begin{aligned} c(y) &\leq x \leq d(y) & \frac{y}{4} - \frac{1}{2} &\leq x \leq \frac{y}{2} \\ a &\leq y \leq b & 6 &\leq y \leq 10 \end{aligned}$$

b) Sketch the range of integration

c) If necessary, manipulate the equation of $x = c(y)$ to make $y$ the subject of the equation such as $y = \mathcal{C}(x)$.

For example $x = \dfrac{y}{4} - \dfrac{1}{2}$ is changed to $y = 4x + 2$ and $x = \frac{y}{2}$ is changed to $y = 2x$

d) Find out the range of $y$ when $x$ is fixed such as

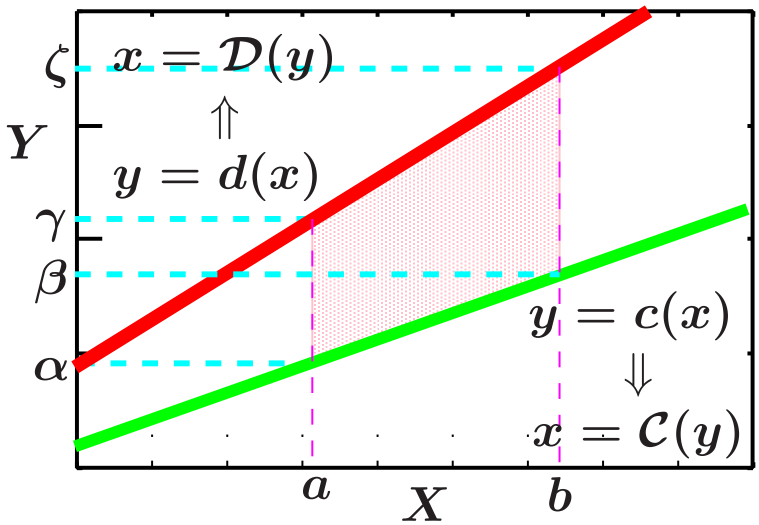
$$\begin{array}{lll} a \leq y \leq \mathcal{C}(x) \quad \text{for} \quad \alpha \leq x \leq \beta & 6 \leq y \leq 4x + 2 & \text{for} \quad 1 \leq x \leq 2 \\ a \leq y \leq b \quad \text{for} \quad \beta \leq x \leq \gamma & 6 \leq y \leq 10 & \text{for} \quad 2 \leq x \leq 3 \\ \mathcal{D}(x) \leq y \leq b \quad \text{for} \quad \gamma \leq x \leq \zeta & 2x \leq y \leq 10 & \text{for} \quad 3 \leq x \leq 5 \end{array}$$

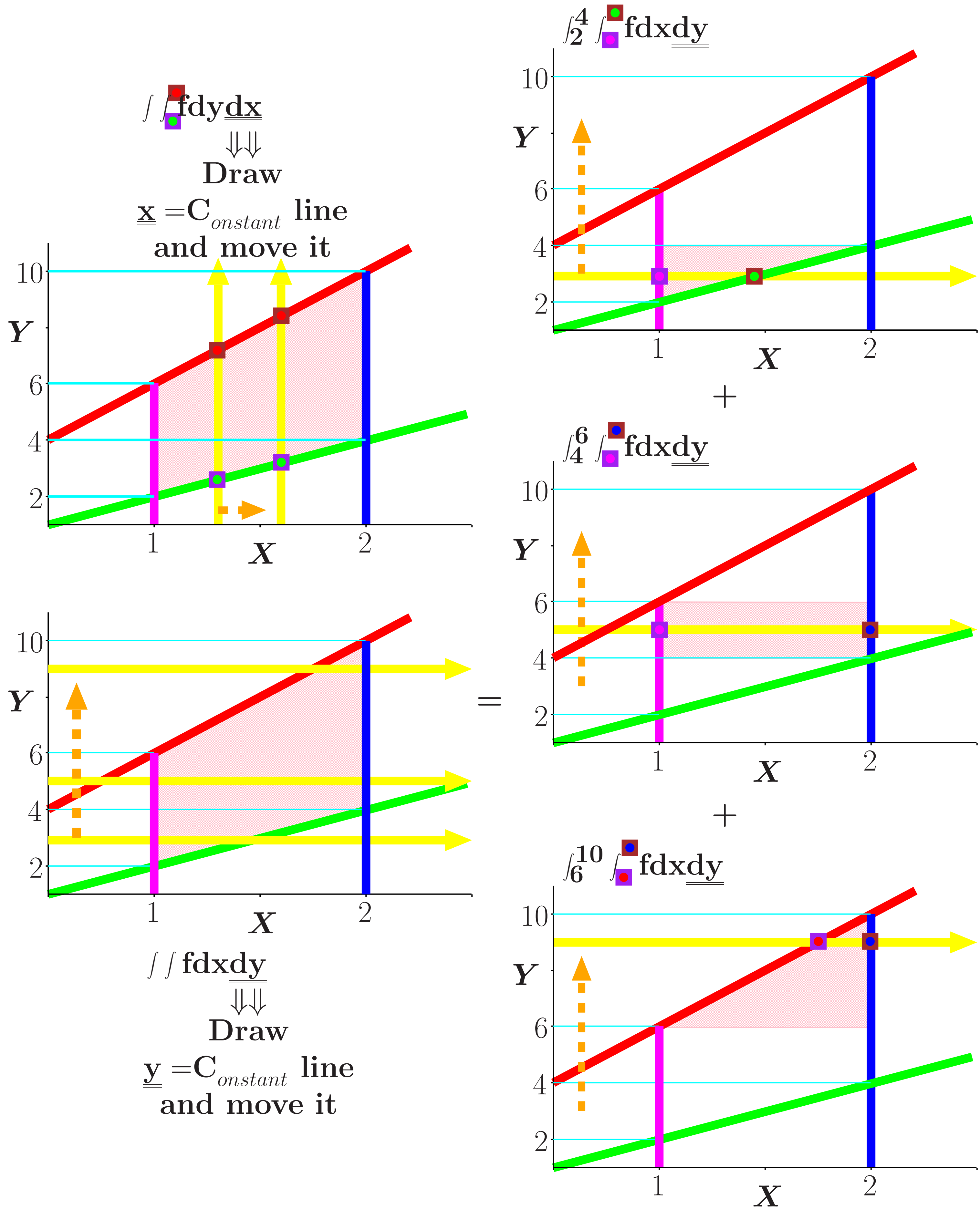
e) Rewrite the integral such as

$$I = \int_{\alpha}^{\beta} \int_{a}^{\mathcal{C}(x)} f(x,y)dydx + \int_{\beta}^{\gamma} \int_{a}^{b} f(x,y)dydx + \int_{\gamma}^{\zeta} \int_{\mathcal{D}(x)}^{b} f(x,y)dydx$$

$$I = \int_1^2 \int_6^{4x+2} f(x,y)dydx + \int_2^3 \int_6^{10} f(x,y)dydx + \int_3^5 \int_{2x}^{10} f(x,y)dydx$$

- $$I = \int_a^b \int_{c(x)}^{d(x)} f(x,y)dydx \qquad I = \int_1^2 \int_{2x}^{4x+2} f(x,y)dydx \text{ where } c(x) = 2x, \qquad d(x) = 4x+2$$




∫∫ fdydx
⇓⇓
Draw
x =C onstant line
and move it
∫∫ fdxdy
⇓⇓
Draw
y =C onstant line
and move it
∫ 2 4 ∫ fdxdy
+
∫ 4 6 ∫ fdxdy
+
∫ 6 10 ∫ fdxdy
=
Y
X
10
6
4
2
1
2

a) Find the original range of integration

$$\begin{aligned} c(x) \leq y \leq d(x) & \qquad 2x \leq y \leq 4x+2 \\ a \leq x \leq b & \qquad 1 \leq x \leq 2 \end{aligned}$$

b) Sketch the range of integration
c) If necessary, manipulate the question of $y = c(x)$ to make $x$ the subject of the equation such as $x = \mathcal{C}(y)$.
   For example $y = 2x$ is changed to $x = \dfrac{y}{2}$ and $y = 4x + 2$ is changed to $x = \dfrac{y}{4} - \dfrac{1}{2}$.
d) Find out the range of $x$ when $y$ is constant such as

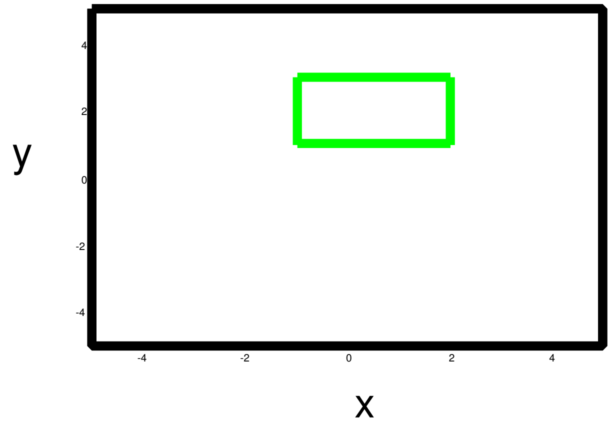
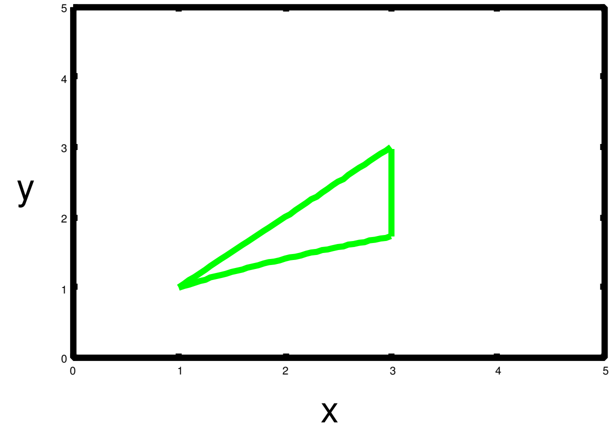
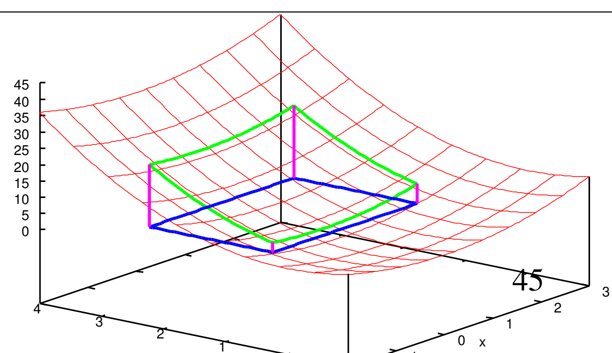
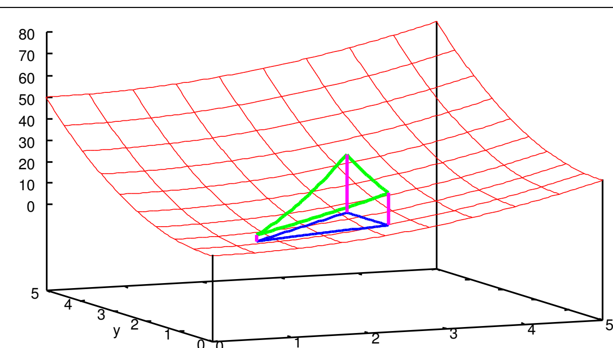
$$\begin{aligned} a \leq x \leq \mathcal{C}(y) \ \text{ for } \ \alpha \leq y \leq \beta & \qquad 1 \leq x \leq \tfrac{y}{2} \ \text{ for } \ 2 \leq y \leq 4 \\ a \leq x \leq b \ \text{ for } \ \beta \leq y \leq \gamma & \qquad 1 \leq x \leq 2 \ \text{ for } \ 4 \leq y \leq 6 \\ \mathcal{D}(y) \leq x \leq b \ \text{ for } \ \gamma \leq y \leq \zeta & \qquad \tfrac{y}{4} - \tfrac{1}{2} \leq x \leq 2 \ \text{ for } \ 6 \leq y \leq 10 \end{aligned}$$

e) Rewrite the integral such as

$$I = \int_{\alpha}^{\beta} \int_{a}^{\mathcal{C}(y)} f(x,y)dxdy + \int_{\beta}^{\gamma} \int_{a}^{b} f(x,y)dxdy + \int_{\gamma}^{\zeta} \int_{\mathcal{D}(y)}^{b} f(x,y)dxdy$$

$$I = \int_{2}^{4} \int_{1}^{\frac{y}{2}} f(x,y)dxdy + \int_{4}^{6} \int_{1}^{2} f(x,y)dxdy + \int_{6}^{10} \int_{\frac{y}{4}-\frac{1}{2}}^{2} f(x,y)dxdy$$

11) Implicit multiple integration $I = \iint_D f(x,y)dA$
   a) Sketch the integral region
   b) Set the range for $x$ and $y$
      The range of either $x$ or $y$ should be fixed without any variables.
   c) Set the variable with fixed limits as the second integral variable and the other one as the first integral variable.
   d) Do the first integral
   e) Do the second integral

| | | |
|---|---|---|
| Function given | $I=\iint_D x^2+2y^2dA$ | $I=\iint_D x^2+2y^2dA$ |
| Ingegral region given | $D$ is the region bounded by $-1\le x\le 2$ and $1\le y\le 3$. | $D$ is the region bounded by $1\le x\le 3$ and $\sqrt{x}\le y\le x$. |
| Sketch the integral region |   |   |
| Set the range of $x$ and $y$ | $-1\le x\le 2$,$1\le y\le 3$ | $1\le x\le 3$, $\sqrt{x}\le y\le x$ |
| Set the variable with fixed limits as the second integral variable and the other one as the first integral variable. | $I=\int_1^3\int_{-1}^2 x^2+2y^2dxdy=\int_{-1}^2\int_1^3 x^2+2y^2dydx$ | $I=\int_1^3\int_{\sqrt{x}}^x x^2+2y^2dydx$ |
| Do the first integral | $I=\int_1^3\int_{-1}^2 x^2+2y^2dxdy$ $=\int_1^3[\frac{x^3}{3}+2y^2x]_{-1}^2dy$ $=\int_1^3[\frac{8}{3}+4y^2-\frac{1}{3}+2y^2]dy$ $=\int_1^3[\frac{7}{3}+6y^2]dy$ | $I=\int_1^3\int_{\sqrt{x}}^x x^2+2y^2dydx$ $=\int_1^3[x^2y+\frac{2}{3}y^3]_{\sqrt{x}}^x dx$ $=\int_1^3[x^3+\frac{2}{3}x^3-x^{2.5}-\frac{2}{3}x^{1.5}]dx$ $=\int_1^3[\frac{5}{3}x^3-x^{2.5}-\frac{2}{3}x^{1.5}]dx$ |
| Do the second integral | $I=\int_1^3[\frac{7}{3}+6y^2]dy$ $=[\frac{7}{3}y+2y^3]_1^3$ $=7+54-\frac{7}{3}-2=\frac{170}{3}$ | $I=\int_1^3[\frac{5}{3}x^3-x^{2.5}-\frac{2}{3}x^{1.5}]dx$ $=[\frac{5}{12}x^4-\frac{1}{3.5}x^{3.5}-\frac{2}{7.5}x^{2.5}]_1^3$ $=\frac{80\cdot 5}{12}+\frac{1-3^{3.5}}{3.5}+2\frac{1-3^{2.5}}{7.5}$ |
| What you got is the volume ; the red curve is $z=x^2+2y^2$ |  |  |

## Integrals of common functions.

Some are very similar to the fundamental functions for differentiation. So please do not mix up!, especially signs such as $+$ or $-$.

$$n \neq -1 \quad \text{and} \quad \int kx^n dx = \frac{1}{n+1} \cdot kx^{n+1} + c \tag{63}$$

$$n = -1 \quad \text{and} \quad \int kx^n dx = \int \frac{k}{x} dx = k \ln |x| + c \tag{64}$$

$$\int \cos kx dx = \frac{1}{k} \sin kx + c \tag{65}$$

$$\int \sin kx dx = -\frac{1}{k} \cos kx + c \tag{66}$$

$$\int \tan kx dx = -\frac{1}{k} \ln |\cos kx| + c \tag{67}$$

$$\int \mathfrak{e}^{kx} dx = \frac{1}{k} \mathfrak{e}^{kx} + c \tag{68}$$

$$\int a^{kx} dx = \frac{a^{kx}}{k \ln a} + c (a > 0) \tag{69}$$

$$\int \cos^2(kx) dx = \frac{1}{2k}(kx + \sin(kx)\cos(kx)) \tag{70}$$

$$\int \frac{1}{\cos^2(kx)} dx = \frac{\tan kx}{k} \tag{71}$$

$$\int \frac{1}{\sin^2(kx)} dx = -\frac{1}{k \tan kx} \tag{72}$$

$$\int \sin^2(kx) dx = \frac{1}{2k}(kx - \sin(kx)\cos(kx)) \tag{73}$$

$$\int \ln kx dx = x \ln kx - x \tag{74}$$

$$\int \frac{dx}{\sqrt{x^2 - k^2}} = \cosh^{-1}\left(\frac{x}{k}\right) \tag{75}$$

$$\int \frac{dx}{\sqrt{x^2 + k^2}} = \sinh^{-1}\left(\frac{x}{k}\right) \tag{76}$$

$$\int \frac{dx}{\sqrt{k^2 - x^2}} = \sin^{-1}\left(\frac{x}{k}\right) \tag{77}$$

$$\int \frac{dx}{x^2 + k^2} = \frac{1}{k} \tan^{-1}\left(\frac{x}{k}\right) \tag{78}$$

Proof of Equation (71)

$$t \triangleq \frac{1}{\tan(kx)} \quad ; \quad \therefore \tan(kx) = \frac{1}{t} \quad ; \quad \therefore \frac{d\{\tan(kx)\}}{dx} = \frac{d\left\{\frac{1}{t}\right\}}{dx} \quad ; \quad \therefore \frac{k}{\cos^2(kx)} = \frac{d\{t\}}{dx}\frac{d\left\{\frac{1}{t}\right\}}{dt}$$

$$\therefore \frac{k}{\cos^2(kx)} = -\frac{1}{t^2}\frac{d\{t\}}{dx} \quad ; \quad \therefore \frac{k}{\cos^2(kx)}dx = -\frac{1}{t^2}dt \quad ; \quad \therefore dx = -\frac{\cos^2(kx)}{kt^2}dt$$

$$\int \frac{1}{\cos^2(kx)}dx = \int \frac{1}{\cos^2(kx)}\left(-\frac{\cos^2(kx)}{kt^2}dt\right) = -\frac{1}{k}\int\left(\frac{1}{t^2}dt\right) = -\frac{1}{k}(-t^{-1}) = \frac{1}{k}\cdot\frac{1}{t} = \frac{1}{k}\cdot\tan(kx) = \frac{\tan(kx)}{k}$$

Proof of Equation (72)

$$t \triangleq \frac{1}{\tan(kx)} \quad ; \quad \therefore \tan(kx) = \frac{1}{t} \quad ; \quad \therefore \frac{d\{\tan(kx)\}}{dx} = \frac{d\left\{\frac{1}{t}\right\}}{dx} \quad ; \quad \therefore \frac{k}{\cos^2(kx)} = \frac{d\{t\}}{dx}\frac{d\left\{\frac{1}{t}\right\}}{dt}$$

$$\therefore \frac{k}{\cos^2(kx)} = -\frac{1}{t^2}\frac{d\{t\}}{dx} \quad ; \quad \therefore \frac{k}{\cos^2(kx)}dx = -\frac{1}{t^2}dt \quad ; \quad \therefore dx = -\frac{\cos^2(kx)}{kt^2}dt \quad ; \quad \int \frac{1}{\cos^2(kx)}dx$$

$$= \int \frac{1}{\sin^2(kx)}\left(-\frac{\cos^2(kx)}{kt^2}dt\right) = -\int \frac{\cos^2(kx)}{\sin^2(kx)}\left(\frac{1}{kt^2}dt\right) = -\int \frac{1}{\tan^2(kx)}\left(\frac{1}{kt^2}dt\right)$$

$$= -\int t^2\left(\frac{1}{kt^2}dt\right) = -\int\left(\frac{1}{k}dt\right) = -\frac{t}{k} = -\frac{1}{k\tan(kx)}$$

## VII. Key points on sequences and series

### Key points

1) Sequences and Series

   a) Arithmetic progressions. Consider a sequence that starts at $r$ and we add $d$ each time. This forms the Arithemtic series as follows.

$$\begin{aligned} a_1 &= r \\ a_2 &= r + d \\ a_3 &= r + 2d \\ a_4 &= r + 3d \\ &\cdots \\ a_n &= r + (n-1)d \end{aligned}$$

   Here $d$ is the difference or common difference between successive terms. The sum of an arthimetic progression is as follows.

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + a_4 + a_5 + a_n \\ S_n &= r + (r+d) + (r+2d) + \cdots + r + (n-1)d \end{aligned}$$

$$S_n = rn + \frac{n(n-1)d}{2} \tag{79}$$

   b) Geometric progressions. Suppose we let the first term equal $a$ and times each successive term by $r$ then we get.

$$\begin{aligned} a_1 &= a \\ a_2 &= ar \\ a_3 &= ar^2 \\ a_4 &= ar^3 \\ a_5 &= ar^4 \\ &\cdots \\ a_n &= ar^{n-1} \end{aligned}$$

   To find the sum of this progression to $n$ terms, we sum all the terms up until $n$.

$$S_n = a + ar + ar^2 + ar^3 + ar^4, + \cdots + ar^{n-1}$$

   Since $r \cdot S_n$ is written as

$$rS_n = ar + ar^2 + ar^3 + ar^4, + \cdots + ar^{n-1} + ar^n$$

   Using these two equations, we calculate $S_n - rS_n$ as follows:

$$S_n - rS_n = a - ar^n$$

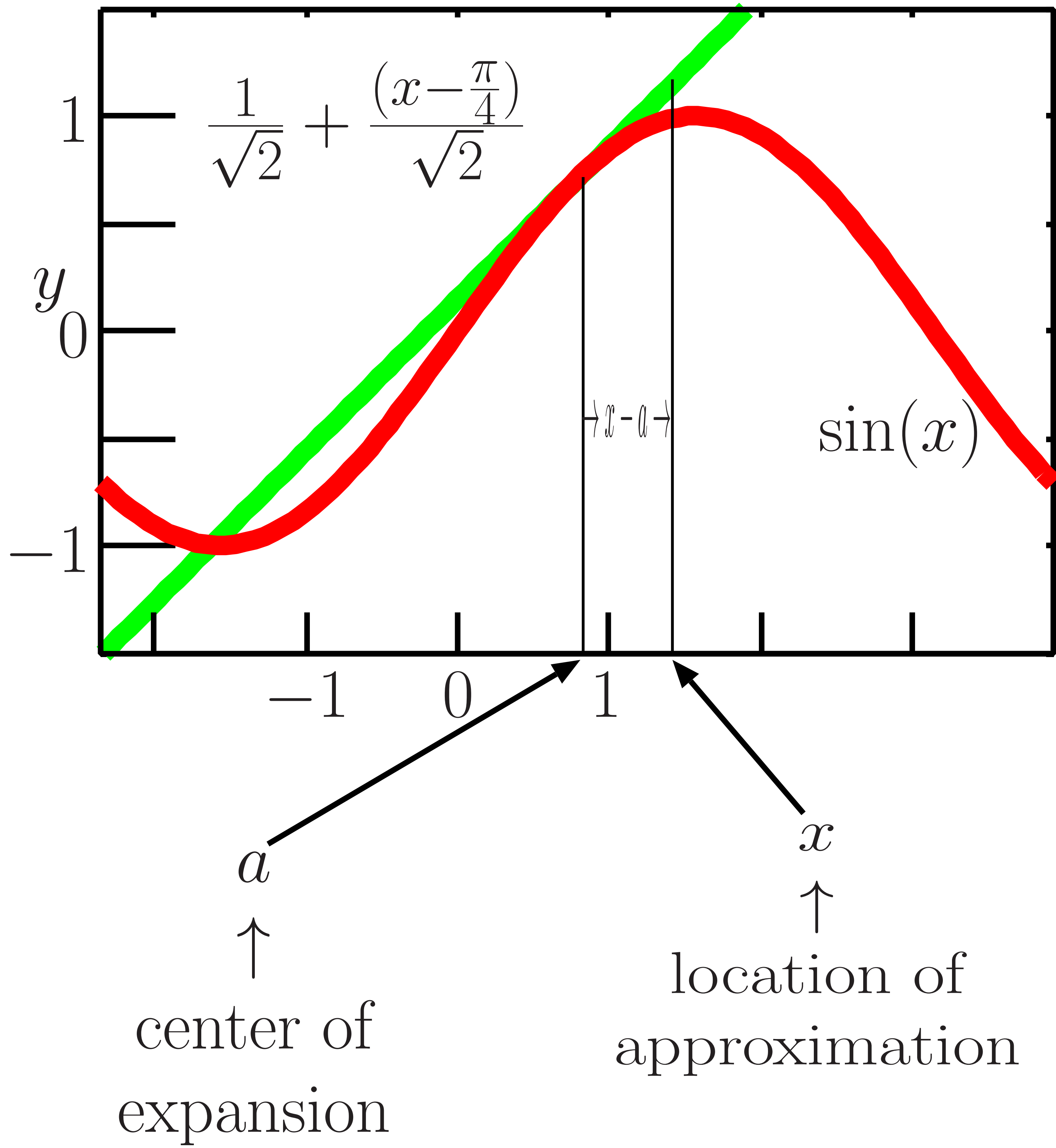
   This leads to :

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r} \tag{80}$$

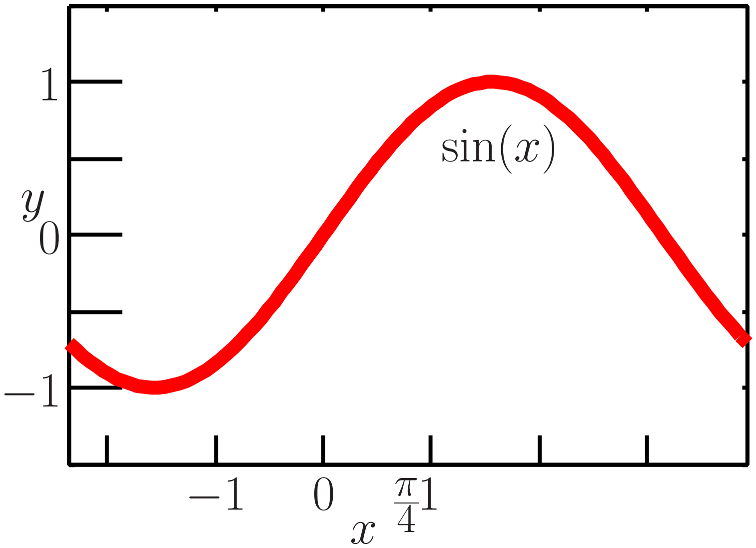
   If $-1 < r < 1$ therefore the sum to infinity of an geomteric series is given by the following

$$S_\infty = \frac{a}{1 - r} \tag{81}$$

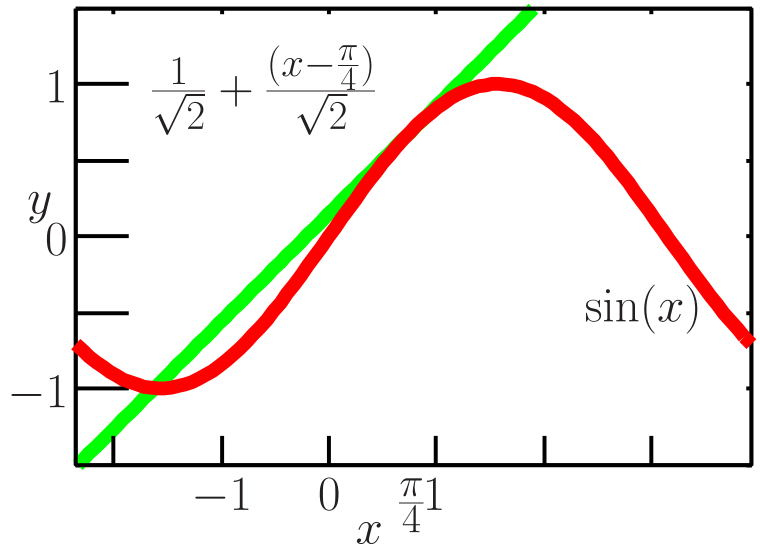
2) Taylor polynomial with one variable,*e.g.*, $x$. This is the example of one-dimensional Taylor series expansion as there is only one variable in the equation.



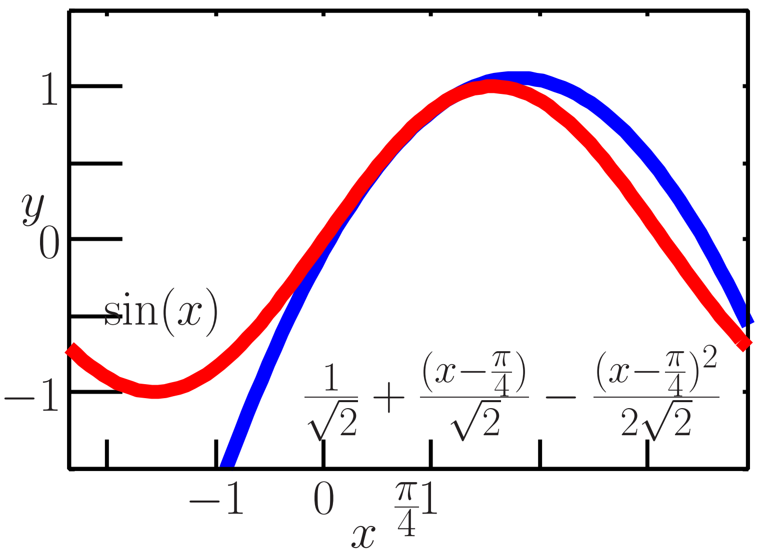

More information at https://www.scss.tcd.ie/Rozenn.Dahyot/CS1BA1/T2007_04_10_CS1BA1.pdf



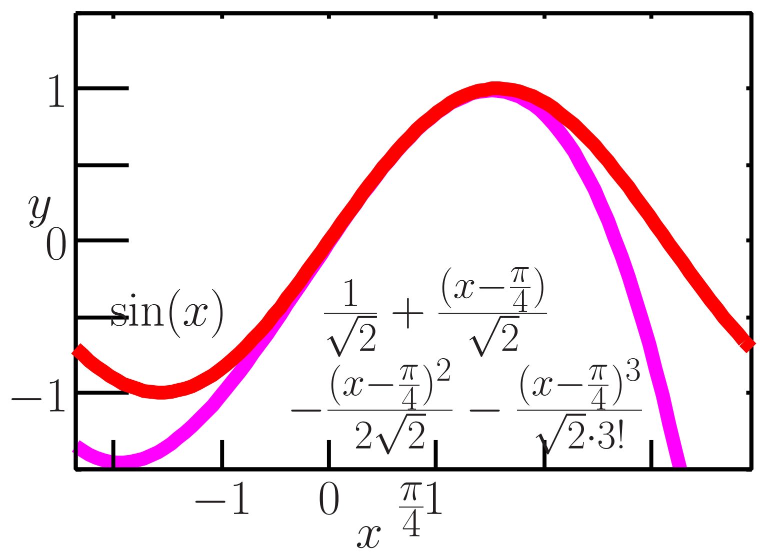
sin(x)



Linear approximation at $x = \frac{\pi}{4}$
degree one



Quadratic approximation at $x = \frac{\pi}{4}$
degree two



Cubic approximation at $x = \frac{\pi}{4}$
degree three

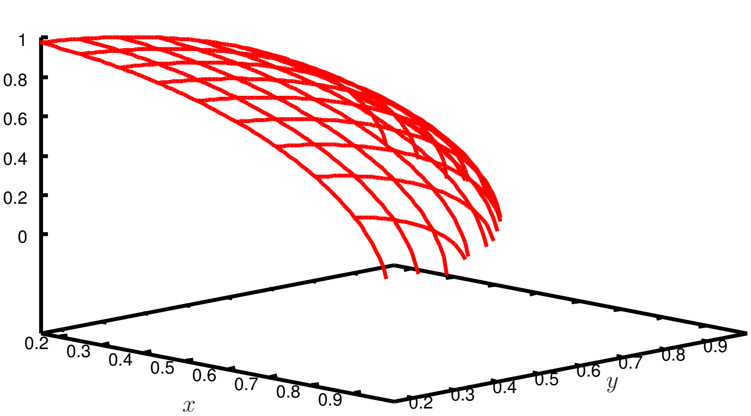
A Taylor series is a series expansion of a function about a point. The Taylor polynomial approximates/expresses the part of the function around $x = a$ using several polynomials.
A one-dimensional Taylor series is an expansion of a real function $f(x)$ about the point at $x = a$ up to degree $n$ ($|x - a| \ll 1$) which is given by

$$f(x) = f(a) + (x-a)\left.\frac{\partial f}{\partial x}\right|_{x=a} + \frac{(x-a)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=a} + \frac{(x-a)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=a} + \cdots + \frac{(x-a)^n}{n!}\left.\frac{\partial^n f}{\partial x^n}\right|_{x=a} \quad (82)$$

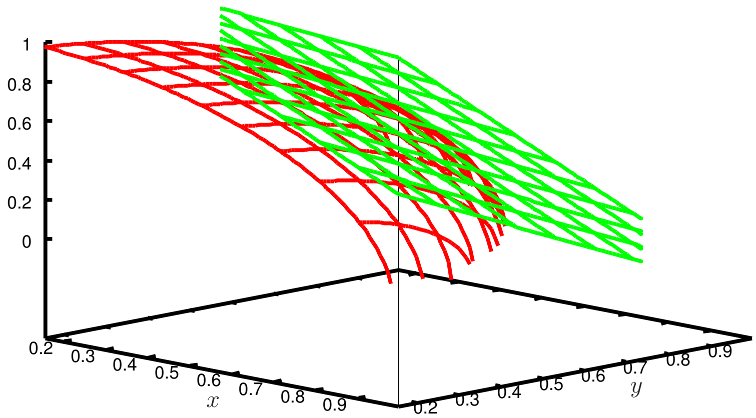
If $a = 0$, the expansion is known as a Maclaurin series.
In the end, in order to obtain the taylor series

a) Obtain $\frac{\partial f}{\partial x}$ , $\frac{\partial^2 f}{\partial x^2}$, $\ldots$ $\frac{\partial^n f}{\partial x^n}$
b) Substitute $x = a$ into $f(x)$, $\frac{\partial f}{\partial x}$ , $\frac{\partial^2 f}{\partial x^2}$, $\ldots$ $\frac{\partial^n f}{\partial x^n}$
c) Put all of them into Equation (82).

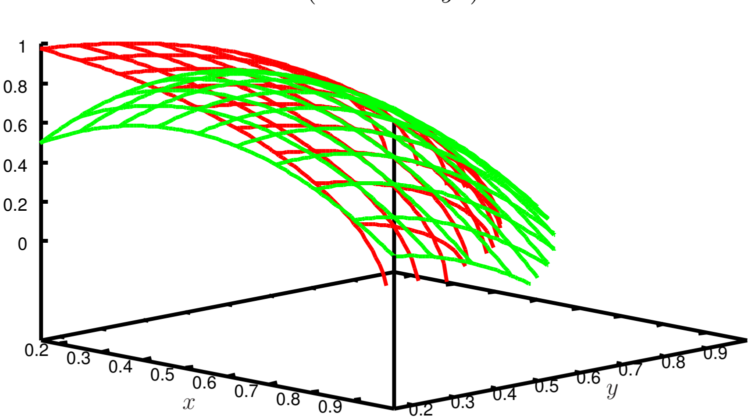
3) Taylor polynomial with two variable,*e.g.*, $x$ and $y$. This is the example of two-dimensional Taylor series expansion as there are two variables in the equation.



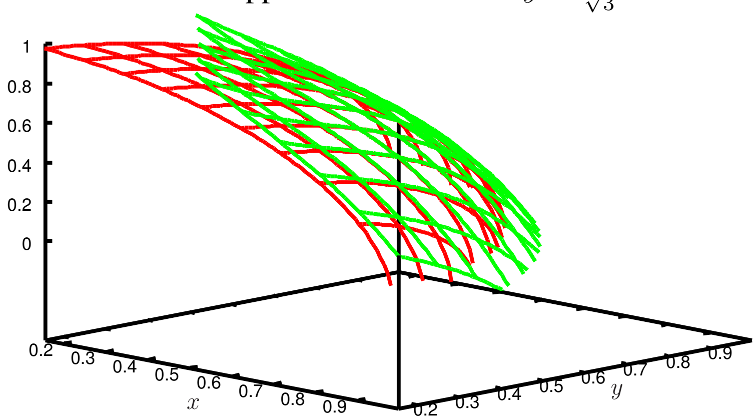

$z = (1 - x^2 - y^2)^{\frac{1}{2}}$




Linear approximation at $x = y = \frac{1}{\sqrt{3}}$




Quadratic approximation




Cubic approximation

The taylor series for two variables is very similar to that of one variable. The same method is used to find the series.
The Taylor polynomial approximates/expresses the part of the function <u>around</u> $(x, y) = (a, b)$ using several polynomials.
The Taylor series expansion about the point at $(x, y) = (a, b)$, where $a$ and $b$ are known constants, up to and including terms of degree three in $x - a$ and $y - b$ ($|x - a| \ll 1$ and $|y - b| \ll 1$) is expressed as

$$
\begin{aligned}
f(x,y) = \; & f(a,b) + (x-a)\frac{d\{f(x,y)\}}{dx}\bigg|_{\substack{x=a\\y=b}} + (y-b)\frac{d\{f(x,y)\}}{dy}\bigg|_{\substack{x=a\\y=b}} \\
& + \frac{1}{2!}\Bigg[(x-a)^2\frac{d^2f(x,y)}{dx^2}\bigg|_{\substack{x=a\\y=b}} + 2(x-a)(y-b)\frac{\partial^2 f(x,y)}{\partial y\partial x}\bigg|_{\substack{x=a\\y=b}} \\
& \qquad + (y-b)^2\frac{\partial^2 f(x,y)}{\partial y^2}\bigg|_{\substack{x=a\\y=b}}\Bigg] \\
& + \frac{1}{3!}\Bigg[(x-a)^3\frac{\partial^3 f(x,y)}{\partial x^3}\bigg|_{\substack{x=a\\y=b}} + 3(x-a)^2(y-b)\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\bigg|_{\substack{x=a\\y=b}} \\
& \qquad + 3(x-a)(y-b)^2\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\bigg|_{\substack{x=a\\y=b}} + (y-b)^3\frac{\partial^3 f(x,y)}{\partial y^3}\bigg|_{\substack{x=a\\y=b}}\Bigg]
\end{aligned}
\tag{83}
$$

$$= f(a,b) + (x-a)\frac{d\{f(x,y)\}}{dx}\bigg|_{\substack{x=a\\y=b}} + (y-b)\frac{d\{f(x,y)\}}{dy}\bigg|_{\substack{x=a\\y=b}}$$

$$+\frac{1}{2!}\left[ {}_2C_0(x-a)^2\frac{d^2f(x,y)}{dx^2}\bigg|_{\substack{x=a\\y=b}} +_2 C_1(x-a)(y-b)\frac{\partial^2 f(x,y)}{\partial y\partial x}\bigg|_{\substack{x=a\\y=b}} +_2C_2((y-b)^2\frac{\partial^2 f(x,y)}{\partial y^2}\bigg|_{\substack{x=a\\y=b}} \right]$$

$$+\frac{1}{3!}\left[ {}_3C_0(x-a)^3\frac{\partial^3 f(x,y)}{\partial x^3}\bigg|_{\substack{x=a\\y=b}} +_3 C_1(x-a)^2(y-b)\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\bigg|_{\substack{x=a\\y=b}} +_3C_2(x-a)(y-b)^2\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\bigg|_{\substack{x=a\\y=b}} +_3 C_3(y-b)^3\frac{\partial^3 f(x,y)}{\partial y^3}\bigg|_{\substack{x=a\\y=b}} \right]$$

$$+\frac{1}{4!}\left[ \sum_{m=0}^{4} {}_4C_m(x-a)^{4-m}(y-b)^m\frac{\partial^4 f(x,y)}{\partial y^m\partial x^{4-m}}\bigg|_{\substack{x=a\\y=b}} \right]$$

$$+\frac{1}{5!}\left[ \sum_{m=0}^{5} {}_5C_m(x-a)^{5-m}(y-b)^m\frac{\partial^5 f(x,y)}{\partial y^m\partial x^{5-m}}\bigg|_{\substack{x=a\\y=b}} \right]$$

$$+\frac{1}{6!}\left[ \sum_{m=0}^{6} {}_6C_m(x-a)^{6-m}(y-b)^m\frac{\partial^6 f(x,y)}{\partial y^m\partial x^{6-m}}\bigg|_{\substack{x=a\\y=b}} \right]\cdots$$

In the end, in order to obtain the taylor series

a) Obtain $\frac{d\{f(x,y)\}}{dx}$,$\frac{d\{f(x,y)\}}{dy}$ and if you need the second degree, then obtain $\frac{d^2f(x,y)}{dx^2}$,$\frac{\partial^2 f(x,y)}{\partial y\partial x}$,$\frac{\partial^2 f(x,y)}{\partial y^2}$ as well, and if you need the third degree, then obtain $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y\partial x^2}$,$\frac{\partial^3 f(x,y)}{\partial y^2\partial x}$,$\frac{\partial^3 f(x,y)}{\partial y^3}$ as well.

b) Substitute $x=a$ and $y=b$ into $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$,$\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y\partial x^2}$,$\frac{\partial^3 f(x,y)}{\partial y^2\partial x}$,$\frac{\partial^3 f(x,y)}{\partial y^3}$

c) Put all of them into Equation (83).

## VIII. Key points on ordinary differential equations

### Key points

1) The solution of the equation $\frac{d\left\{y\right\}}{dx} = f(x)g(y)$ may be found from separating the variables and integrating

$$\int \frac{1}{g(y)} dy = \int f(x)dx \tag{84}$$

Procedure:
   a) Allocate $f(x)$ and $g(x)$
   b) Calculate

$$\int \frac{1}{g(y)} dy = \int f(x)dx$$

2) When the differential equation can be written as $\frac{d\left\{y\right\}}{dx} + P(x)y = Q(x)$ then the answer is

$$y = \frac{1}{\Phi(x)}\left[\int \Phi(x)Q(x)dx + c\right] \tag{85}$$

where

$$\Phi(x) = \mathfrak{e}^{\int P(x)dx} \tag{86}$$

Procedure:
   a) Allocate $P(x)$ and $Q(x)$
   b) Calculate $A = \int P(x)dx$
   c) Obtain $\Phi(x) = \mathfrak{e}^{A}$
   d) Calculate $B = \int \Phi(x)Q(x)dx$
   e) Obtain the general solution $y = \frac{1}{\Phi(x)}\left[B + c\right]$
   f) Apply the condition to $y = \frac{1}{\Phi(x)}\left[B + c\right]$ in order to find out $c$ and thus the particular solution

Proof:

When we multiply $\frac{d\left\{y\right\}}{dx} + P(x)y = Q(x)$ with $\Phi(x)$, we get:
$\Phi(x)\frac{d\left\{y\right\}}{dx} + \Phi(x)P(x)y = \Phi(x)Q(x)$. Since,

$$\begin{aligned}\frac{d\left\{\Phi(x)\right\}}{dx} &= \frac{d\left\{\mathfrak{e}^{\int P(x)dx}\right\}}{dx}\\ &= \mathfrak{e}^{\int P(x)dx}\frac{d\left\{\int P(x)dx\right\}}{dx}\\ &= \mathfrak{e}^{\int P(x)dx}P(x)\\ &= \Phi(x)P(x),\end{aligned}$$

$$\begin{aligned}\Phi(x)Q(x) &= \Phi(x)\frac{d\left\{y\right\}}{dx} + \Phi(x)P(x)y\\ &= \Phi(x)\frac{d\left\{y\right\}}{dx} + \frac{d\left\{\Phi(x)\right\}}{dx}y\\ &= \frac{d\left\{y\Phi(x)\right\}}{dx}\end{aligned}$$

because $\frac{d\{y\}}{dx} + P(x)y = Q(x)$ and $\frac{d\{\Phi(x)\}}{dx} = \Phi(x)P(x)$.
When we integrate $\frac{d\{y\Phi(x)\}}{dx} = \Phi(x)Q(x)$ with respect to $x$,

$$\int \frac{d\{y\Phi(x)\}}{dx}dx = \int \Phi(x)Q(x)dx$$
$$\therefore y\Phi(x) = \int \Phi(x)Q(x)dx + c$$
$$\therefore y = \frac{1}{\Phi(x)}\left[\int \Phi(x)Q(x)dx + c\right]$$

3) When $f$ can be written as a function of $y/x \triangleq z$, the solution of the equation $\frac{d\{y\}}{dx} = f(y/x)$ may be found as

$$\int \frac{dz}{f(z)-z} = \int \frac{1}{x}dx = \ln x + c \tag{87}$$

Procedure:
 a) Find $f(\frac{y}{x})$
 b) Calculate
$$\int \frac{dz}{f(z)-z} \triangleq g(z)$$
 c) Set $\ln(x) + c = g(z)$
 d) Replace $z$ with $\frac{y}{x}$ so that $\ln(x) + c = g(\frac{y}{x})$ is the answer

Proof: $y/x \triangleq z$ can be written as $y = zx$. Thus $\frac{d\{y\}}{dx} = \frac{d\{z\}}{dx}x + z\frac{d\{x\}}{dx} = x\frac{d\{z\}}{dx} + z$. Thus $\frac{d\{y\}}{dx} = f(y/x) = f(z)$ can be written as

$$x\frac{d\{z\}}{dx} + z = f(z)$$
$$\therefore x\frac{d\{z\}}{dx} = f(z) - z$$
$$\therefore \frac{1}{x}dx = \frac{1}{f(z)-z}dz$$
$$\therefore \int \frac{1}{f(z)-z}dz = \int \frac{1}{x}dx = \ln x + c$$

4) When the differential equation can be written as $f(x,y)dx + g(x,y)dy = 0$ and if

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{g(x,y)\}}{dx}, \tag{88}$$

then there is a function $U(x,y)$ which satisfies

$$\begin{aligned} dU(x,y) &= \frac{d\{U(x,y)\}}{dx}dx + \frac{d\{U(x,y)\}}{dy}dy \\ &\equiv f(x,y)dx + g(x,y)dy = 0 \end{aligned} \tag{89}$$

$dU(x,y) = 0$ gives

$$U(x,y) = c \tag{90}$$

which is the answer. In order to find $U(x,y)$, we first perform

$$U(x,y)=\int f(x,y)dx+h(y) \tag{91}$$

then we find $h(y)$ from

$$\begin{aligned}\frac{d\left\{U(x,y)\right\}}{dy}&=\frac{d\left\{\int f(x,y)dx+h(y)\right\}}{dy}\\&=g(x,y)\end{aligned} \tag{92}$$

The alternative approach to obtain $U(x,y)$ is

$$U(x,y)=\int_{x_0}^{x} f(x,y)dx+\int_{y_0}^{y} g(x_0,y)dy \tag{93}$$

where $x_0$ and $y_0$ are arbitrary constants. Please be aware of $g(\underline{x_0},y)$ which is not $g(\underline{x},y)$
$x_0$ and $y_0$ can be added into $c$ in Equation (90) as they are arbitrary constants.
Procedure:

a) Allocate $f(x,y)$ and $g(x,y)$
b) Confirm

$$\frac{d\left\{f(x,y)\right\}}{dy}=\frac{d\left\{g(x,y)\right\}}{dx}$$

c) Apply $\int_{x_0}^{x} f(x,y)dx+\int_{y_0}^{y} g(x_0,y)dy=c$
d) Merge all the terms which have $x_0$ and $y_0$

Proof: Let's assume there is a function

$$U(x,y)=\int_{x_0}^{x} f(x,y)dx+\int_{y_0}^{y} g(x_0,y)dy=c \qquad ①$$

When you calculate $\int_{x_0}^{x} f(x,y)dx$, you assume $y$ is a constant and let it be $y_0$. Thus we can write

$$\int f(x,y)dx\equiv\int f(x,y_0)dx\triangleq F(x,y_0) \qquad ②$$

In the similar way we can write

$$\int g(x_0,y)dy\triangleq G(x_0,y) \qquad ③$$

By putting ② and ③ into ①, we get

$$\begin{aligned}&U(x,y)\\=F(x,y_0)-F(x_0,y_0)+G(x_0,y)-G(x_0,y_0)=c& \qquad ④\end{aligned}$$

Since $U(x,y)=c$ from ①, we can write

$$\partial U(x,y)=\frac{d\left\{U(x,y)\right\}}{dx}dx+\frac{d\left\{U(x,y)\right\}}{dy}dy=0 \qquad ⑤$$

Using ④, we obtain $\frac{d\left\{U(x,y)\right\}}{dx}$ and $\frac{d\left\{U(x,y)\right\}}{dy}$ as follows:

$$\frac{d\left\{U(x,y)\right\}}{dx}=f(x,y_0) \qquad ⑥$$
$$\frac{d\left\{U(x,y)\right\}}{dy}=g(x_0,y) \qquad ⑦$$

By putting ⑥ and ⑦ into ⑤, we get

$$\frac{d\{U(x,y)\}}{dx}dx + \frac{d\{U(x,y)\}}{dy}dy$$
$$= f(x,y_0)dx + g(x_0,y)dy = 0 \qquad ⑥$$

Now since

$$\frac{d\{f(x,y_0)\}}{dy} = \frac{d\{g(x_0,y)\}}{dx}(=0) \qquad ⑦$$

we can conclude that ① satisfies ⑥ and ⑦. In other words, when ⑥ and ⑦ are given, we can say ① is valid.

5) The solution of Jean Bernoulli equation

$$\frac{d\{y\}}{dx} + p(x)y = q(x)y^{\alpha} \qquad (\alpha \neq 0,1) \tag{94}$$

is obtained by solving

$$\frac{d\{Y\}}{dx} + (1-\alpha)p(x)Y = (1-\alpha)q(x) \tag{95}$$

where

$$Y = y^{1-\alpha}. \tag{96}$$

In other words, $Y$ ( $= y^{1-\alpha}$, be aware that this is not $y$ but $Y$!!) is obtained from $Y = \frac{1}{\Phi(x)}\left[\int \Phi(x)Q(x)dx + c\right]$ where $\Phi(x) = \mathfrak{e}^{\int P(x)dx}$ and $P(x) = (1-\alpha)p(x)$ and $Q(x) = (1-\alpha)q(x)$. The steps to the solution are:

a) allocate $p(x)$ and $q(x)$
b) identify the value of $\alpha$
c) allocate $P(x) = (1-\alpha)p(x)$ and $Q(x) = (1-\alpha)q(x)$
d) calculate $\int P(x)dx$
e) calculate $\Phi(x) = \mathfrak{e}^{\int P(x)dx}$
f) calculate $y^{1-\alpha} = \frac{1}{\Phi(x)}[\int \Phi(x)Q(x)dx + c]$

Proof:

$$\frac{d\{y\}}{dx} + p(x)y = q(x)y^{\alpha}$$
$$\therefore y^{-\alpha}\frac{d\{y\}}{dx} + p(x)y \cdot y^{-\alpha} = q(x)$$
$$\therefore y^{-\alpha}\frac{d\{y\}}{dx} + p(x)y^{1-\alpha} = q(x)$$

Since

$$\frac{d\{y^{1-\alpha}\}}{dx} = \frac{d\{y^{1-\alpha}\}}{dy}\frac{d\{y\}}{dx}$$
$$= (1-\alpha)y^{1-\alpha-1}\frac{d\{y\}}{dx}$$
$$= (1-\alpha)y^{-\alpha}\frac{d\{y\}}{dx}$$
$$\therefore \frac{1}{1-\alpha}\frac{d\{y^{1-\alpha}\}}{dx} = y^{-\alpha}\frac{d\{y\}}{dx}$$

we can manipulate the equation as follows:

$$
\begin{aligned}
y^{-\alpha}\frac{d\{y\}}{dx} + p(x)y^{1-\alpha} &= q(x) \\
\therefore \frac{1}{1-\alpha}\frac{d\{y^{1-\alpha}\}}{dx} + p(x)y^{1-\alpha} &= q(x) \\
\therefore \frac{d\{y^{1-\alpha}\}}{dx} + (1-\alpha)p(x)y^{1-\alpha} & \\
&= (1-\alpha)q(x) \\
\therefore \frac{d\{Y\}}{dx} + (1-\alpha)p(x)Y = (1-\alpha)q(x) &
\end{aligned}
$$

The answer can be obtained from Equation (85) where

$$P(x) = (1-\alpha)p(x) \tag{97}$$

$$Q(x) = (1-\alpha)q(x) \tag{98}$$

6) Clairaut type

$$y = x\frac{d\{y\}}{dx} + f\left(\frac{d\{y\}}{dx}\right) \tag{99}$$

can be solved as follows:

a) Allocate $f(\frac{d\{y\}}{dx})$
b) Write down the general solution of

$$y = ax + f(a)$$

which is the answer!. State $a$ is a constant value.
c) Differentiate

$$y = ax + f(a)$$

with respect to $a$
d) Express $a$ as a function of $x$, let's say $a = g(x)$
e) Insert $a = g(x)$ into the general solution to get a particular solution of

$$y = x \cdot g(x) + f(g(x))$$

Proof:

$$
\begin{aligned}
\frac{d\{y\}}{dx} &= \frac{d\left\{x\frac{d\{y\}}{dx} + f\left(\frac{d\{y\}}{dx}\right)\right\}}{dx} \\
&= \frac{d\{x\}}{dx}\frac{d\{y\}}{dx} + x\frac{d^2y}{dx^2} + \frac{d\left\{f\left(\frac{d\{y\}}{dx}\right)\right\}}{dx} \\
&= \frac{d\{y\}}{dx} + x\frac{d^2y}{dx^2} + \frac{\partial\left\{f\left(\frac{d\{y\}}{dx}\right)\right\}}{\partial\left\{\frac{d\{y\}}{dx}\right\}}\frac{d\left\{\frac{d\{y\}}{dx}\right\}}{dx} \\
&= \frac{d\{y\}}{dx} + x\frac{d^2y}{dx^2} + \frac{\partial\left\{f\left(\frac{d\{y\}}{dx}\right)\right\}}{\partial\left\{\frac{d\{y\}}{dx}\right\}}\frac{d^2y}{dx^2}
\end{aligned}
$$

$$\therefore 0 = x\frac{d^2y}{dx^2} + \frac{\partial\left\{f\left(\dfrac{d\{y\}}{dx}\right)\right\}}{\partial\left\{\dfrac{d\{y\}}{dx}\right\}}\frac{d^2y}{dx^2}$$

$$\therefore 0 = \left(x + \frac{\partial\left\{f\left(\dfrac{d\{y\}}{dx}\right)\right\}}{\partial\left\{\dfrac{d\{y\}}{dx}\right\}}\right)\frac{d^2y}{dx^2}$$

Thus we obtain

$$\frac{d^2y}{dx^2} = 0$$

or

$$x + \frac{\partial\left\{f\left(\dfrac{d\{y\}}{dx}\right)\right\}}{\partial\left\{\dfrac{d\{y\}}{dx}\right\}} = 0$$

From $\frac{d^2y}{dx^2} = 0$ we obtain

$$\begin{aligned}
\frac{d^2y}{dx^2} &= 0\\
\therefore \frac{d\left\{\dfrac{d\{y\}}{dx}\right\}}{dx} &= 0\\
\therefore \partial(\frac{d\{y\}}{dx}) &= 0\cdot\partial x\\
\therefore \int d(\frac{d\{y\}}{dx}) = \int &0\cdot dx\\
\therefore \frac{d\{y\}}{dx} &= a\\
\therefore dy &= a\cdot dx\\
\therefore \int dy = \int &a\cdot dx\\
\therefore y &= ax+b\\
\therefore \frac{d\{y\}}{dx} = \frac{d\{ax+b\}}{dx} &= a
\end{aligned}$$

where $a$ and $b$ are the arbitrary constants. Substituting $y = ax + b$ and $\frac{d\{y\}}{dx} = a$ into the original equation, we get

$$\begin{aligned}
y &= x\frac{d\{y\}}{dx} + f\left(\frac{d\{y\}}{dx}\right)\\
\therefore ax+b &= x\cdot a + f(a)\\
\therefore b &= f(a)
\end{aligned}$$

Therefore

$$y = ax + f(a) \tag{100}$$

is a general solution with an arbitrary constant of $a$. Furthermore, when we take the differentiation of the equation with respect to $a$, we get

$$\frac{\partial\{y\}}{\partial a} = \frac{\partial\{ax + f(a)\}}{\partial a}$$
$$\therefore 0 = \frac{\partial\{ax\}}{\partial a} + \frac{\partial\{f(a)\}}{\partial a}$$
$$\therefore 0 = x + \frac{\partial\{f(a)\}}{\partial a}$$

We solve the equation for $a$. Let's assume $a = A(x)$ satisfies $x + \dfrac{\partial\{f(a)\}}{\partial a} = 0$. The resultant expression of $a$ using $x$, which is $A(x)$ is put into $y = ax + f(a)$ to obtain a particular solution of Equation (101).

$$y = A(x) \cdot x + f(A(x)) \tag{101}$$

7) In order to solve second order differential equations

$$\frac{d^2y}{dx^2} + v\frac{d\{y\}}{dx} + wy = r(x), \tag{102}$$

where $v$, $w$ are the constant values,

a) Production of an auxiliary equation by forcing $r(x)$ to 0
By substituting

$$\frac{d^2y}{dx^2} = \lambda^2, \frac{d\{y\}}{dx} = \lambda, y = \lambda^0 = 1 \tag{103}$$

into the original original equation, forcing $r(x)$ to zero, we solve the auxiliary equation of

$$\lambda^2 + v\lambda + w = 0 \tag{104}$$

and we obtain the answers $\lambda = \alpha$ and $\beta$.

b) Set complementary function as follows:

i) $\alpha$ and $\beta$ are real and $\alpha \neq \beta$
Set the complementary function $Y_1(x)$ as

$$Y_1(x) = a\mathfrak{e}^{\alpha x} + b\mathfrak{e}^{\beta x} \tag{105}$$

where $a$, $b$ are constant value which is found from the initial condition.

ii) $\alpha$ and $\beta$ are real and $\alpha = \beta$
Set the complementary function $Y_1(x)$ as

$$Y_1(x) = a\mathfrak{e}^{\alpha x} + bx\mathfrak{e}^{\alpha x} \tag{106}$$

iii) $\alpha$ and $\beta$ are complex numbers and $p \pm \jmath q$ (where $p, q$ are real)
Set the complementary function $Y_1(x)$ as

$$Y_1(x) = \mathfrak{e}^{px}(a\cos qx + b\sin qx) \tag{107}$$

c) Check the characteristics of $r(x)$ and set the particular integral

i) $r(x)$ is proportional to $\mathfrak{e}^{cx}$, where $c$ is a constant value

A) $\alpha \neq c$ and $\beta \neq c$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = g\mathfrak{e}^{cx} \tag{108}$$

where $g$ is a constant value which is found from Equation (102).

B) $\alpha = c$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = gx^k\mathfrak{e}^{cx} \tag{109}$$

where $k$ is 1 or 2 or 3 ...

ii) $r(x)$ is $n$th order polynomial

A) $\alpha \neq 0$ and $\beta \neq 0$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = \sum_{m=0}^{n} g_m x^m \tag{110}$$

where $g_m$ is a constant value which is found from Equation (102).

B) $\alpha = 0$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = x^k \left( \sum_{m=0}^{n} g_m x^m \right) \tag{111}$$

where $k$ is 1 or 2 or 3 ...

iii) $r(x)$ is in the form of $P(x)\mathfrak{e}^{cx}$ where $P(x)$ is the $n$th order polynomial.

A) $\alpha \neq c$ and $\beta \neq c$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = \mathfrak{e}^{cx} \left( \sum_{m=0}^{n} g_m x^m \right) \tag{112}$$

where $g_m$ is a constant value which is found from Equation (102).

B) $\alpha = c$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = \mathfrak{e}^{cx} x^k \left( \sum_{m=0}^{n} g_m x^m \right) \tag{113}$$

where $k$ is 1 or 2 or 3 ...

iv) $r(x)$ is the combination of $\cos \omega x$ and $\sin \omega x$

A) $\alpha \neq \pm \jmath\omega$ and $\beta \neq \pm \jmath\omega$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = g \cos \omega x + h \sin \omega x \tag{114}$$

where $g$ and $h$ are constant values which is found from Equation (102).

B) $\alpha = \pm \jmath\omega$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = x^k (g \cos \omega x + h \sin \omega x) \tag{115}$$

where $k$ is 1 or 2 or 3 ...

v) $r(x)$ is the combination of $\mathfrak{e}^{cx} \cos \omega x$ and $\mathfrak{e}^{cx} \sin \omega x$

A) $\alpha \neq c \pm \jmath\omega$ and $\beta \neq c \pm \jmath\omega$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = \mathfrak{e}^{cx} (g \cos \omega x + h \sin \omega x) \tag{116}$$

where $g$ and $h$ are constant values which is found from Equation (102).

B) $\alpha = c \pm \jmath\omega$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = x^k \mathfrak{e}^{cx} (g \cos \omega x + h \sin \omega x) \tag{117}$$

where $k$ is 1 or 2 or 3 ...

d) Find the constant values $g$ and $h$ by

$$\frac{d^2Y_2(x)}{dx^2} + v\frac{d\{Y_2(x)\}}{dx} + wY_2(x) = r(x) \tag{118}$$

e) Get the general solution of The general solution is $y = Y_1(x) + Y_2(x)$ leaving $a$ and $b$ unknown.

f) Find the constant values $a$ and $b$

Usually there are initial conditions for $y(0)$ and $\frac{d\{y\}}{dx}|_{x=0}$. Using these conditions, $a$ and $b$ are found.

g) The particular solution is $y = Y_1(x) + Y_2(x)$.

Summary Procedure of 2nd order ODE $\frac{d^2y}{dx^2} + v\frac{d\{y\}}{dx} + wy = r(x)$

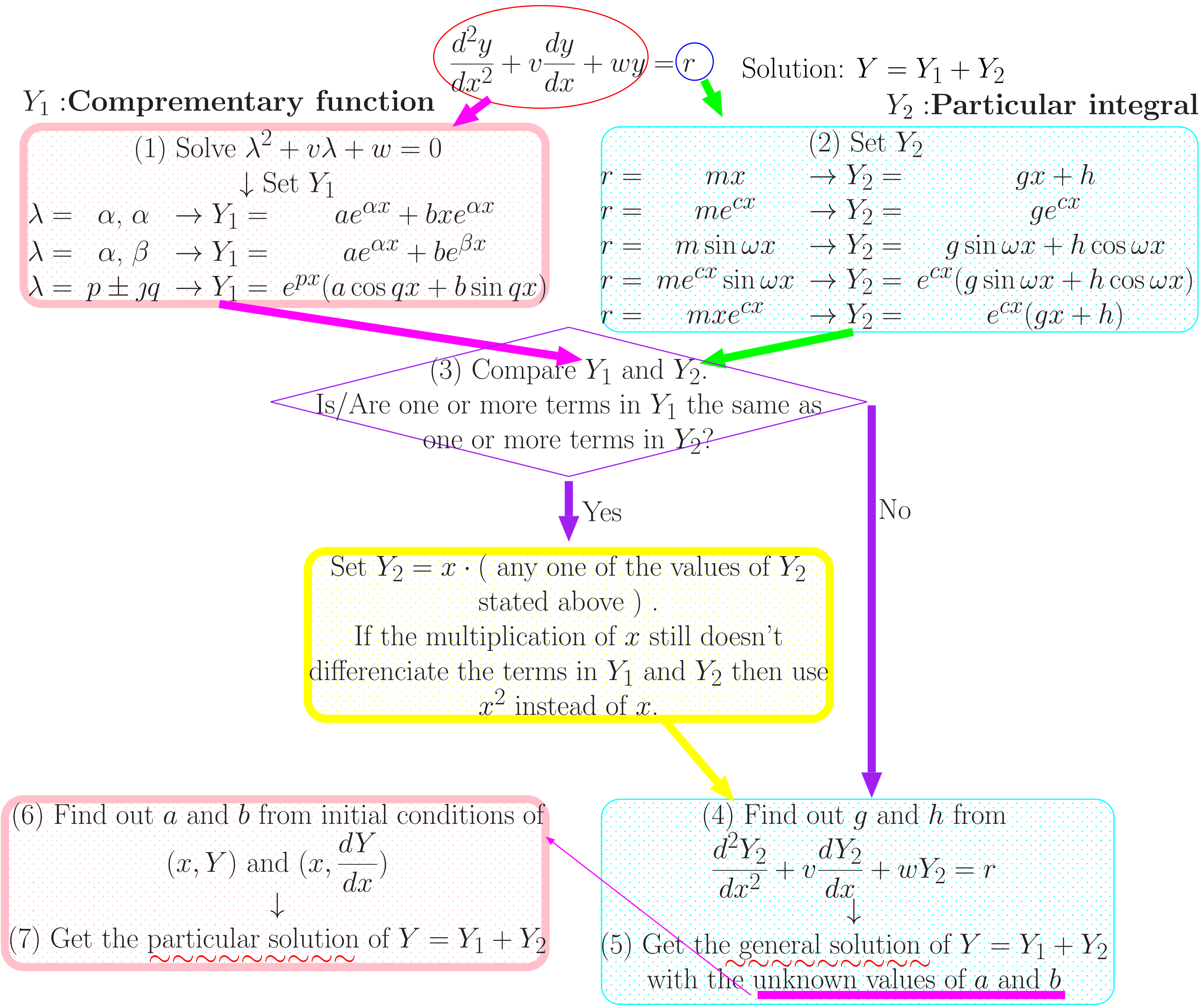
a) Produce and solve an auxiliary equation by setting $r(x) = 0$
b) Set the complementary function $Y_1(x)$ with the unknown variables $a$ and $b$
c) Set particular integral $Y_2(x)$ with the unknown variables $g$ and $h$
d) Find $g$ and $h$ from $\frac{d^2Y_2(x)}{dx^2} + v\frac{d\{Y_2(x)\}}{dx} + wY_2(x) = r(x)$
e) Get the general solution $y = Y_1(x) + Y_2(x)$ with unknown $a$ and $b$
f) Find $a$ and $b$ using the initial conditions
g) Get the particular solution $y = Y_1(x) + Y_2(x)$ with known $a$ and $b$

8) Lookup table for 2nd order ODE

| $r(x)$ | particular integral $Y_2(x)$ |
|---|---|
| $\mathfrak{e}^{cx}, \alpha \neq c, \beta \neq c$ | $g\mathfrak{e}^{cx}$ |
| $\mathfrak{e}^{cx}, \alpha = c$ | $gx^k\mathfrak{e}^{cx}$ |
| $\sum_{m=0}^{n} \rho_m x^m, \alpha \neq 0, \beta \neq 0$ | $\sum_{m=0}^{n} g_m x^m$ |
| $\sum_{m=0}^{n} \rho_m x^m, \alpha = 0$ | $x^k \left(\sum_{m=0}^{n} g_m x^m\right)$ |
| $\mathfrak{e}^{cx}\sum_{m=0}^{n} \rho_m x^m, \alpha \neq c, \beta \neq c$ | $\mathfrak{e}^{cx}\sum_{m=0}^{n} g_m x^m$ |
| $\mathfrak{e}^{cx}\sum_{m=0}^{n} \rho_m x^m, \alpha = c$ | $x^k\mathfrak{e}^{cx}\sum_{m=0}^{n} g_m x^m$ |
| $\rho_1 \cos\omega x + \rho_2 \sin\omega x, \alpha \neq \pm\jmath\omega, \beta \neq \pm\jmath\omega$ | $g\cos\omega x + h\sin\omega x$ |
| $\rho_1 \cos\omega x + \rho_2 \sin\omega x, \alpha = \pm\jmath\omega$ | $x^k(g\cos\omega x + h\sin\omega x)$ |
| $\mathfrak{e}^{cx}(\rho_1 \cos\omega x + \rho_2 \sin\omega x), \alpha \neq c \pm \jmath\omega, \beta \neq c \pm \jmath\omega$ | $\mathfrak{e}^{cx}(g\cos\omega x + h\sin\omega x)$ |
| $\mathfrak{e}^{cx}(\rho_1 \cos\omega x + \rho_2 \sin\omega x), \alpha = c \pm \jmath\omega$ | $x^k\mathfrak{e}^{cx}(g\cos\omega x + h\sin\omega x)$ |

TABLE I
PARTICULAR INTEGRAL FOR THE SECOND ORDER ODE

9) Summary for 2nd order ODE



10) Summary for 1st order ODE

| Equation type | Procedure to follow |
|---|---|
| $\frac{d\{y\}}{dx} = f(x)g(y)$ | a) Allocate $f(x)$ and $g(x)$<br>b) Calculate $$\int \frac{1}{g(y)} dy = \int f(x)dx$$ |
| $\frac{d\{y\}}{dx} = f(\frac{y}{x})$ | a) Find $f(\frac{y}{x})$<br>b) Calculate $$\int \frac{dz}{f(z) - z} \triangleq g(z)$$<br>c) Set $\ln(x) + c = g(z)$<br>d) Replace $z$ with $\frac{y}{x}$ so that $\ln(x) + c = g(\frac{y}{x})$ is the answer |
| $\frac{d\{y\}}{dx} = -\frac{f(x,y)}{g(x,y)}$ | a) Allocate $f(x, y)$ and $g(x, y)$<br>b) Confirm $$\frac{d\{f(x,y)\}}{dy} = \frac{d\{g(x,y)\}}{dx}$$<br>c) Apply $\int_{x_0}^{x} f(x,y)dx + \int_{y_0}^{y} g(x_0, y)dy = c$<br>d) Merge all the terms which have $x_0$ and $y_0$ |
| $\frac{d\{y\}}{dx} = -P(x)y + Q(x)$ | a) Allocate $P(x)$ and $Q(x)$<br>b) Calculate $\int P(x)dx$<br>c) Calculate $\Phi(x) = \mathfrak{e}^{\int P(x)dx}$<br>d) Calculate $y = \frac{1}{\Phi(x)} \left[ \int \Phi(x)Q(x)dx + c \right]$ |
| $\frac{d\{y\}}{dx} = -p(x)y + q(x)y^{\alpha}$ | a) allocate $p(x)$ and $q(x)$<br>b) identify the value of $\alpha$<br>c) allocate $P(x) = (1-\alpha)p(x)$ and $Q(x) = (1-\alpha)q(x)$<br>d) calculate $\int P(x)dx$<br>e) calculate $\Phi(x) = \mathfrak{e}^{\int P(x)dx}$<br>f) calculate $y^{1-\alpha} = \frac{1}{\Phi(x)}[\int \Phi(x)Q(x)dx + c]$ |

| | |
|---|---|
| $$\frac{d\{y\}}{dx} = \frac{y}{x} + \frac{1}{x} f\left(\frac{d\{y\}}{dx}\right)$$ | a) Allocate $f(\frac{d\{y\}}{dx})$<br>b) Write down the general solution of<br>$$y = ax + f(a)$$<br>which is the answer!. State $a$ is a constant value.<br>c) Differentiate<br>$$y = ax + f(a)$$<br>with respect to $a$<br>d) Express $a$ as a function of $x$, let's say $a = g(x)$<br>e) Insert $a = g(x)$ into the general solution to get a particular solution of<br>$$y = x \cdot g(x) + f(g(x))$$ |

1) **DAY1**

2) Find $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, when $f(x,y) = \frac{1}{\sin(x)} + \frac{1}{\cos(y)}$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{(\sin(x))^{-1} + (\cos(y))^{-1}\}}{dx} = \frac{d\{(\sin(x))^{-1}\}}{dx} = -(\sin(x))^{-2}\cos(x)$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{(\sin(x))^{-1} + (\cos(y))^{-1}\}}{dy} = \frac{d\{(\cos(y))^{-1}\}}{dy} = -(\cos(y))^{-2}(-\sin(y)) = (\cos(y))^{-2}\sin(y)$$

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\{-(\sin(x))^{-2}\cos(x)\}}{dx} = \frac{d\{-(\sin(x))^{-2}\}}{dx}\cos(x) - (\sin(x))^{-2}\frac{d\{\cos(x)\}}{dx}$$

$$= 2(\sin(x))^{-3}\cos^2(x) + (\sin(x))^{-2}\sin(x) = 2(\sin(x))^{-3}\cos^2(x) + (\sin(x))^{-1}$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dx} = \frac{d\{(\cos(y))^{-2}\sin(y)\}}{dx} = 0$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{(\cos(y))^{-2}\sin(y)\}}{dy} = \frac{d\{(\cos(y))^{-2}\}}{dy}\sin(y) + (\cos(y))^{-2}\frac{d\{\sin(y)\}}{dy}$$

$$= 2(\cos(y))^{-3}\sin^2(y) + (\cos(y))^{-2}\cos(y) = 2(\cos(y))^{-3}\sin^2(y) + (\cos(y))^{-1}$$

3) Find $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y \partial x^2}$, $\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$ when
$f(x,y) = \frac{1}{1-x} + \ln y$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\left\{\frac{1}{1-x} + \ln y\right\}}{dx} = \frac{d\{(1-x)^{-1}\}}{dx} + \frac{d\{\ln y\}}{dx}$$

$$= \frac{d\{u\}}{dx}\frac{\partial\{u^{-1}\}}{\partial u} + 0(\because u \triangleq 1-x) = \frac{d\{1-x\}}{dx}(-u^{-2}) = -(-u^{-2}) = u^{-2} = (1-x)^{-2}$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\left\{\frac{1}{1-x} + \ln y\right\}}{dy}$$

$$= \frac{d\left\{\frac{1}{1-x}\right\}}{dy} + \frac{d\{\ln y\}}{dy} = \frac{1}{y}$$

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\{(1-x)^{-2}\}}{dx}$$

$$= \frac{d\{u\}}{dx}\frac{\partial\{u^{-2}\}}{\partial u} = \frac{d\{1-x\}}{dx}(-2u^{-3}) = -(-2u^{-3}) = 2(1-x)^{-3}$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dx} = \frac{d\left\{\frac{1}{y}\right\}}{dx} = 0 \;;\quad \frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{y^{-1}\}}{dy} = -y^{-2}$$

$$\frac{\partial^3 f(x,y)}{\partial x^3} = \frac{d\left\{\frac{d^2 f(x,y)}{dx^2}\right\}}{dx} = \frac{d\{2(1-x)^{-3}\}}{dx}$$

$$= \frac{d\{u\}}{dx}\frac{\partial\{2u^{-3}\}}{\partial u} = \frac{d\{1-x\}}{dx}(-6u^{-4}) = -(-6u^{-4}) = 6(1-x)^{-4}$$

$$\frac{\partial^3 f(x,y)}{\partial y \partial x^2} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y \partial x}\right\}}{dx} = \frac{d\{0\}}{dx} = 0 \ ; \ \frac{\partial^3 f(x,y)}{\partial y^2 \partial x} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dx} = \frac{d\{-y^{-2}\}}{dx} = 0$$

$$\frac{\partial^3 f(x,y)}{\partial y^3} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dy} = \frac{d\{-y^{-2}\}}{dy} = 2y^{-3}$$

4) Find $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$,$\frac{\partial^3 f(x,y)}{\partial y \partial x^2}$,$\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$ when
$f(x,y) = \log_2 x + \cos y$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{\log_2 x + \cos y\}}{dx} = \frac{d\{\log_2 x\}}{dx} + \frac{d\{\cos y\}}{dx} = \frac{d\left\{\frac{\ln x}{\ln 2}\right\}}{dx} + 0 = \frac{1}{\ln 2}\frac{d\{\ln x\}}{dx} = \frac{1}{x \ln 2}$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{\log_2 x + \cos y\}}{dy} = \frac{d\{\log_2 x\}}{dy} + \frac{d\{\cos y\}}{dy} = -\sin y$$

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\left\{\frac{1}{x \ln 2}\right\}}{dx} = \frac{1}{\ln 2}\frac{d\{x^{-1}\}}{dx} = -\frac{1}{\ln 2}x^{-2}$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dx} = \frac{d\{-\sin y\}}{dx} = 0 \ ; \ \frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{-\sin y\}}{dy} = -\cos y$$

$$\frac{\partial^3 f(x,y)}{\partial x^3} = \frac{d\left\{\frac{d^2 f(x,y)}{dx^2}\right\}}{dx} = \frac{d\left\{-\frac{1}{\ln 2}x^{-2}\right\}}{dx} = -\frac{1}{\ln 2}\frac{d\{x^{-2}\}}{dx} = \frac{2}{\ln 2}x^{-3}$$

$$\frac{\partial^3 f(x,y)}{\partial y \partial x^2} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y \partial x}\right\}}{dx} = \frac{d\{0\}}{dx} = 0 \ ; \ \frac{\partial^3 f(x,y)}{\partial y^2 \partial x} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dx} = \frac{d\{-\cos y\}}{dx} = 0$$

$$\frac{\partial^3 f(x,y)}{\partial y^3} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dy} = \frac{d\{-\cos y\}}{dy} = \sin y$$

5) Find $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$,$\frac{\partial^3 f(x,y)}{\partial y \partial x^2}$,$\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$ when
$f(x,y) = \cos(x^2 y)$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{\cos(x^2 y)\}}{dx} = \frac{d\{A\}}{dx}\frac{d\{\cos(A)\}}{dA}(\because A \triangleq x^2 y) = \frac{d\{x^2 y\}}{dx}(-\sin(A)) = -2xy\sin(x^2 y)$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{\cos(x^2 y)\}}{dy} = \frac{d\{A\}}{dy}\frac{d\{\cos(A)\}}{dA}(\because A \triangleq x^2 y) = \frac{d\{x^2 y\}}{dy}(-\sin(A)) = -x^2\sin(x^2 y)$$

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\{-2xy\sin(x^2 y)\}}{dx}$$

$$= \frac{d\{-2xy\}}{dx}\sin(x^2 y) - 2xy\frac{d\{\sin(x^2 y)\}}{dx} = -2y\sin(x^2 y) - 2xy\frac{d\{A\}}{dx}\frac{d\{\sin(A)\}}{dA}$$

$$= -2y\sin(x^2y) - 2xy\frac{d\{x^2y\}}{dx}\cos(A) = -2y\sin(x^2y) - 2xy(2xy)\cos(x^2y)$$
$$= -2y\sin(x^2y) - (2xy)^2\cos(x^2y)$$

$$\frac{\partial^2 f(x,y)}{\partial y\partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dx} = \frac{d\{-x^2\sin(x^2y)\}}{dx} = \frac{d\{-x^2\}}{dx}\sin(x^2y) - x^2\frac{d\{\sin(x^2y)\}}{dx}$$
$$= -2x\sin(x^2y) - x^2(2xy)\cos(x^2y) = -2x\sin(x^2y) - 2x^3y\cos(x^2y)$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{-x^2\sin(x^2y)\}}{dy}$$
$$= -x^2\frac{d\{A\}}{dy}\frac{d\{\sin(A)\}}{dA} = -x^2\frac{d\{x^2y\}}{dy}\cos(A) = -x^4\cos(x^2y)$$

$$\frac{\partial^3 f(x,y)}{\partial x^3} = \frac{d\left\{\frac{d^2 f(x,y)}{dx^2}\right\}}{dx} = \frac{d\{-2y\sin(x^2y) - (2xy)^2\cos(x^2y)\}}{dx}$$
$$= -2y\frac{d\{\sin(x^2y)\}}{dx} + \frac{d\{-(2xy)^2\}}{dx}\cos(x^2y) - (2xy)^2\frac{d\{\cos(x^2y)\}}{dx}$$
$$= -2y(2xy)\cos(x^2y) - 4(2x)y^2\cos(x^2y) - (2xy)^2(2xy)(-\sin(x^2y))$$
$$= -4xy^2\cos(x^2y) - 8xy^2\cos(x^2y) + (2xy)^3\sin(x^2y)$$

$$\frac{\partial^3 f(x,y)}{\partial y\partial x^2} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y\partial x}\right\}}{dx} = \frac{d\{-2x\sin(x^2y) - 2x^3y\cos(x^2y)\}}{dx}$$
$$= \frac{d\{-2x\}}{dx}\sin(x^2y) - 2x\frac{d\{\sin(x^2y)\}}{dx} + \frac{d\{-2x^3y\}}{dx}\cos(x^2y) - 2x^3y\frac{d\{\cos(x^2y)\}}{dx}$$
$$= -2\sin(x^2y) - 2x(2xy)\cos(x^2y) - 6x^2y\cos(x^2y) + 2x^3y(2xy)\sin(x^2y)$$
$$= -2\sin(x^2y) - (2x)^2y\cos(x^2y) - 6x^2y\cos(x^2y) + 4x^4y^2\sin(x^2y)$$

$$\frac{\partial^3 f(x,y)}{\partial y^2\partial x} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dx} = \frac{d\{-x^4\cos(x^2y)\}}{dx} = \frac{d\{-x^4\}}{dx}\cos(x^2y) - x^4\frac{d\{\cos(x^2y)\}}{dx}$$
$$= -4x^3\cos(x^2y) - x^4(2xy)(-\sin(x^2y)) = -4x^3\cos(x^2y) + 2x^5y\sin(x^2y)$$

$$\frac{\partial^3 f(x,y)}{\partial y^3} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dy} = \frac{d\{-x^4\cos(x^2y)\}}{dy} = -x^4\frac{d\{\cos(x^2y)\}}{dy} = -x^4x^2(-\sin(x^2y)) = x^6\sin(x^2y)$$

6) Find $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y\partial x^2}$, $\frac{\partial^3 f(x,y)}{\partial y^2\partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$ when
$f(x,y) = x\ln(x^2+y)$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{x\ln(x^2+y)\}}{dx} = \frac{d\{x\}}{dx}\ln(x^2+y) + x\frac{d\{\ln(x^2+y)\}}{dx}$$
$$= \ln(x^2+y) + x\frac{d\{A\}}{dx}\frac{d\{\ln(A)\}}{dA}(\because A \triangleq x^2+y) = \ln(x^2+y) + x\frac{d\{x^2+y\}}{dx}\frac{1}{A}$$
$$= \ln(x^2+y) + x(2x)\frac{1}{x^2+y} = \ln(x^2+y) + \frac{2x^2}{x^2+y}$$
$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{x\ln(x^2+y)\}}{dy} = x\frac{1}{x^2+y} = \frac{x}{x^2+y}$$

$$\frac{d^2f(x,y)}{dx^2}=\frac{d\left\{\dfrac{d\left\{f(x,y)\right\}}{dx}\right\}}{dx}=\frac{d\left\{\ln(x^2+y)+\frac{2x^2}{x^2+y}\right\}}{dx}$$

$$=\frac{d\left\{\ln(x^2+y)\right\}}{dx}+\frac{d\left\{\frac{2x^2+2y-2y}{x^2+y}\right\}}{dx}=\frac{2x}{x^2+y}+\frac{d\left\{2-2\frac{y}{x^2+y}\right\}}{dx}$$

$$=\frac{2x}{x^2+y}-2y(-1)(x^2+y)^{-2}(2x)=\frac{2x}{x^2+y}+4xy(x^2+y)^{-2}$$

$$=2x(x^2+y)^{-1}+4xy(x^2+y)^{-2}=2x(x^2+y)(x^2+y)^{-2}+4xy(x^2+y)^{-2}$$

$$=(2x^3+2xy)(x^2+y)^{-2}+4xy(x^2+y)^{-2}=(2x^3+6xy)(x^2+y)^{-2}$$

$$\frac{\partial^2f(x,y)}{\partial y\partial x}=\frac{d\left\{\dfrac{d\left\{f(x,y)\right\}}{dy}\right\}}{dx}=\frac{d\left\{\frac{x}{x^2+y}\right\}}{dx}=\frac{d\left\{x(x^2+y)^{-1}\right\}}{dx}$$

$$=\frac{d\left\{x\right\}}{dx}(x^2+y)^{-1}+x\frac{d\left\{(x^2+y)^{-1}\right\}}{dx}=(x^2+y)^{-1}+x(-1)(x^2+y)^{-2}(2x)=(x^2+y)^{-1}-2x^2(x^2+y)^{-2}$$

$$=(x^2+y)(x^2+y)^{-2}-2x^2(x^2+y)^{-2}=(x^2+y-2x^2)(x^2+y)^{-2}=(y-x^2)(x^2+y)^{-2}$$

$$\frac{\partial^2f(x,y)}{\partial y^2}=\frac{d\left\{\dfrac{d\left\{f(x,y)\right\}}{dy}\right\}}{dy}=\frac{d\left\{\frac{x}{x^2+y}\right\}}{dy}=x(-1)(x^2+y)^{-2}=-x(x^2+y)^{-2}$$

$$\frac{\partial^3f(x,y)}{\partial x^3}=\frac{d\left\{\dfrac{d^2f(x,y)}{dx^2}\right\}}{dx}=\frac{d\left\{(2x^3+6xy)(x^2+y)^{-2}\right\}}{dx}$$

$$=\frac{d\left\{2x^3+6xy\right\}}{dx}(x^2+y)^{-2}+(2x^3+6xy)\frac{d\left\{(x^2+y)^{-2}\right\}}{dx}$$

$$=(6x^2+6y)(x^2+y)^{-2}+(2x^3+6xy)(-2)(x^2+y)^{-3}(2x)$$

$$=(6x^2+6y)(x^2+y)^{-2}-4x(2x^3+6xy)(x^2+y)^{-3}$$

$$\frac{\partial^3f(x,y)}{\partial y\partial x^2}=\frac{d\left\{\dfrac{\partial^2f(x,y)}{\partial y\partial x}\right\}}{dx}=\frac{d\left\{(y-x^2)(x^2+y)^{-2}\right\}}{dx}$$

$$=\frac{d\left\{y-x^2\right\}}{dx}(x^2+y)^{-2}+(y-x^2)\frac{d\left\{(x^2+y)^{-2}\right\}}{dx}$$

$$=-2x(x^2+y)^{-2}+(y-x^2)(-2)(x^2+y)^{-3}(2x)=-2x(x^2+y)^{-2}-4x(y-x^2)(x^2+y)^{-3}$$

$$\frac{\partial^3f(x,y)}{\partial y^2\partial x}=\frac{d\left\{\dfrac{\partial^2f(x,y)}{\partial y^2}\right\}}{dx}=\frac{d\left\{-x(x^2+y)^{-2}\right\}}{dx}$$

$$=\frac{d\left\{-x\right\}}{dx}(x^2+y)^{-2}-x\frac{d\left\{(x^2+y)^{-2}\right\}}{dx}=-(x^2+y)^{-2}-x(-2)(x^2+y)^{-3}(2x)$$

$$=-(x^2+y)^{-2}+(2x)^2(x^2+y)^{-3}$$

$$\frac{\partial^3f(x,y)}{\partial y^3}=\frac{d\left\{\dfrac{\partial^2f(x,y)}{\partial y^2}\right\}}{dy}=\frac{d\left\{-x(x^2+y)^{-2}\right\}}{dy}=-x(-2)(x^2+y)^{-3}=2x(x^2+y)^{-3}$$

7) Find $\frac{d\left\{f(x,y)\right\}}{dx}$, $\frac{d\left\{f(x,y)\right\}}{dy}$, $\frac{d^2f(x,y)}{dx^2}$, $\frac{\partial^2f(x,y)}{\partial y\partial x}$, $\frac{\partial^2f(x,y)}{\partial y^2}$, $\frac{\partial^3f(x,y)}{\partial x^3}$, $\frac{\partial^3f(x,y)}{\partial y\partial x^2}$, $\frac{\partial^3f(x,y)}{\partial y^2\partial x}$, and $\frac{\partial^3f(x,y)}{\partial y^3}$ when

$f(x, y) = x^4 - y^5 + x^6 y^7$

$$\frac{d\left\{f(x,y)\right\}}{dx} = \frac{d\left\{x^4 - y^5 + x^6y^7\right\}}{dx} = 4x^3 + 6x^5y^7$$

$$\frac{d\left\{f(x,y)\right\}}{dy} = \frac{d\left\{x^4 - y^5 + x^6y^7\right\}}{dy} = -5y^4 + 7x^6y^6$$

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{\dfrac{d\left\{f(x,y)\right\}}{dx}\right\}}{dx} = \frac{d\left\{4x^3 + 6x^5y^7\right\}}{dx} = 12x^2 + 30x^4y^7$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{d\left\{\dfrac{d\left\{f(x,y)\right\}}{dy}\right\}}{dx} = \frac{d\left\{-5y^4 + 7x^6y^6\right\}}{dx} = 42x^5y^6$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\dfrac{d\left\{f(x,y)\right\}}{dy}\right\}}{dy} = \frac{d\left\{-5y^4 + 7x^6y^6\right\}}{dy} = -20y^3 + 42x^6y^5$$

$$\frac{\partial^3 f(x,y)}{\partial x^3} = \frac{d\left\{\dfrac{d^2 f(x,y)}{dx^2}\right\}}{dx} = \frac{d\left\{12x^2 + 30x^4y^7\right\}}{dx} = 24x + 120x^3y^7$$

$$\frac{\partial^3 f(x,y)}{\partial y \partial x^2} = \frac{d\left\{\dfrac{\partial^2 f(x,y)}{\partial y \partial x}\right\}}{dx} = \frac{d\left\{42x^5y^6\right\}}{dx} = 210x^4y^6$$

$$\frac{\partial^3 f(x,y)}{\partial y^2 \partial x} = \frac{d\left\{\dfrac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dx} = \frac{d\left\{-20y^3 + 42x^6y^5\right\}}{dx} = 252x^5y^5$$

$$\frac{\partial^3 f(x,y)}{\partial y^3} = \frac{d\left\{\dfrac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dy} = \frac{d\left\{-20y^3 + 42x^6y^5\right\}}{dy} = -60y^2 + 210x^6y^4$$

**DAY2**

8) Find $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$, and $\frac{\partial^2 f(x,y)}{\partial y^2}$ when $f(x,y) = 3^{\frac{x}{y}}$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\left\{3^{\frac{x}{y}}\right\}}{dx} = \frac{d\left\{(3^{\frac{1}{y}})^x\right\}}{dx} = \frac{d\{A^x\}}{dx}(\because A \triangleq 3^{\frac{1}{y}}) = A^x \ln A$$
$$= (3^{\frac{1}{y}})^x \ln 3^{\frac{1}{y}} = 3^{\frac{x}{y}}\frac{\ln 3}{y}$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\left\{3^{\frac{x}{y}}\right\}}{dy} = \frac{d\{3^B\}}{dy}(\because B \triangleq \frac{x}{y}) = \frac{\partial 3^B}{\partial B}\frac{\partial B}{\partial y}$$
$$= 3^B \ln 3\frac{\partial\left(\frac{x}{y}\right)}{\partial y} = 3^B \ln 3x\frac{\partial\left(\frac{1}{y}\right)}{\partial y}$$
$$= \left(3^B \ln 3\right)\cdot x\frac{\partial\left(y^{-1}\right)}{\partial y} = 3^B x(-y^{-2})\ln 3 = -\frac{3^{\frac{x}{y}}x}{y^2}\ln 3$$

$$\frac{d^2f(x,y)}{dx^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\left\{3^{\frac{x}{y}}\frac{\ln 3}{y}\right\}}{dx} = \frac{\ln 3}{y}\frac{d\left\{3^{\frac{x}{y}}\right\}}{dx} = \frac{\ln 3}{y}3^{\frac{x}{y}}\frac{\ln 3}{y} = 3^{\frac{x}{y}}\left(\frac{\ln 3}{y}\right)^2$$

$$\frac{\partial^2 f(x,y)}{\partial y\partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dy} = \frac{d\left\{3^{\frac{x}{y}}\frac{\ln 3}{y}\right\}}{dy} = \frac{d\left\{3^{\frac{x}{y}}\right\}}{dy}\frac{\ln 3}{y} + 3^{\frac{x}{y}}\frac{d\left\{\frac{\ln 3}{y}\right\}}{dy}$$
$$= \frac{d\left\{3^{\frac{x}{y}}\right\}}{dy}\frac{\ln 3}{y} + 3^{\frac{x}{y}}\frac{d\left\{\frac{\ln 3}{y}\right\}}{dy}$$
$$= -\frac{3^{\frac{x}{y}}x}{y^2}\ln 3\cdot\frac{\ln 3}{y} + 3^{\frac{x}{y}}\frac{d\{y^{-1}\}}{dy}\ln 3 = -\frac{3^{\frac{x}{y}}x}{y^2}\ln 3\cdot\frac{\ln 3}{y} - 3^{\frac{x}{y}}y^{-2}\ln 3$$
$$= -\frac{3^{\frac{x}{y}}x}{y^3}(\ln 3)^2 - \frac{3^{\frac{x}{y}}\ln 3}{y^2}$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\left\{-\frac{3^{\frac{x}{y}}x}{y^2}\ln 3\right\}}{dy} = \frac{d\left\{-3^{\frac{x}{y}}x\cdot y^{-2}\ln 3\right\}}{dy} = -x\ln 3\frac{d\left\{3^{\frac{x}{y}}\cdot y^{-2}\right\}}{dy}$$
$$= -x\ln 3(\frac{d\left\{3^{\frac{x}{y}}\right\}}{dy}\cdot y^{-2} + 3^{\frac{x}{y}}\frac{d\{y^{-2}\}}{dy}) = -x\ln 3(-\frac{3^{\frac{x}{y}}x}{y^2}\ln 3\cdot y^{-2} + 3^{\frac{x}{y}}\cdot(-2)y^{-3})$$
$$= x\ln 3\left(\frac{3^{\frac{x}{y}}x\ln 3}{y^4} + 2\cdot 3^{\frac{x}{y}}y^{-3}\right)$$

9) Find $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y\partial x^2}$, $\frac{\partial^3 f(x,y)}{\partial y^2\partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$ when $f(x,y) = \mathfrak{e}^{2x}3^y$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{\mathfrak{e}^{2x}3^y\}}{dx} = 3^y\frac{d\{\mathfrak{e}^{2x}\}}{dx} = 2\cdot 3^y\mathfrak{e}^{2x}\ ;\ \frac{d\{f(x,y)\}}{dy} = \frac{d\{\mathfrak{e}^{2x}3^y\}}{dy} = \mathfrak{e}^{2x}\frac{d\{3^y\}}{dy} = \mathfrak{e}^{2x}3^y\ln 3$$

$$\frac{d^2f(x,y)}{dx^2} = \frac{d\left\{\dfrac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\{2\cdot 3^y\mathfrak{e}^{2x}\}}{dx} = 2\cdot 3^y\frac{d\{\mathfrak{e}^{2x}\}}{dx} = 4\cdot 3^y\mathfrak{e}^{2x}$$

$$\frac{\partial^2 f(x,y)}{\partial y\partial x} = \frac{d\left\{\dfrac{d\{f(x,y)\}}{dy}\right\}}{dx} = \frac{d\{\mathfrak{e}^{2x}3^y\ln 3\}}{dx} = 3^y\ln 3\frac{d\{\mathfrak{e}^{2x}\}}{dx} = 2\cdot 3^y\mathfrak{e}^{2x}\ln 3$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\dfrac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{\mathfrak{e}^{2x}3^y\ln 3\}}{dy} = \mathfrak{e}^{2x}\ln 3\frac{d\{3^y\}}{dy} = \mathfrak{e}^{2x}(\ln 3)^2 3^y$$

$$\frac{\partial^3 f(x,y)}{\partial x^3} = \frac{d\left\{\dfrac{d^2f(x,y)}{dx^2}\right\}}{dx} = \frac{d\{4\cdot 3^y\mathfrak{e}^{2x}\}}{dx} = 4\cdot 3^y\frac{d\{\mathfrak{e}^{2x}\}}{dx} = 8\cdot 3^y\mathfrak{e}^{2x}$$

$$\frac{\partial^3 f(x,y)}{\partial y\partial x^2} = \frac{d\left\{\dfrac{\partial^2 f(x,y)}{\partial y\partial x}\right\}}{dx} = \frac{d\{2\cdot 3^y\mathfrak{e}^{2x}\ln 3\}}{dx} = 2\cdot 3^y\ln 3\frac{d\{\mathfrak{e}^{2x}\}}{dx} = 4\cdot 3^y\ln 3\mathfrak{e}^{2x}$$

$$\frac{\partial^3 f(x,y)}{\partial y^2\partial x} = \frac{d\left\{\dfrac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dx} = \frac{d\{\mathfrak{e}^{2x}(\ln 3)^2 3^y\}}{dx} = (\ln 3)^2 3^y\frac{d\{\mathfrak{e}^{2x}\}}{dx} = 2(\ln 3)^2 3^y\mathfrak{e}^{2x}$$

$$\frac{\partial^3 f(x,y)}{\partial y^3} = \frac{d\left\{\dfrac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dy} = \frac{d\{\mathfrak{e}^{2x}(\ln 3)^2 3^y\}}{dy} = \mathfrak{e}^{2x}(\ln 3)^2\frac{d\{3^y\}}{dy} = \mathfrak{e}^{2x}(\ln 3)^3 3^y$$

10) Find $\dfrac{d\{f(x,y)\}}{dx}$, $\dfrac{d\{f(x,y)\}}{dy}$, $\dfrac{d^2f(x,y)}{dx^2}$, $\dfrac{\partial^2 f(x,y)}{\partial y\partial x}$, $\dfrac{\partial^2 f(x,y)}{\partial y^2}$, when
$f(x,y) = \dfrac{x}{1+x^3+y^3}$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{x(1+x^3+y^3)^{-1}\}}{dx} = \frac{d\{x\}}{dx}(1+x^3+y^3)^{-1} + x\frac{d\{(1+x^3+y^3)^{-1}\}}{dx}$$
$$= (1+x^3+y^3)^{-1} - x(1+x^3+y^3)^{-2}(3x^2) = (1+x^3+y^3)^{-1} - 3x^3(1+x^3+y^3)^{-2}$$
$$= (1+x^3+y^3)(1+x^3+y^3)^{-2} - 3x^3(1+x^3+y^3)^{-2} = (1-2x^3+y^3)(1+x^3+y^3)^{-2}$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{x(1+x^3+y^3)^{-1}\}}{dy} = \frac{d\{x\}}{dy}(1+x^3+y^3)^{-1} + x\frac{d\{(1+x^3+y^3)^{-1}\}}{dy}$$
$$= -x(1+x^3+y^3)^{-2}(3y^2) = -3xy^2(1+x^3+y^3)^{-2}$$

$$\frac{d^2f(x,y)}{dx^2} = \frac{d\left\{\dfrac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\{(1-2x^3+y^3)(1+x^3+y^3)^{-2}\}}{dx}$$
$$= \frac{d\{(1-2x^3+y^3)\}}{dx}(1+x^3+y^3)^{-2} + (1-2x^3+y^3)\frac{d\{(1+x^3+y^3)^{-2}\}}{dx}$$
$$= -6x^2(1+x^3+y^3)^{-2} - 2(1-2x^3+y^3)(1+x^3+y^3)^{-3}(3x^2)$$

$$\frac{\partial^2 f(x,y)}{\partial y\partial x} = \frac{d\left\{\dfrac{d\{f(x,y)\}}{dx}\right\}}{dy} = \frac{d\{(1-2x^3+y^3)(1+x^3+y^3)^{-2}\}}{dy}$$
$$= \frac{d\{(1-2x^3+y^3)\}}{dy}(1+x^3+y^3)^{-2} + (1-2x^3+y^3)\frac{d\{(1+x^3+y^3)^{-2}\}}{dy}$$
$$= 3y^2(1+x^3+y^3)^{-2} - 2(1-2x^3+y^3)(1+x^3+y^3)^{-3}(3y^2)$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{-3xy^2(1+x^3+y^3)^{-2}\}}{dy}$$
$$= \frac{d\{-3xy^2\}}{dy}(1+x^3+y^3)^{-2} - 3xy^2\frac{d\{(1+x^3+y^3)^{-2}\}}{dy}$$
$$= -6xy(1+x^3+y^3)^{-2} + 6xy^2(1+x^3+y^3)^{-3}(3y^2)$$

11) Find $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y \partial x^2}$, $\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$ when
$f(x,y) = x^2 \sin(y)$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{x^2\sin(y)\}}{dx} = \frac{d\{x^2\}}{dx}\sin(y) = 2x\sin(y)$$
$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{x^2\sin(y)\}}{dy} = x^2\frac{d\{\sin(y)\}}{dy} = x^2\cos(y)$$
$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\{2x\sin(y)\}}{dx} = \frac{d\{2x\}}{dx}\sin(y) = 2\sin(y)$$
$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dx} = \frac{d\{x^2\cos(y)\}}{dx} = \frac{d\{x^2\}}{dx}\cos(y) = 2x\cos(y)$$
$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{x^2\cos(y)\}}{dy} = x^2\frac{d\{\cos(y)\}}{dy} = -x^2\sin(y)$$
$$\frac{\partial^3 f(x,y)}{\partial x^3} = \frac{d\left\{\frac{d^2 f(x,y)}{dx^2}\right\}}{dx} = \frac{d\{2\sin(y)\}}{dx} = 0$$
$$\frac{\partial^3 f(x,y)}{\partial y \partial x^2} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y \partial x}\right\}}{dx} = \frac{d\{2x\cos(y)\}}{dx} = \frac{d\{2x\}}{dx}\cos(y) = 2\cos(y)$$
$$\frac{\partial^3 f(x,y)}{\partial y^2 \partial x} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dx} = \frac{d\{-x^2\sin(y)\}}{dx} = \frac{d\{-x^2\}}{dx}\sin(y) = -2x\sin(y)$$
$$\frac{\partial^3 f(x,y)}{\partial y^3} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dy} = \frac{d\{-x^2\sin(y)\}}{dy} = -x^2\frac{d\{\sin(y)\}}{dy} = -x^2\cos(y)$$

12) Find $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, when $f(x,y) = \sin\left(\frac{x}{y}\right)$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\left\{\sin\left(\frac{x}{y}\right)\right\}}{dx} = \frac{d\{\sin(t)\}}{dx}(\because t \triangleq \frac{x}{y}) = \frac{d\{t\}}{dx}\frac{d\{\sin(t)\}}{dt}$$
$$= \frac{d\left\{\frac{x}{y}\right\}}{dx}\cos(t) = \frac{1}{y}\frac{d\{x\}}{dx}\cos(t) = y^{-1}\cos\left(\frac{x}{y}\right)$$

$$\frac{d\left\{f(x,y)\right\}}{dy} = \frac{d\left\{\sin\left(\frac{x}{y}\right)\right\}}{dy} = \frac{d\left\{\sin(t)\right\}}{dy}(\because t \triangleq \frac{x}{y}) = \frac{d\left\{t\right\}}{dy}\frac{d\left\{\sin(t)\right\}}{dt}$$

$$= \frac{d\left\{\frac{x}{y}\right\}}{dy}\cos(t) = x\frac{d\left\{y^{-1}\right\}}{dy}\cos(t) = -x\cdot y^{-2}\cos\left(\frac{x}{y}\right)$$

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{\frac{d\left\{f(x,y)\right\}}{dx}\right\}}{dx} = \frac{d\left\{y^{-1}\cos\left(\frac{x}{y}\right)\right\}}{dx}$$

$$= y^{-1}\frac{d\left\{\cos\left(\frac{x}{y}\right)\right\}}{dx} = y^{-1}\frac{d\left\{\cos(t)\right\}}{dx}(\because t \triangleq \frac{x}{y})$$

$$= y^{-1}\frac{d\left\{t\right\}}{dx}\frac{d\left\{\cos(t)\right\}}{dt} = y^{-1}\frac{d\left\{x\cdot y^{-1}\right\}}{dx}\cdot(-\sin(t))$$

$$= y^{-2}\frac{d\left\{x\right\}}{dx}\cdot(-\sin(t)) = y^{-2}\left(-\sin\left(\frac{x}{y}\right)\right)$$

$$\frac{\partial^2 f(x,y)}{\partial y\partial x} = \frac{d\left\{\frac{d\left\{f(x,y)\right\}}{dy}\right\}}{dx} = \frac{d\left\{-x\cdot y^{-2}\cos\left(\frac{x}{y}\right)\right\}}{dx} = \frac{d\left\{-x\cdot y^{-2}\right\}}{dx}\cos\left(\frac{x}{y}\right) - xy^{-2}\frac{d\left\{\cos\left(\frac{x}{y}\right)\right\}}{dx}$$

$$= -y^{-2}\frac{d\left\{x\right\}}{dx}\cos\left(\frac{x}{y}\right) - xy^{-2}\frac{d\left\{\cos(t)\right\}}{dx}(\because t \triangleq \frac{x}{y}) = -y^{-2}\cos\left(\frac{x}{y}\right) - xy^{-2}\frac{d\left\{t\right\}}{dx}\frac{d\left\{\cos(t)\right\}}{dt}$$

$$= -y^{-2}\cos\left(\frac{x}{y}\right) - xy^{-2}\frac{d\left\{x\cdot y^{-1}\right\}}{dx}(-\sin(t)) = -y^{-2}\cos\left(\frac{x}{y}\right) - xy^{-3}\frac{d\left\{x\right\}}{dx}(-\sin(t))$$

$$= -y^{-2}\cos\left(\frac{x}{y}\right) + xy^{-3}\sin\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\frac{d\left\{f(x,y)\right\}}{dy}\right\}}{dy} = \frac{d\left\{-x\cdot y^{-2}\cos\left(\frac{x}{y}\right)\right\}}{dy} = \frac{d\left\{-x\cdot y^{-2}\right\}}{dy}\cos\left(\frac{x}{y}\right) - xy^{-2}\frac{d\left\{\cos(t)\right\}}{dy}$$

$$= -x\frac{d\left\{y^{-2}\right\}}{dy}\cos\left(\frac{x}{y}\right) - xy^{-2}\frac{d\left\{t\right\}}{dy}\frac{d\left\{\cos(t)\right\}}{dt} = -x(-2)y^{-3}\cos\left(\frac{x}{y}\right) - xy^{-2}\frac{d\left\{xy^{-1}\right\}}{dy}(-\sin(t))$$

$$= -x(-2)y^{-3}\cos\left(\frac{x}{y}\right) - xy^{-2}(-1)xy^{-2}(-\sin(t)) = 2xy^{-3}\cos\left(\frac{x}{y}\right) - x^2y^{-4}\sin\left(\frac{x}{y}\right)$$

**DAY3**

13) Express $\dfrac{d\{f\}}{ds}$ and $\dfrac{d\{f\}}{dt}$ using $s$ and $t$ when

$$f = x + \log_3 y$$
$$x = t - s$$
$$y = \frac{t}{s}$$

Since $f$ is the function of $x$ and $y$ we can express $\dfrac{d\{f\}}{ds}$ and $\dfrac{d\{f\}}{dt}$ as follows:

$$\frac{d\{f\}}{ds} = \frac{d\{x\}}{ds}\frac{d\{f\}}{dx} + \frac{d\{y\}}{ds}\frac{d\{f\}}{dy}$$
$$\frac{d\{f\}}{dt} = \frac{d\{x\}}{dt}\frac{d\{f\}}{dx} + \frac{d\{y\}}{dt}\frac{d\{f\}}{dy}$$

Thus we need $\dfrac{d\{f\}}{dx}, \dfrac{d\{f\}}{dy}, \dfrac{d\{x\}}{ds}, \dfrac{d\{y\}}{ds}, \dfrac{d\{x\}}{dt}, \dfrac{d\{y\}}{dt}$.

$$\frac{d\{f\}}{dx} = \frac{d\{x + \log_3 y\}}{dx} = \frac{d\{x\}}{dx} + \frac{d\left\{\frac{\ln y}{\ln 3}\right\}}{dx} = 1$$
$$\frac{d\{f\}}{dy} = \frac{d\{x + \log_3 y\}}{dy} = \frac{d\{x\}}{dy} + \frac{d\left\{\frac{\ln y}{\ln 3}\right\}}{dy} = \frac{1}{y\ln 3}$$
$$\frac{d\{x\}}{ds} = \frac{d\{t - s\}}{ds} = \frac{d\{t\}}{ds} - \frac{d\{s\}}{ds} = -1$$
$$\frac{d\{x\}}{dt} = \frac{d\{t - s\}}{dt} = \frac{d\{t\}}{dt} - \frac{d\{s\}}{dt} = 1$$
$$\frac{d\{y\}}{ds} = \frac{d\left\{\frac{t}{s}\right\}}{ds} = t\frac{d\left\{\frac{1}{s}\right\}}{ds} = -\frac{t}{s^2}$$
$$\frac{d\{y\}}{dt} = \frac{d\left\{\frac{t}{s}\right\}}{dt} = \frac{1}{s}\frac{d\{t\}}{dt} = \frac{1}{s}$$

Therefore

$$\frac{d\{f\}}{ds} = \frac{d\{x\}}{ds}\frac{d\{f\}}{dx} + \frac{d\{y\}}{ds}\frac{d\{f\}}{dy} = -1\cdot 1 - \frac{t}{s^2}\cdot\frac{1}{y\ln 3} = -1 - \frac{t}{s^2 y\ln 3} \qquad ①$$

Since the answer should not have $y$ we substitute $y = \frac{t}{s}$ into ① as follows:

$$-1 - \frac{t}{s^2 y\ln 3} = -1 - \frac{t}{s^2\cdot\dfrac{t}{s}\ln 3} = -1 - \frac{1}{s\ln 3}$$

In the same way we can obtain $\dfrac{d\{f\}}{dt}$ as follows:

$$\frac{d\{f\}}{dt} = \frac{d\{x\}}{dt}\frac{d\{f\}}{dx} + \frac{d\{y\}}{dt}\frac{d\{f\}}{dy} = 1\cdot 1 + \frac{1}{s}\cdot\frac{1}{y\ln 3} = 1 + \frac{1}{sy\ln 3} \qquad ②$$

Since the answer should not have $y$ we substitute $y = \frac{t}{s}$ into ② as follows:

$$1 + \frac{1}{sy\ln 3} = 1 + \frac{1}{s\cdot\dfrac{t}{s}\ln 3} = 1 + \frac{1}{t\ln 3}$$

14) Express $\dfrac{\partial\{f\}}{\partial u}$ using $s$, $t$ and $u$ when

$$f = 3^{\frac{x}{y}}$$
$$x = s + u$$
$$y = stu$$

Since $f$ is the function of $x$, and $y$ we can express $\dfrac{d\{f\}}{ds}$ and $\dfrac{d\{f\}}{dt}$ as follows:

$$\frac{\partial\{f\}}{\partial u} = \frac{\partial\{x\}}{\partial u}\frac{d\{f\}}{dx} + \frac{\partial\{y\}}{\partial u}\frac{d\{f\}}{dy}$$

Thus we need $\dfrac{d\{f\}}{dx}$,$\dfrac{d\{f\}}{dy}$, $\dfrac{\partial\{x\}}{\partial u}$, and $\dfrac{\partial\{y\}}{\partial u}$.

$$\frac{d\{f\}}{dx} = \frac{d\left\{3^{\frac{x}{y}}\right\}}{dx} = \frac{d\left\{(3^{\frac{1}{y}})^x\right\}}{dx} = \frac{d\{A^x\}}{dx}(\because A \triangleq 3^{\frac{1}{y}}) = A^x \ln A = (3^{\frac{1}{y}})^x \ln 3^{\frac{1}{y}} = 3^{\frac{x}{y}}\frac{\ln 3}{y}$$

$$\frac{d\{f\}}{dy} = \frac{d\left\{3^{\frac{x}{y}}\right\}}{dy} \quad ; \quad \therefore \frac{d\{\ln(f)\}}{dy} = \frac{d\left\{\ln(3^{\frac{x}{y}})\right\}}{dy} \quad ; \quad \therefore \frac{d\{f\}}{dy}\frac{d\{\ln(f)\}}{df} = \frac{d\left\{\frac{x}{y}\ln 3\right\}}{dy}$$

$$\therefore \frac{d\{f\}}{dy}\cdot\frac{1}{f} = x\ln 3\frac{d\left\{\frac{1}{y}\right\}}{dy} \quad ; \quad \therefore \frac{d\{f\}}{dy}\cdot\frac{1}{f} = x\ln 3\cdot(-\frac{1}{y^2})$$

$$\therefore \frac{d\{f\}}{dy} = -f\cdot\frac{x\ln 3}{y^2} = -3^{\frac{x}{y}}\cdot\frac{x\ln 3}{y^2} = -\frac{3^{\frac{x}{y}}x\ln 3}{y^2}$$

$$\frac{\partial\{x\}}{\partial u} = \frac{\partial\{s+u\}}{\partial u} = \frac{\partial\{s\}}{\partial u} + \frac{\partial\{u\}}{\partial u} = 1$$

$$\frac{\partial\{y\}}{\partial u} = \frac{\partial\{stu\}}{\partial u} = st\frac{\partial\{u\}}{\partial u} = st$$

Therefore

$$\frac{\partial\{f\}}{\partial u} = \frac{\partial\{x\}}{\partial u}\frac{d\{f\}}{dx} + \frac{\partial\{y\}}{\partial u}\frac{d\{f\}}{dy} = 3^{\frac{x}{y}}\frac{\ln 3}{y} - st\frac{3^{\frac{x}{y}}x\ln 3}{y^2} \qquad ①$$

Since the answer should not have $x$, and $y$, we substitute $x = s + u$ and $y = stu$ into ① as follows:

$$3^{\frac{x}{y}}\frac{\ln 3}{y} - st\frac{3^{\frac{x}{y}}x\ln 3}{y^2} = 3^{\frac{s+u}{stu}}\frac{\ln 3}{stu} - st\frac{3^{\frac{s+u}{stu}}(s+u)\ln 3}{(stu)^2}$$
$$= \frac{3^{\frac{s+u}{stu}}\ln 3}{stu} - \frac{st(s+u)3^{\frac{s+u}{stu}}\ln 3}{s^2t^2u^2} = \frac{3^{\frac{s+u}{stu}}\ln 3}{stu} - \frac{(s+u)3^{\frac{s+u}{stu}}\ln 3}{stu^2}$$

15) What is $(3x+6)\cdot(0) + 3\cdot(-1) + \ln 3$ ?

$$(3x+6)\cdot(0) + 3\cdot(-1) + \ln 3 = \ln 3 - 3(\because a\cdot 0 = 0)$$

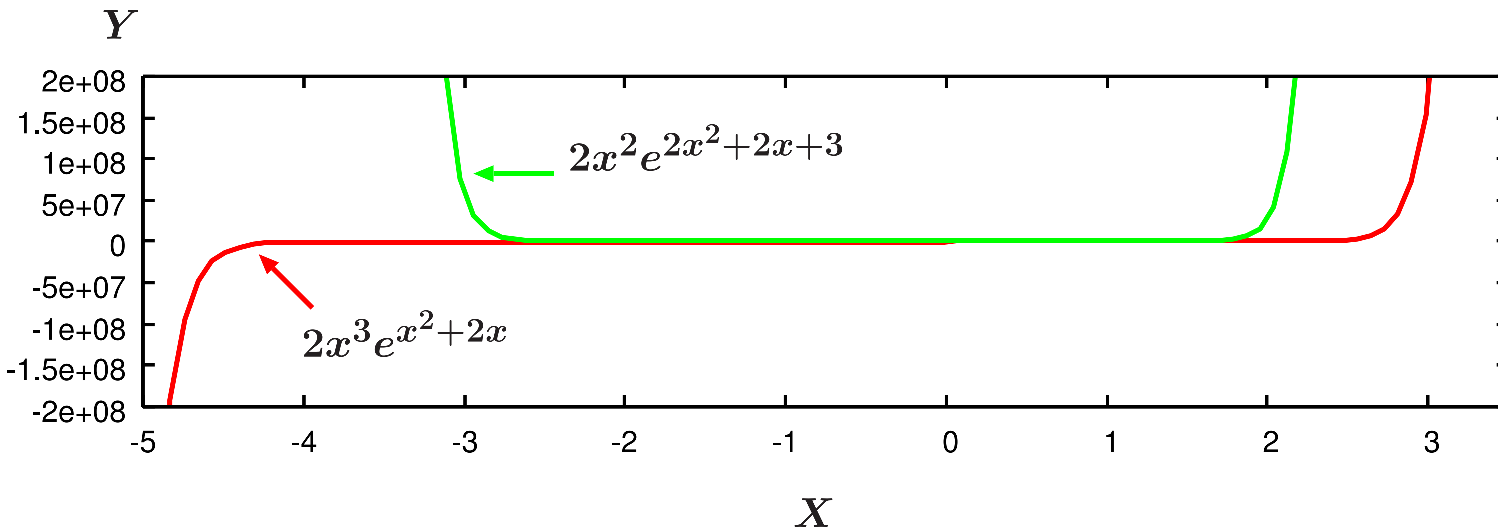
16) Differentiate $y = \ln x$

$$y = \ln x \;\; ; \;\; \frac{d\{y\}}{dx} = \frac{1}{x}$$

17) Evaluate $\sqrt{8 \cdot 9 \div 2 - 11} - 51$

$$\sqrt{8 \cdot 9 \div 2 - 11} - 51 = \sqrt{72 \div 2 - 11} - 51 = \sqrt{36 - 11} - 51 = \sqrt{25} - 51 = 5 - 51 = -46$$

18) Find $\frac{d\{y\}}{dx}$ of $y = 2x^3 \mathfrak{e}^{x^2+2x}$.




First we split the function in to $f(x)$ and $g(x)$. Letting $f(x) = 2x^3$ and $g(x) = \mathfrak{e}^{x^2+2x}$. Then applying the chain rule to $g(x)$.

Let $u = x^2 + 2x$. Therefore $g(x) = \mathfrak{e}^u$ We can pre-calculate $\frac{\partial\{g(x)\}}{\partial u}, \frac{d\{u\}}{dx}$ as follows:

$$\frac{\partial\{g(x)\}}{\partial u} = \frac{\partial\{\mathfrak{e}^u\}}{\partial u} = \mathfrak{e}^u \quad ; \quad \frac{d\{u\}}{dx} = 2x + 2$$

Thus

$$\frac{d\{g(x)\}}{dx} = \frac{d\{u\}}{dx} \cdot \frac{\partial\{g(x)\}}{\partial u} = (2x+2) \cdot \mathfrak{e}^u = (2x+2) \cdot \mathfrak{e}^{x^2+2x} (\because u \triangleq x^2 + 2x)$$

Simultaneously

$$f(x) = 2x^3 \quad ; \quad \therefore \frac{d\{f(x)\}}{dx} = 6x^2$$

Now by substituting in to the product rule we get:

$$\begin{aligned}\frac{d\{y\}}{dx} = f(x) \cdot \frac{d\{g(x)\}}{dx} + g(x) \cdot \frac{d\{f(x)\}}{dx} &= 2x^3 \cdot (2x+2)\mathfrak{e}^{x^2+2x} + \mathfrak{e}^{x^2+2x} \cdot 6x^2 \\ &= 2x^2 \mathfrak{e}^{x^2+2x}(x(2x+2)+3) = 2x^2 \mathfrak{e}^{x^2+2x}(2x^2+2x+3)\end{aligned}$$

19) Express $\frac{d\{f\}}{ds}$ using $s$ and $t$ when

$$\begin{aligned} f &= \mathfrak{e}^{xyz} \\ x &= \log_4 st \\ y &= \sin(st) \\ z &= st \end{aligned}$$

Since $f$ is the function of $x$, $y$, $z$, we can express $\frac{d\{f\}}{ds}$ as follows:

$$\frac{d\{f\}}{ds} = \frac{d\{x\}}{ds}\frac{d\{f\}}{dx} + \frac{d\{y\}}{ds}\frac{d\{f\}}{dy} + \frac{d\{z\}}{ds}\frac{d\{f\}}{dz}$$

Thus we need $\frac{d\{f\}}{dx}, \frac{d\{f\}}{dy}, \frac{d\{f\}}{dz}, \frac{d\{x\}}{ds}, \frac{d\{y\}}{ds}, \frac{d\{z\}}{ds}$.

$$\frac{d\{f\}}{dx} = \frac{d\{\mathfrak{e}^{xyz}\}}{dx} = yz\mathfrak{e}^{xyz}$$
$$\frac{d\{f\}}{dy} = \frac{d\{\mathfrak{e}^{xyz}\}}{dy} = xz\mathfrak{e}^{xyz}$$
$$\frac{d\{f\}}{dz} = \frac{d\{\mathfrak{e}^{xyz}\}}{dz} = xy\mathfrak{e}^{xyz}$$
$$\frac{d\{x\}}{ds} = \frac{d\{\log_4 st\}}{ds} = \frac{d\left\{\frac{\ln st}{\ln 4}\right\}}{ds} = \frac{d\left\{\frac{\ln s + \ln t}{\ln 4}\right\}}{ds} = \frac{\frac{1}{s}}{\ln 4} = \frac{1}{s\ln 4} = \frac{1}{2s\ln 2}$$
$$\frac{d\{y\}}{ds} = \frac{d\{\sin(st)\}}{ds} = t\cos(st)$$
$$\frac{d\{z\}}{ds} = \frac{d\{st\}}{ds} = t$$

Therefore

$$\frac{d\{f\}}{ds} = \frac{d\{x\}}{ds}\frac{d\{f\}}{dx} + \frac{d\{y\}}{ds}\frac{d\{f\}}{dy} + \frac{d\{z\}}{ds}\frac{d\{f\}}{dz}$$
$$= \frac{1}{2s\ln 2}\cdot yz\mathfrak{e}^{xyz} + t\cos(st)\cdot xz\mathfrak{e}^{xyz} + t\cdot xy\mathfrak{e}^{xyz} = \mathfrak{e}^{xyz}\left(\frac{yz}{2s\ln 2} + xzt\cos(st) + xyt\right)$$

Since the answer should not have $x$, $y$, and $z$, we substitute $x$, $y$ and $z$ with $x = \log_4 st$, $y = \sin(st)$, and $z = st$ as follows:

$$\mathfrak{e}^{xyz}\left(\frac{yz}{2s\ln 2} + xzt\cos(st) + xyt\right) = \mathfrak{e}^{st\sin(st)\log_4 st}\left(\frac{st\sin(st)}{2s\ln 2} + st^2\log_4(st)\cos(st) + t\sin(st)\log_4(st)\right)$$
$$= \mathfrak{e}^{st\sin(st)\log_4 st}\left(\frac{t\sin(st)}{2\ln 2} + st^2\log_4(st)\cos(st) + t\sin(st)\log_4(st)\right)$$

20) Differentiate $f(x) = 2^{3x+1}\ln(5x-11)$ with respect to $x$.

If we let $f(x) = 2^{3x+1}$ and $g(x) = \ln(5x-11)$ using Equation (39) we get,

$$\frac{d\{f(x)\}}{dx} = 2^{3x+1}\frac{d\{\ln(5x-11)\}}{dx} + \frac{d\{2^{3x+1}\}}{dx}\ln(5x-11)$$

We now need to work out $\frac{d\{f(x)\}}{dx}$ and $\frac{d\{g(x)\}}{dx}$. First of all, we work out $\frac{d\{\ln(5x-11)\}}{dx}$.
Using the chain rule, When $u = 5x - 11$,

$$\frac{d\{\ln(5x-11)\}}{dx} = \frac{d\{\ln(u)\}}{dx} = \frac{d\{u\}}{dx}\frac{\partial\{(\ln u)\}}{\partial u}$$
$$= \frac{d\{5x-11\}}{dx}\frac{\partial\{\ln u\}}{\partial u} = 5\frac{1}{u} = \frac{5}{5x-11}.$$

Now we need to work out $\frac{d\{2^{3x+1}\}}{dx}$. Let $v = 3x+1$. Then we obtain
$\frac{d\{2^{3x+1}\}}{dx} = \frac{d\{2^v\}}{dx} = \frac{d\{v\}}{dx}\frac{\partial\{(2^v)\}}{\partial v}$.Using the chain rule we have to work out $\frac{\partial\{(2^v)\}}{\partial v}$. Let's assume

$g(x) = 2^v$. If we apply the natural logarithm to both sides of this equation $g(x) = 2^v$, we get $\ln(g(x)) = \ln(2^v) = v\ln(2)$. When we differentiate both sides of this equation $\ln(g(x)) = v\ln(2)$, we get

$$\frac{\partial\{\ln(g(x))\}}{\partial v} = \frac{\partial\{v\ln(2)\}}{\partial v} \quad ; \quad \therefore \frac{\partial\{\ln(g(x))\}}{\partial v} = \frac{\partial\{g(x)\}}{\partial v}\frac{\partial\{\ln g(x)\}}{\partial\{g(x)\}} = \frac{\partial\{v\ln(2)\}}{\partial v}$$

$$\therefore \frac{\partial\{g(x)\}}{\partial v}\frac{1}{g(x)} = \ln(2) \quad ; \quad \therefore \frac{\partial\{g(x)\}}{\partial v} = g(x)\ln(2) = 2^v\ln(2)$$

$$\therefore \frac{d\{2^{3x+1}\}}{dx} = \frac{d\{v\}}{dx}\frac{\partial\{2^v\}}{\partial v} = \frac{d\{v\}}{dx}2^v\ln(2) = \frac{d\{3x+1\}}{dx}2^v\ln(2) = 3\cdot 2^v\ln(2) = 3\cdot 2^{3x+1}\ln(2)$$

Thus we found out that

$$\frac{d\{\ln(5x-11)\}}{dx} = \frac{5}{5x-11} \quad ; \quad \frac{d\{2^{3x+1}\}}{dx} = 3\cdot 2^{3x+1}\ln(2)$$

Therefore by substituting in to the product rule we get,

$$\frac{d\{f(x)\}}{dx} = 2^{3x+1}\frac{d\{\ln(5x-11)\}}{dx} + \frac{d\{2^{3x+1}\}}{dx}\ln(5x-11) = 2^{3x+1}\frac{5}{5x-11} + 3\cdot 2^{3x+1}\ln(2)\ln(5x-11)$$

Or using the alternative method:
Let $y = f(x) = 2^{3x+1}\ln(5x-11)$, $v = 2^{3x+1}$ and $u = \ln(5x-11)$. To find the differential of $y$ with respect to $x$ we must use the product rule which is

$$\frac{d\{y\}}{dx} = v\cdot\frac{d\{u\}}{dx} + u\cdot\frac{d\{v\}}{dx}$$

Now let's find $\frac{d\{v\}}{dx}$. Start by taking $\ln$ of both sides of $v = 2^{3x+1}$

$$\ln v = \ln(2^{(3x+1)}) = (3x+1)\cdot\ln 2 (\because \ln b^a = a\ln b)$$

$$\therefore \frac{1}{v}\frac{d\{v\}}{dx} = 3\ln 2 \quad ; \quad \therefore \frac{d\{v\}}{dx} = 3\ln 2\cdot v \quad ; \quad \therefore \frac{d\{v\}}{dx} = 3\ln 2\cdot 2^{3x+1}(\because v \triangleq 2^{3x+1})$$

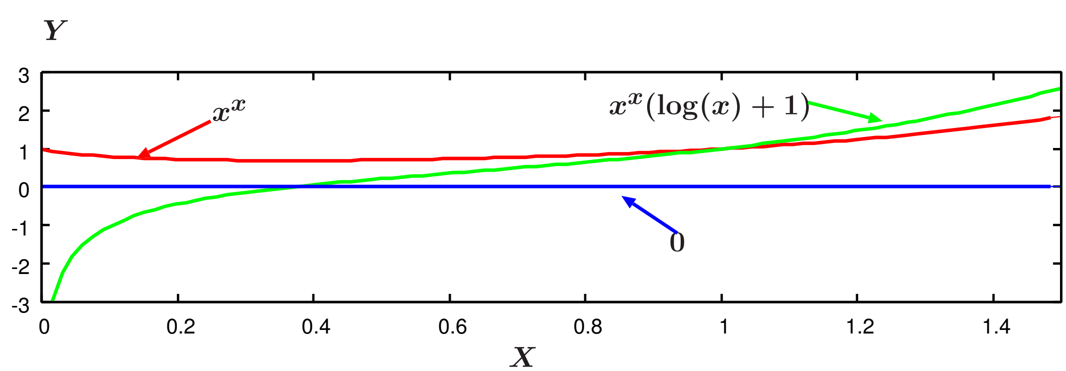
Now let's find $\frac{d\{u\}}{dx}$. Let $g \triangleq 5x-11$, $u = \ln g \because g = 5x-11$. Using the chain rule

$$\frac{d\{u\}}{dx} = \frac{\partial u}{\partial g}\cdot\frac{d\{g\}}{dx} \therefore \frac{d\{u\}}{dx} = 5\cdot\frac{1}{g}(\because \frac{d\{g\}}{dx} = 5, \frac{\partial u}{\partial g} = \frac{1}{g}) \therefore \frac{d\{u\}}{dx} = \frac{5}{5x-11}(\because g \triangleq 5x-11)$$

Now substituting into the product rule, we get

$$\frac{d\{y\}}{dx} = 2^{3x+1}\cdot\frac{5}{5x-11} + 3\ln 2\cdot 2^{3x+1}\cdot\ln(5x-11) = \frac{5\cdot 2^{3x+1}}{5x-11} + 3\ln 2\cdot 2^{3x+1}\cdot\ln(5x-11)$$

21) Differentiate $f(x,y) = \ln(x^2+y^3)$ with respect to $y$.



$u \triangleq x^2 + y^3$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{\ln(x^2+y^3)\}}{dy} = \frac{d\{\ln(u)\}}{dy} = \frac{d\{u\}}{dy}\frac{\partial\{\ln(u)\}}{\partial u} = \frac{d\{x^2+y^3\}}{dy}\frac{\partial\{\ln(u)\}}{\partial u} = 3y^2\frac{1}{u} = \frac{3y^2}{x^2+y^3}$$

22) Differentiate $f(x,y) = \ln(x^2+y^3)$ with respect to $x$.

Let $u = x^2 + y^3$ therefore $\frac{d\{u\}}{dx} = 2x$ and $f(x,y) = \ln u$

$$\frac{d\{f(x,y)\}}{dx} = \frac{\partial f(x,y)}{\partial u}\cdot\frac{d\{u\}}{dx} = \frac{1}{u}\cdot 2x = \frac{2x}{x^2+y^3}$$

23) Differentiate $f(x) = x^x$ with respect to $x$.

let $y = x^x$ therefore take $\ln$'s of both sides.

$$\ln y = \ln x^x \;\; ; \quad \therefore \ln y = x\ln x$$

The differentiation of both sides gives you

$$\frac{1}{y}\frac{d\{y\}}{dx} = \frac{d\{x\ln x\}}{dx}$$

Let $u = x$ and $v = \ln x$. Then $\frac{d\{u\}}{dx} = 1$ and $\frac{d\{v\}}{dx} = \frac{1}{x}$. Using the product rule we get.

$$\frac{d\{x\ln x\}}{dx} = \frac{d\{uv\}}{dx} = v\cdot\frac{d\{u\}}{dx} + u\frac{d\{v\}}{dx} = \ln x\cdot 1 + x\cdot\frac{1}{x} = \ln x + 1$$

$$\therefore \frac{1}{y}\frac{d\{y\}}{dx} = \ln x + 1 \;\; ; \quad \therefore \frac{d\{y\}}{dx} = y\cdot(\ln x + 1) = x^x\cdot(\ln x + 1)(\because y \triangleq x^x)$$

24) Express $\frac{d\{f\}}{ds}$ and $\frac{d\{f\}}{dt}$ using $s$ and $t$ when

$$f = 3^z\cos x\log_2(y)$$
$$x = st$$
$$y = s - t$$
$$z = \frac{s}{t}$$

Since $f$ is the function of $x$, $y$, $z$, we can express $\frac{d\{f\}}{ds}$ and $\frac{d\{f\}}{dt}$ as follows:

$$\frac{d\{f\}}{ds} = \frac{d\{x\}}{ds}\frac{d\{f\}}{dx} + \frac{d\{y\}}{ds}\frac{d\{f\}}{dy} + \frac{d\{z\}}{ds}\frac{d\{f\}}{dz}$$
$$\frac{d\{f\}}{dt} = \frac{d\{x\}}{dt}\frac{d\{f\}}{dx} + \frac{d\{y\}}{dt}\frac{d\{f\}}{dy} + \frac{d\{z\}}{dt}\frac{d\{f\}}{dz}$$

Thus we need $\frac{d\{f\}}{dx}, \frac{d\{f\}}{dy}, \frac{d\{f\}}{dz}, \frac{d\{x\}}{ds}, \frac{d\{y\}}{ds}, \frac{d\{z\}}{ds}, \frac{d\{x\}}{dt}, \frac{d\{y\}}{dt}, \frac{d\{z\}}{dt}$.

$$\frac{d\{f\}}{dx} = \frac{d\{3^z \cos x \log_2(y)\}}{dx} = 3^z \log_2(y)\frac{d\{\cos x\}}{dx} = -3^z \log_2(y) \sin x$$

$$\frac{d\{f\}}{dy} = \frac{d\{3^z \cos x \log_2(y)\}}{dy} = 3^z \cos x \frac{d\left\{\frac{\ln(y)}{\ln 2}\right\}}{dy} = \frac{3^z \cos x}{y \ln 2}$$

$$\frac{d\{f\}}{dz} = \frac{d\{3^z \cos x \log_2(y)\}}{dz} = \cos x \log_2(y)\frac{d\{3^z\}}{dz} = \cos x \log_2(y) 3^z \ln 3$$

$$\frac{d\{x\}}{ds} = \frac{d\{st\}}{ds} = t\frac{d\{s\}}{ds} = t \quad ; \quad \frac{d\{x\}}{dt} = \frac{d\{st\}}{dt} = s\frac{d\{t\}}{dt} = s$$

$$\frac{d\{y\}}{ds} = \frac{d\{s-t\}}{ds} = 1 \quad ; \quad \frac{d\{y\}}{dt} = \frac{d\{s-t\}}{dt} = -1$$

$$\frac{d\{z\}}{ds} = \frac{d\left\{\frac{s}{t}\right\}}{ds} = \frac{1}{t}\frac{d\{s\}}{ds} = \frac{1}{t} \quad ; \quad \frac{d\{z\}}{dt} = \frac{d\left\{\frac{s}{t}\right\}}{dt} = s\frac{d\left\{\frac{1}{t}\right\}}{dt} = -\frac{s}{t^2}$$

Using those results, we get

$$\frac{d\{f\}}{ds} = \frac{d\{x\}}{ds}\frac{d\{f\}}{dx} + \frac{d\{y\}}{ds}\frac{d\{f\}}{dy} + \frac{d\{z\}}{ds}\frac{d\{f\}}{dz} = -t \cdot 3^z \log_2(y) \sin x + \frac{3^z \cos x}{y \ln 2} + \frac{\cos x \log_2(y) 3^z \ln 3}{t} \qquad ①$$

Since the answer should not have $x$, $y$, and $z$, we substitute $x = st$, $y = s - t$, and $z = \frac{s}{t}$ into ① as follows:

$$-t \cdot 3^{\frac{s}{t}} \log_2(s-t) \sin(st) + \frac{3^{\frac{s}{t}} \cos(st)}{(s-t)\ln 2} + \frac{\cos(st) \log_2(s-t) 3^{\frac{s}{t}} \ln 3}{t}$$

In the same way we can obtain $\frac{d\{f\}}{dt}$ as follows:

$$\frac{d\{f\}}{dt} = \frac{d\{x\}}{dt}\frac{d\{f\}}{dx} + \frac{d\{y\}}{dt}\frac{d\{f\}}{dy} + \frac{d\{z\}}{dt}\frac{d\{f\}}{dz} = -s \cdot 3^z \log_2(y) \sin x - \frac{3^z \cos x}{y \ln 2} - \frac{s \cos x \log_2(y) 3^z \ln 3}{t^2} \qquad ②$$

Since the answer should not have $x$, $y$, and $z$, we substitute $x = st$, $y = s - t$, and $z = \frac{s}{t}$ into ② as follows:

$$-s \cdot 3^{\frac{s}{t}} \log_2(s-t) \sin(st) - \frac{3^{\frac{s}{t}} \cos(st)}{(s-t)\ln 2} - \frac{s \cos(st) \log_2(s-t) 3^{\frac{s}{t}} \ln 3}{t^2}$$

25) Express $\frac{\partial\{f\}}{\partial u}$ using $s$, $t$, and $u$ when

$$f = x + 2^{yz}$$
$$x = \frac{t}{u}$$
$$y = s - t - u$$
$$z = su$$

Since $f$ is the function of $x$, $y$, $z$, we can express $\frac{d\{f\}}{ds}$, $\frac{d\{f\}}{dt}$ and $\frac{\partial\{f\}}{\partial u}$ as follows:

$$\frac{d\{f\}}{ds} = \frac{d\{x\}}{ds}\frac{d\{f\}}{dx} + \frac{d\{y\}}{ds}\frac{d\{f\}}{dy} + \frac{d\{z\}}{ds}\frac{d\{f\}}{dz}$$
$$\frac{d\{f\}}{dt} = \frac{d\{x\}}{dt}\frac{d\{f\}}{dx} + \frac{d\{y\}}{dt}\frac{d\{f\}}{dy} + \frac{d\{z\}}{dt}\frac{d\{f\}}{dz}$$
$$\frac{\partial\{f\}}{\partial u} = \frac{\partial\{x\}}{\partial u}\frac{d\{f\}}{dx} + \frac{\partial\{y\}}{\partial u}\frac{d\{f\}}{dy} + \frac{\partial\{z\}}{\partial u}\frac{d\{f\}}{dz}$$

Now we just need to find out $\frac{\partial\{f\}}{\partial u}$ Thus we need $\frac{d\{f\}}{dx}$,$\frac{d\{f\}}{dy}$,$\frac{d\{f\}}{dz}$, $\frac{\partial\{x\}}{\partial u}$,$\frac{\partial\{y\}}{\partial u}$, and $\frac{\partial\{z\}}{\partial u}$.

$$\frac{d\{f\}}{dx}=\frac{d\{x+2^{yz}\}}{dx}=\frac{d\{x\}}{dx}+\frac{d\{2^{yz}\}}{dx}=1$$

$$\frac{d\{f\}}{dy}=\frac{d\{x+2^{yz}\}}{dy}=\frac{d\{x\}}{dy}+\frac{d\{2^{yz}\}}{dy}=\frac{d\{(2^z)^y\}}{dy}=\frac{d\{A^y\}}{dy}(\because A\triangleq 2^z)=A^y\ln A=(2^z)^y\ln 2^z=z2^{yz}\ln 2$$

$$\frac{d\{f\}}{dz}=\frac{d\{x+2^{yz}\}}{dz}=\frac{d\{x\}}{dz}+\frac{d\{2^{yz}\}}{dz}=\frac{d\{A^z\}}{dz}(\because A\triangleq 2^y)=A^z\ln A=(2^y)^z\ln 2^y=y2^{yz}\ln 2$$

$$\frac{\partial\{x\}}{\partial u}=\frac{\partial\left\{\frac{t}{u}\right\}}{\partial u}=t\frac{\partial\left\{\frac{1}{u}\right\}}{\partial u}=-\frac{t}{u^2}$$

$$\frac{\partial\{y\}}{\partial u}=\frac{\partial\{s-t-u\}}{\partial u}=\frac{\partial\{s\}}{\partial u}+\frac{\partial\{-t\}}{\partial u}+\frac{\partial\{-u\}}{\partial u}=0+0-1=-1$$

$$\frac{\partial\{z\}}{\partial u}=\frac{\partial\{su\}}{\partial u}=s\frac{\partial\{u\}}{\partial u}=s$$

Therefore

$$\frac{\partial\{f\}}{\partial u}=\frac{\partial\{x\}}{\partial u}\frac{d\{f\}}{dx}+\frac{\partial\{y\}}{\partial u}\frac{d\{f\}}{dy}+\frac{\partial\{z\}}{\partial u}\frac{d\{f\}}{dz}=-\frac{t}{u^2}-z2^{yz}\ln 2+sy2^{yz}\ln 2 \qquad ①$$

Since the answer should not have $y$, and $z$, we substitute $y=s-t-u$, and $z=su$ into ① as follows:

$$-\frac{t}{u^2}-z2^{yz}\ln 2+sy2^{yz}\ln 2=-\frac{t}{u^2}-su2^{(s-t-u)su}\ln 2+s(s-t-u)2^{(s-t-u)su}\ln 2$$

26) Simplify $\ln x\cdot 1+x\cdot\frac{1}{x}$.

$$\ln x\cdot 1+x\cdot\frac{1}{x}=\ln x+\frac{x}{x}=\ln x+1$$

27) Solve these equations $7x+y=11$ and $35x+2y=9$.

$$7x+y=11 \qquad ①$$
$$35x+2y=9 \qquad ②$$

① $\times 2$ gives:

$$14x+2y=22 \qquad ③$$

③-② gives us

$$14x-35x+2y-2y=22-9\ ;\quad \therefore -21x=13\ ;\quad \therefore x=\frac{-13}{21}$$

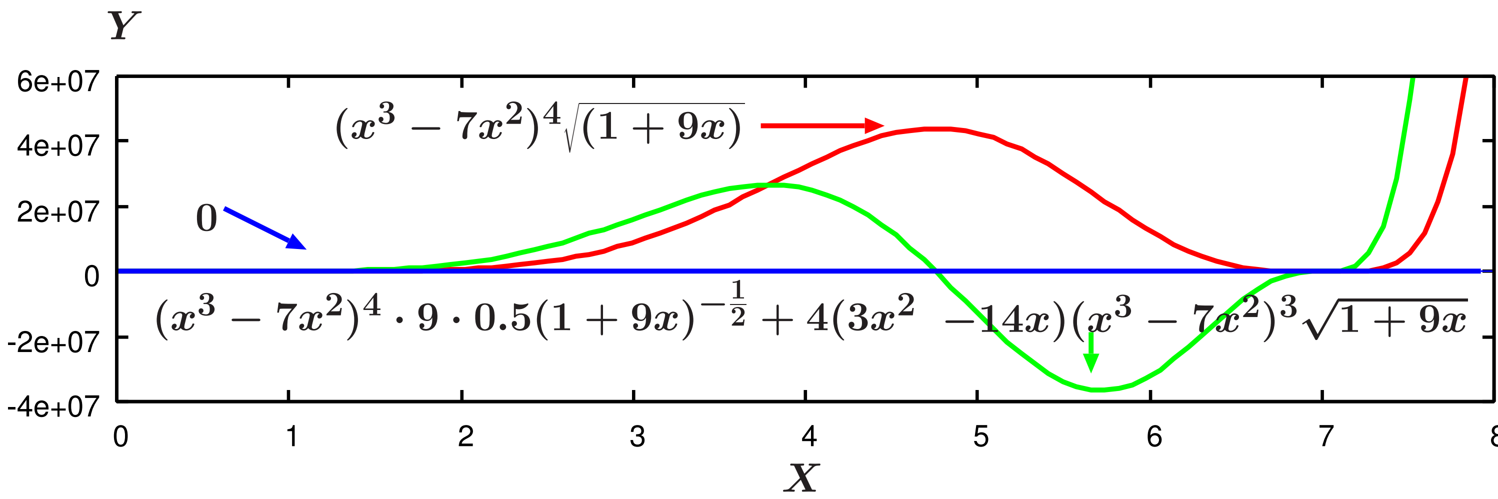
Finally we get

$$y=11-7x=11-7\cdot\left(\frac{-13}{21}\right)=11+\frac{91}{21}=\frac{11\cdot 21+91}{21}=\frac{322}{21}=\frac{46}{3}$$

28) What is $\frac{3}{7}-\frac{1}{14}$.

$$\frac{3}{7}-\frac{1}{14}=\frac{3\times 2}{7\times 2}-\frac{1}{14}=\frac{6}{14}-\frac{1}{14}=\frac{6-1}{14}=\frac{5}{14}$$

29) Differentiate $f(x)=(x^3-7x^2)^4(1+9x)^{\frac{1}{2}}$ with regard to $x$.




$$\frac{d\{f(x)\}}{dx}=(x^3-7x^2)^4\frac{d\left\{(1+9x)^{\frac{1}{2}}\right\}}{dx}+\frac{d\{(x^3-7x^2)^4\}}{dx}(1+9x)^{\frac{1}{2}}$$

In order to find out $\frac{d\left\{(1+9x)^{\frac{1}{2}}\right\}}{dx}$, $u\triangleq 1+9x$.

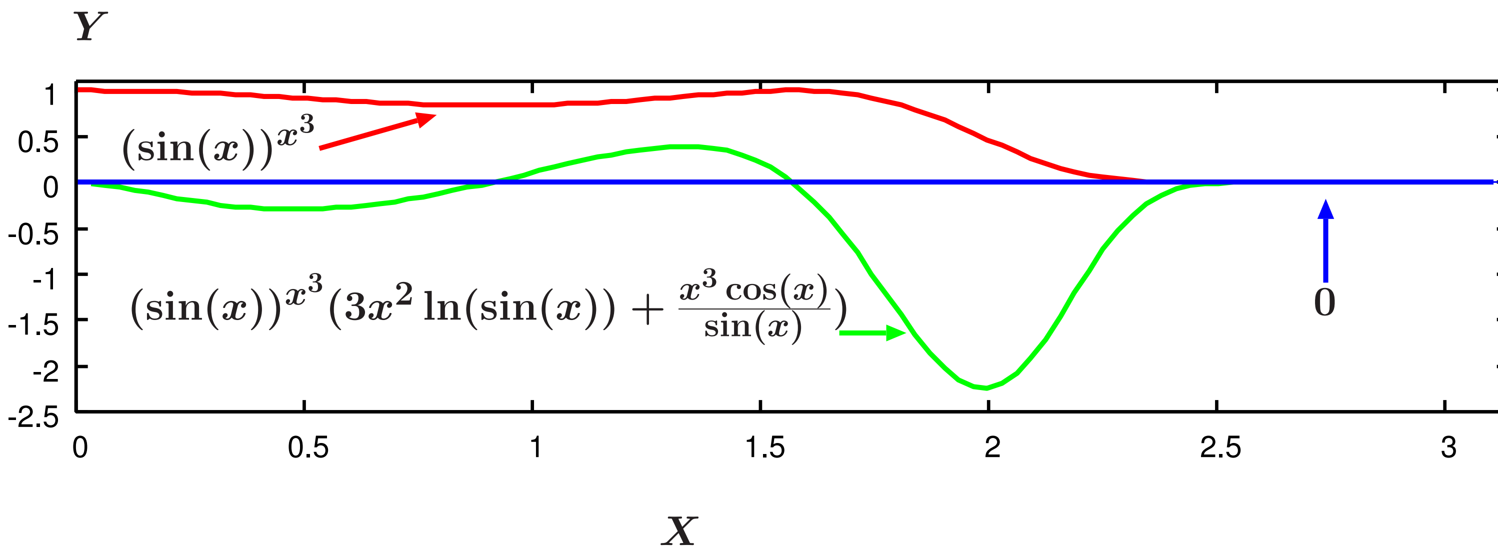
$$\therefore \frac{d\left\{(1+9x)^{\frac{1}{2}}\right\}}{dx}=\frac{d\left\{u^{\frac{1}{2}}\right\}}{dx}=\frac{d\{u\}}{dx}\frac{\partial\{u^{\frac{1}{2}}\}}{\partial u}=\frac{d\{1+9x\}}{dx}\cdot\frac{1}{2}\cdot u^{-\frac{1}{2}}=9\cdot\frac{1}{2}\cdot(1+9x)^{-\frac{1}{2}}$$

In order to find out $\frac{d\{(x^3-7x^2)^4\}}{dx}$, by letting $v\triangleq x^3-7x^2$

$$\frac{d\{(x^3-7x^2)^4\}}{dx}=\frac{d\{v^4\}}{dx}=\frac{d\{v\}}{dx}\frac{\partial\{v^4\}}{\partial v}=\frac{d\{x^3-7x^2\}}{dx}\cdot 4\cdot v^3=(3x^2-14x)\cdot 4\cdot(x^3-7x^2)^3$$

$$\therefore \frac{d\{f(x)\}}{dx}=(x^3-7x^2)^4\frac{d\left\{(1+9x)^{\frac{1}{2}}\right\}}{dx}+\frac{d\{(x^3-7x^2)^4\}}{dx}(1+9x)^{\frac{1}{2}}$$
$$=(x^3-7x^2)^4\cdot 9\cdot\frac{1}{2}\cdot(1+9x)^{-\frac{1}{2}}+(3x^2-14x)\cdot 4\cdot(x^3-7x^2)^3\cdot(1+9x)^{\frac{1}{2}}$$

30) Differentiate $f(x)=(\sin x)^{x^3}$ with regard to $x$.




By applying the natural logarithm to both sides of the equation we get

$$\ln f(x) = \ln(\sin x)^{x^3} = x^3 \ln(\sin x)$$

By differentiating both sides of this equation we get

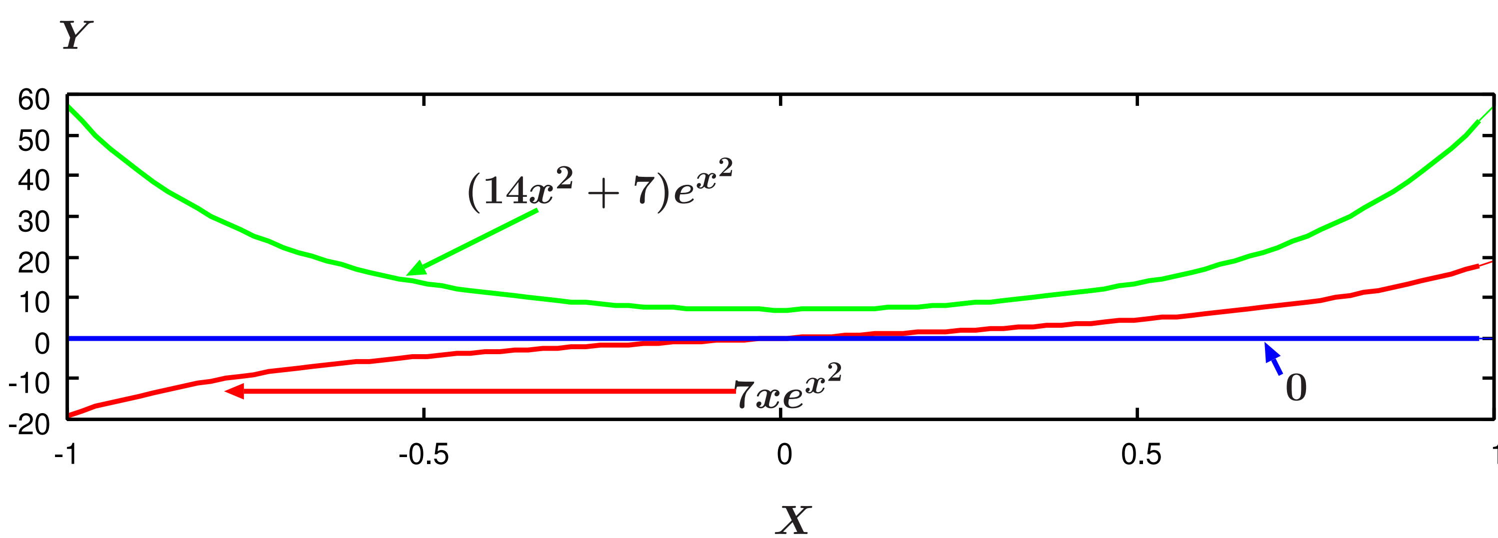
$$\frac{d\{\ln f(x)\}}{dx} = \frac{d\{x^3\ln(\sin x)\}}{dx}$$

$$\therefore \frac{d\{f(x)\}}{dx}\frac{\partial\{\ln f(x)\}}{\partial\{f(x)\}} = \frac{d\{x^3\}}{dx}\ln(\sin x) + x^3\frac{d\{\ln(\sin x)\}}{dx}$$

$$\therefore \frac{d\{f(x)\}}{dx}\frac{1}{f(x)} = 3x^2\ln(\sin x) + x^3\frac{d\{\sin x\}}{dx}\frac{\partial\{\ln(\sin x)\}}{\partial\{\sin x\}} = 3x^2\ln(\sin x) + x^3\cos x\frac{1}{\sin x}$$

$$\therefore \frac{d\{f(x)\}}{dx} = f(x)\left(3x^2\ln(\sin x) + x^3\cos x\frac{1}{\sin x}\right) = (\sin x)^{x^3}\left(3x^2\ln(\sin x) + x^3\frac{\cos x}{\sin x}\right)$$

31) Differentiate $f(x) = 7x\mathrm{e}^{x^2}$ with regard to $x$ and express $\frac{d\{f(x)\}}{dx}$ using $f(x)$ and $x$ (*i.e.,* produce a differential equation).



$$\frac{d\{f(x)\}}{dx} = 7x\frac{d\left\{\mathfrak{e}^{x^2}\right\}}{dx} + \frac{d\{7x\}}{dx}\mathfrak{e}^{x^2}$$

When $x^2$ is replaced with $u$,

$$\frac{\partial \mathfrak{e}^{x^2}}{\partial x} = \frac{d\{\mathfrak{e}^u\}}{dx} = \frac{\partial u}{\partial x}\frac{\partial \mathfrak{e}^u}{\partial u} = \frac{\partial x^2}{\partial x}\cdot\frac{\partial \mathfrak{e}^u}{\partial u} = (2x)\mathfrak{e}^u = 2x\mathfrak{e}^{x^2} \because u \triangleq x^2.$$

Therefore

$$\frac{d\{f(x)\}}{dx} = 7x(2x\mathfrak{e}^{x^2}) + \frac{d\{7x\}}{dx}\mathfrak{e}^{x^2} = (14x^2+7)\mathfrak{e}^{x^2}$$

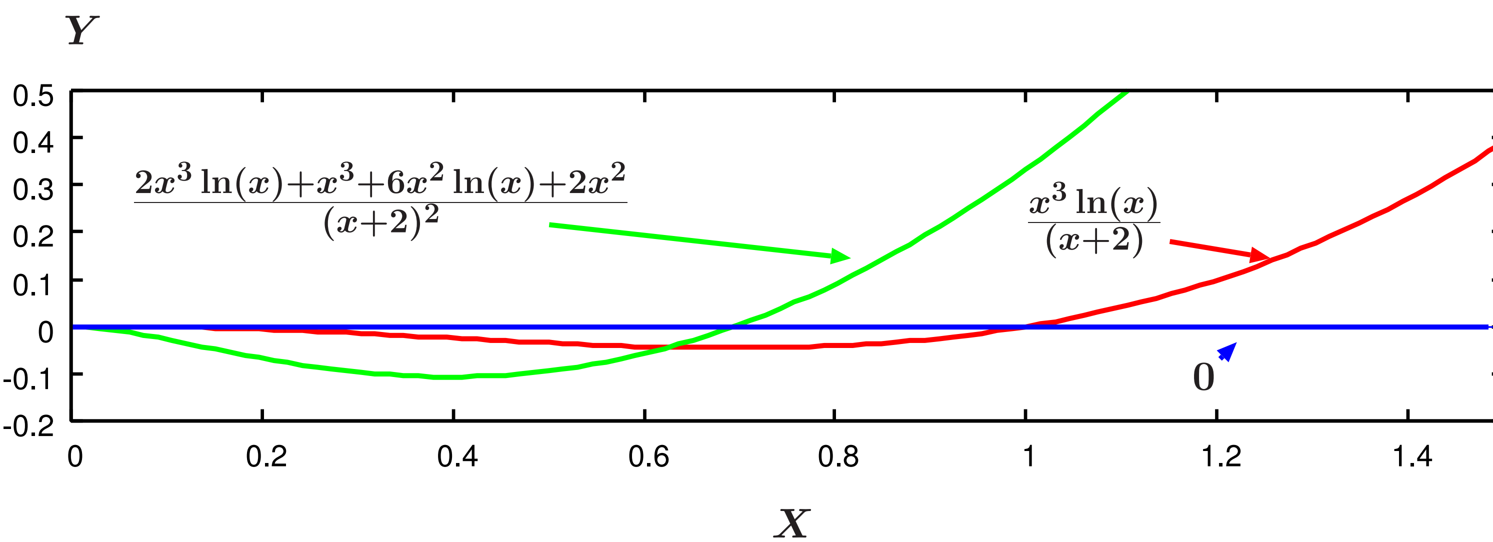
Since $(14x^2+7)\mathfrak{e}^{x^2}$ can be re-written as $14x^2\mathfrak{e}^{x^2} + 7\mathfrak{e}^{x^2}$ or $2x\cdot(7x\mathfrak{e}^{x^2}) + \frac{1}{x}\cdot 7x\mathfrak{e}^{x^2}$,

$$\frac{d\{f(x)\}}{dx} = 2x\cdot(7x\mathfrak{e}^{x^2}) + \frac{1}{x}\cdot 7x\mathfrak{e}^{x^2} = 2xf(x) + \frac{f(x)}{x}(\because f(x) \triangleq 7x\mathfrak{e}^{x^2})$$

Times those by $x$ and re-arrange

$$\therefore x\frac{d\{f(x)\}}{dx} - 2x^2f(x) - f(x) = 0$$

32) Differentiate $f(x) = \frac{x^3\ln x}{x+2}$ with regard to $x$.




$$\frac{d\{f(x)\}}{dx} = \frac{\frac{d\{x^3\ln x\}}{dx}(x+2) - (x^3\ln x)\frac{d\{x+2\}}{dx}}{(x+2)^2} = \frac{\left[\frac{d\{x^3\}}{dx}\ln x + x^3\frac{d\{\ln x\}}{dx}\right](x+2) - (x^3\ln x)\frac{d\{x+2\}}{dx}}{(x+2)^2}$$

$$= \frac{\left[3x^2\ln x + x^3\frac{1}{x}\right](x+2) - (x^3\ln x)\frac{d\{x+2\}}{dx}}{(x+2)^2} = \frac{\left[3x^2\ln x + x^2\right](x+2) - (x^3\ln x)\frac{d\{x+2\}}{dx}}{(x+2)^2}$$

$$= \frac{\left[3x^3\ln x + x^3 + 6x^2\ln x + 2x^2\right] - (x^3\ln x)\cdot 1}{(x+2)^2} = \frac{2x^3\ln x + x^3 + 6x^2\ln x + 2x^2}{(x+2)^2}$$

Or using this alternative method.

Let $y = f(x) = \frac{x^3 \ln x}{x+2}$. Let $u = x^3 \ln(x)$ and $v = x + 2$. To solve this equation we use the quotient rule, which is

$$\frac{d\{y\}}{dx} = \frac{v \cdot \frac{d\{u\}}{dx} - u \cdot \frac{d\{v\}}{dx}}{v^2}$$

Now let's find $\frac{d\{u\}}{dx}$. Let $w = x^3$ and $z = \ln(x)$. Then we get $\frac{d\{w\}}{dx} = 3x^2$, and $\frac{d\{z\}}{dx} = \frac{1}{x}$. Using the Product rule

$$\frac{d\{u\}}{dx} = w \cdot \frac{d\{z\}}{dx} + z \cdot \frac{d\{w\}}{dx} = \frac{x^3}{x} + 3x^2 \cdot \ln x = x^2 + 3x^2 \cdot \ln x$$

And $v = x + 2$ simply becomes $\frac{d\{v\}}{dx} = 1$. Now substituting in to

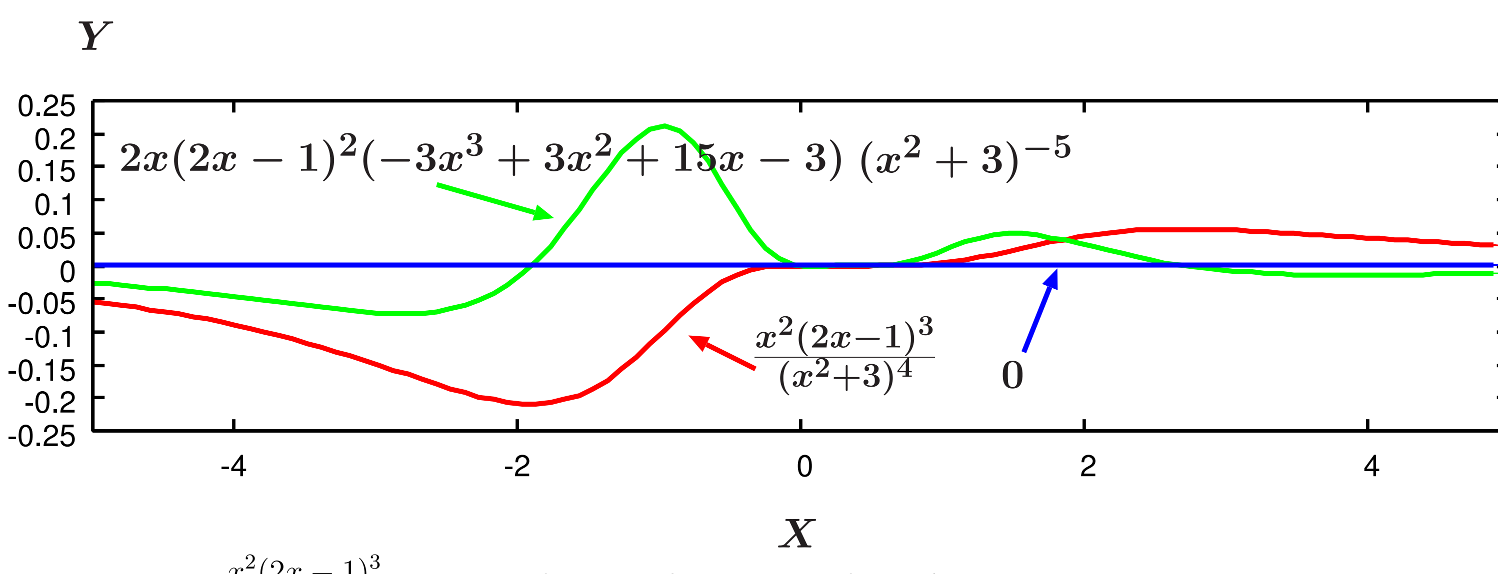
$$\frac{d\{y\}}{dx} = \frac{v \cdot \frac{d\{u\}}{dx} - u \cdot \frac{d\{v\}}{dx}}{v^2}$$

We get

$$\frac{d\{y\}}{dx} = \frac{(x+2) \cdot (x^2 + 3x^2 \cdot \ln x) - x^3 \ln x \cdot 1}{(x+2)^2}$$

$$= \frac{x^3 + 3x^3 \cdot \ln x + 2x^2 + 6x^2 \cdot \ln x - x^3 \cdot \ln x}{(x+2)^2} = \frac{2x^3 \ln x + x^3 + 6x^2 \ln x + 2x^2}{(x+2)^2}$$

33) Differentiate $f(x) = \frac{x^2(2x-1)^3}{(x^2+3)^4}$ with regard to $x$.




Let $y = f(x) = \frac{x^2(2x-1)^3}{(x^2+3)^4}$. Let $u = x^2(2x-1)^3$ and $v = (x^2+3)^4$. To solve this equation we use the quotient rule, which is

$$\frac{d\{y\}}{dx} = \frac{v \cdot \frac{d\{u\}}{dx} - u \cdot \frac{d\{v\}}{dx}}{v^2}$$

Now let's find $\frac{d\{u\}}{dx}$. Let $w = x^2$ and $z = (2x-1)^3$. Then we obtain $\frac{d\{z\}}{dx} = 6(2x-1)^2$ and $\frac{d\{w\}}{dx} = 2x$. Using the Product rule

$$\frac{d\{u\}}{dx} = w \cdot \frac{d\{z\}}{dx} + z \cdot \frac{d\{w\}}{dx} \quad ; \quad \therefore \frac{d\{u\}}{dx} = x^2 \cdot 6(2x-1)^2 + (2x-1)^3 \cdot 2x = 6x^2 \cdot (2x-1)^2 + 2x \cdot (2x-1)^3$$
$$= 2x(2x-1)^2 \cdot (3x + (2x-1)) = 2x(2x-1)^2 \cdot (5x-1)$$

Now let's find $\frac{d\{v\}}{dx}$. Let $k = x^2 + 3$. Then $v = k^4$. Then we get $\frac{\partial v}{\partial k} = 4k^3$, $\frac{d\{k\}}{dx} = 2x$ . Using the chain rule

$$\frac{d\{v\}}{dx} = \frac{\partial v}{\partial k} \cdot \frac{d\{k\}}{dx} = 2x \cdot (4k^3) = 8x(x^2+3)^3 (\because k \triangleq x^2 + 3)$$

Now substituting in to

$$\frac{d\{y\}}{dx} = \frac{v \cdot \frac{d\{u\}}{dx} - u \cdot \frac{d\{v\}}{dx}}{v^2}$$

We get

$$\frac{(x^2+3)^4 \cdot 2x(2x-1)^2 \cdot (5x-1) - x^2(2x-1)^3 \cdot 8x(x^2+3)^3}{((x^2+3)^4)^2}$$
$$= \frac{2x(x^2+3)^3(2x-1)^2\{(x^2+3)(5x-1) - 4x^2(2x-1)\}}{(x^2+3)^8}$$
$$= \frac{2x(x^2+3)^3(2x-1)^2\{-3x^3+3x^2+15x-3\}}{(x^2+3)^8} = \frac{2x(2x-1)^2\{-3x^3+3x^2+15x-3\}}{(x^2+3)^5}$$

34) Express $\frac{d\{f\}}{dt}$ using $s$ and $t$ when

$$f = \frac{\sin(z)\log_5(x)}{3^y}$$
$$x = \mathfrak{e}^s + \cos(t)$$
$$y = \mathfrak{e}^t \log_3 s$$
$$z = \frac{\cos t}{1 + \sin s}$$

Since $f$ is the function of $x$, $y$, $z$, we can express $\frac{d\{f\}}{dt}$ as follows:

$$\frac{d\{f\}}{dt} = \frac{d\{x\}}{dt}\frac{d\{f\}}{dx} + \frac{d\{y\}}{dt}\frac{d\{f\}}{dy} + \frac{d\{z\}}{dt}\frac{d\{f\}}{dz}$$

Thus we need $\frac{d\{f\}}{dx}, \frac{d\{f\}}{dy}, \frac{d\{f\}}{dz}, \frac{d\{x\}}{dt}, \frac{d\{y\}}{dt}, \frac{d\{z\}}{dt}$.

$$\frac{d\{f\}}{dx} = \frac{d\left\{\frac{\sin(z)\log_5(x)}{3^y}\right\}}{dx}$$
$$= \frac{d\left\{\frac{\sin(z)\frac{\ln(x)}{\ln 5}}{3^y}\right\}}{dx} = \frac{\sin(z)}{x \cdot 3^y \cdot \ln 5}$$
$$\frac{d\{f\}}{dy} = \frac{d\left\{\frac{\sin(z)\log_5(x)}{3^y}\right\}}{dy} = \sin(z)\log_5(x)\frac{d\{3^{-y}\}}{dy}$$
$$= \sin(z)\log_5(x)\frac{d\{3^u\}}{dy}(\because u \triangleq -y) = \sin(z)\log_5(x)\frac{d\{u\}}{dy}\frac{\partial\{3^u\}}{\partial u}$$
$$= \sin(z)\log_5(x)\frac{d\{-y\}}{dy}3^u \cdot \ln 3 = \sin(z)\log_5(x)\cdot(-1)3^{-y}\cdot \ln 3 = -\frac{\sin(z)\log_5(x)\cdot \ln 3}{3^y}$$
$$\frac{d\{f\}}{dz} = \frac{d\left\{\frac{\sin(z)\log_5(x)}{3^y}\right\}}{dz} = \frac{\cos(z)\log_5(x)}{3^y}$$
$$\frac{d\{x\}}{dt} = \frac{d\{\mathfrak{e}^s + \cos(t)\}}{dt} = -\sin(t)$$
$$\frac{d\{y\}}{dt} = \frac{d\{\mathfrak{e}^t \log_3 s\}}{dt} = \mathfrak{e}^t \log_3 s$$
$$\frac{d\{z\}}{dt} = \frac{d\left\{\frac{\cos t}{1+\sin s}\right\}}{dt} = \frac{-\sin t}{1+\sin s}$$

$$\frac{d\{f\}}{dt} = \frac{d\{x\}}{dt}\frac{d\{f\}}{dx} + \frac{d\{y\}}{dt}\frac{d\{f\}}{dy} + \frac{d\{z\}}{dt}\frac{d\{f\}}{dz}$$
$$= -\sin(t)\cdot\frac{\sin(z)}{x\cdot 3^y \cdot \ln 5} - \mathfrak{e}^t\log_3(s)\cdot\frac{\sin(z)\log_5(x)\cdot\ln 3}{3^y} - \frac{\sin t}{1+\sin s}\cdot\frac{\cos(z)\log_5(x)}{3^y}$$
$$= 3^{-y}\left\{-\frac{\sin(z)\sin(t)}{x\ln 5} - \mathfrak{e}^t\log_3(s)\sin(z)\log_5(x)\ln 3 - \frac{\sin(t)\cos(z)\log_5(x)}{1+\sin s}\right\}$$
$$= 3^{-\mathfrak{e}^t\log_3 s}\left\{-\frac{\sin(\frac{\cos t}{1+\sin s})\sin(t)}{(\mathfrak{e}^s+\cos(t))\ln 5} - \mathfrak{e}^t\log_3(s)\sin(\frac{\cos t}{1+\sin s})\log_5(\mathfrak{e}^s+\cos(t))\ln 3\right.$$
$$\left. - \frac{\sin(t)\cos(\frac{\cos t}{1+\sin s})\log_5(\mathfrak{e}^s+\cos(t))}{1+\sin s}\right\}$$

**DAY4**

35) For the function of $f(x,y) = 3x^2 + 2xy^2 + x^2y - 4y^2 + 12 - xy - 2x - 5y$

a) find $\frac{d\{f(x,y)\}}{dx}, \frac{d\{f(x,y)\}}{dy}, \frac{d^2 f(x,y)}{dx^2}, \frac{\partial^2 f(x,y)}{\partial y \partial x}$ and $\frac{\partial^2 f(x,y)}{\partial y^2}$

We need to partially differentiate the function $f(x,y)$. For example if we are differentiating with respect to $x$ and the function contains both $x$ and $y$ then we treat $x$ the same as we normally do but $y$ would be treated like a constant. So if you had $yx$, $y$ would be the constant coefficient of $x$ while you were differentiating with respect to $x$. Therefore the answers are as follows.

$$\frac{d\{f(x,y)\}}{dx} = 6x + 2y^2 + 2xy - y - 2; \frac{d\{f(x,y)\}}{dy} = 4xy + x^2 - 8y - x - 5$$
$$\frac{d^2 f(x,y)}{dx^2} = 6 + 2y; \frac{\partial^2 f(x,y)}{\partial y \partial x} = 4y + 2x - 1; \frac{\partial^2 f(x,y)}{\partial y^2} = 4x - 8$$

b) find the gradient $\nabla f$ of the function $f(x,y)$ at the point $(0,1)$ The definition of $\nabla f$ in 3D is

$$\nabla f = \frac{d\{f\}}{dx}\boldsymbol{i} + \frac{d\{f\}}{dy}\boldsymbol{j} + \frac{d\{f\}}{dz}\boldsymbol{k}.$$

Now we handle two dimensional case at the point $(0,1)$

$$\begin{aligned}\nabla f &= \left.\frac{d\{f\}}{dx}\right|_{(x,y)=(0,1)}\boldsymbol{i} + \left.\frac{d\{f\}}{dy}\right|_{(x,y)=(0,1)}\boldsymbol{j} \\ &= \left.(6x + 2y^2 + 2xy - y - 2)\right|_{(x,y)=(0,1)}\boldsymbol{i} + \left.(4xy + x^2 - 8y - x - 5)\right|_{(x,y)=(0,1)}\boldsymbol{j} \\ &= (6\cdot 0 + 2\cdot 1^2 + 2\cdot 0\cdot 1 - 1 - 2)\boldsymbol{i} + (4\cdot 0\cdot 1 + 0^2 - 8\cdot 1 - 0 - 5)\boldsymbol{j} = -\boldsymbol{i} - 13\boldsymbol{j}\end{aligned}$$

So at the point $(x,y) = (0,1)$, the gradient is $(-1,-13)$.

c) find the directional derivative of $f$ at the point $(x,y) = (0,1)$ in the direction of the vector $\boldsymbol{n} = 3\boldsymbol{i} - 4\boldsymbol{j}$.

At the point $(x,y,z) = (0,1)$ the gradient is $-\boldsymbol{i} - 13\boldsymbol{j} \triangleq \boldsymbol{v}$. Now we need to find the magnitude of $\boldsymbol{n}$-directional component of $\boldsymbol{v}$. When the angle between $\boldsymbol{n}$ and $\boldsymbol{v}$ is $\theta$ the magnitude of $\boldsymbol{n}$-directional component of $\boldsymbol{v}$ can be written as $|\boldsymbol{v}|\cos\theta$. As $\boldsymbol{n}\cdot\boldsymbol{v} = |\boldsymbol{n}||\boldsymbol{v}|\cos\theta$, we can obtain the magnitude as

$$|\boldsymbol{v}|\cos\theta = |\boldsymbol{v}|\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}||\boldsymbol{v}|} = \frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}|}$$

The magnitude of $\boldsymbol{n}$ is $|\boldsymbol{n}| = \sqrt{3^2 + (-4)^2} = \sqrt{9+16} = 5$. Therefore

$$\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}|} = \frac{3\cdot(-1) + (-4)\cdot(-13)}{\sqrt{3^2+4^2}} = \frac{52-3}{\sqrt{25}} = \frac{49}{5}$$

d) calculate the approximate change of $f(x,y)$ in moving from the position $(x,y) = (0,1)$ to a new position of $(x,y) = (-0.01, 1.01)$ In case of three dimensional problems, the change of $f$ is

$$df = \frac{d\{f\}}{dx}dx + \frac{d\{f\}}{dy}dy + \frac{d\{f\}}{dz}dz$$

Now we need to find the change of $f$ at $(x,y) = (0,1)$ and $(dx,dy) = (-0.01-0, 1.01-1) = (-0.01, 0.01)$.

$$\begin{aligned}df &= \left.\frac{d\{f\}}{dx}\right|_{(x,y)=(0,1)} dx|_{dx=-0.01} + \left.\frac{d\{f\}}{dy}\right|_{(x,y)=(0,1)} dy|_{dy=0.01} \\ &= -1\cdot(-0.01) - 13\cdot(0.01) = 0.01 - 0.13 = -0.12\end{aligned}$$

In fact $f(0,1) = 3$ and $f(-0.01, 1.01) = 2.8797$. Thus the difference is $-0.120301$.

e) calculate the percentage error in calculating $f(x,y)$ in around the position $(x,y)=(1,-1)$ and there are errors of -1% in $x$ and +0.1% in $y$ In case of three dimensional problems, the percentage error in calculating $f$ is

$$\frac{df}{f}=\frac{\frac{d\{f\}}{dx}dx}{f}+\frac{\frac{d\{f\}}{dy}dy}{f}+\frac{\frac{d\{f\}}{dz}dz}{f}$$

Now we need to find the percentage error in calculating $f$ at $(x,y)=(1,-1)$ and $(\frac{dx}{x},\frac{dy}{y})=(-0.01,0.001)$.

$$\begin{aligned}\frac{df}{f}&=\frac{\frac{d\{f\}}{dx}}{f}\cdot x\Bigg|_{(x,y)=(1,-1)}\cdot\frac{dx}{x}\Bigg|_{\frac{dx}{x}=-0.01}+\frac{\frac{d\{f\}}{dy}}{f}\cdot y\Bigg|_{(x,y)=(1,-1)}\cdot\frac{dy}{y}\Bigg|_{\frac{dy}{y}=0.001}\\
&=\frac{(6x+2y^2+2xy-y-2)x}{3x^2+2xy^2+x^2y-4y^2+12-xy-2x-5y}\Bigg|_{\substack{x=1\\ y=-1}}\cdot(-0.01)\\
&+\frac{(4xy+x^2-8y-x-5)y}{3x^2+2xy^2+x^2y-4y^2+12-xy-2x-5y}\Bigg|_{\substack{x=1\\ y=-1}}\cdot(0.001)\\
&=\frac{6+2(-1)^2-2+1-2}{3+2(-1)^2-1-4(-1)^2+12+1-2+5}\cdot(-0.01)\\
&+\frac{(4(-1)+1+8-1-5)\cdot(-1)}{3+2(-1)^2-1-4\cdot(-1)^2+12+1-2+5}\cdot(0.001)\\
&=\frac{5}{16}\cdot(-0.01)+\frac{1}{16}\cdot(0.001)=\frac{-0.05+0.001}{16}=-0.0030625\end{aligned}$$

Thus the percent error is -0.30625 % whilst the real error is $\frac{15.9512-16}{16}=-0.00305$ because $3x^2+2xy^2+x^2y-4y^2+12-xy-2x-5y\big|_{\substack{x=0.99\\ y=-1.001}}=15.9512$ and

$$3x^2+2xy^2+x^2y-4y^2+12-xy-2x-5y\big|_{\substack{x=1\\ y=-1}}=16$$

36) For the function of $f(x,y,z)=x^3-y^2+z-x^2z^3+yz^2-xy^3+1$

a) find the gradient $\nabla f$ of the function $f(x,y,z)$ at the point $(1,-1,1)$ The definition of $\nabla f$ in 3D is

$$\nabla f=\frac{d\{f\}}{dx}\boldsymbol{i}+\frac{d\{f\}}{dy}\boldsymbol{j}+\frac{d\{f\}}{dz}\boldsymbol{k}.$$

Now we handle three dimensional case at the point $(1,-1,1)$

$$\begin{aligned}\nabla f&=\frac{d\{f\}}{dx}\bigg|_{(x,y,z)=(1,-1,1)}\boldsymbol{i}+\frac{d\{f\}}{dy}\bigg|_{(x,y,z)=(1,-1,1)}\boldsymbol{j}+\frac{d\{f\}}{dz}\bigg|_{(x,y,z)=(1,-1,1)}\boldsymbol{k}\\
&=(3x^2-2xz^3-y^3)\big|_{(x,y,z)=(1,-1,1)}\boldsymbol{i}+(-2y+z^2-3xy^2)\big|_{(x,y,z)=(1,-1,1)}\boldsymbol{j}\\
&\quad+(1-3x^2z^2+2yz)\big|_{(x,y,z)=(1,-1,1)}\boldsymbol{k}\\
&=(3-2-(-1))\boldsymbol{i}+(-(-2)+1-3)\boldsymbol{j}+(1-3+2(-1))\boldsymbol{k}=2\boldsymbol{i}+0\boldsymbol{j}-4\boldsymbol{k}\end{aligned}$$

So at the point $(x, y, z) = (1, -1, 1)$, the gradient is $(2, 0, -4)$.

b) find the directional derivative of $f$ at the point $(x, y, z) = (1, -1, 1)$ in the direction of the vector $\boldsymbol{n} = 3\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$. At the point $(x, y, z) = (1, -1, 1)$ the gradient is $2\boldsymbol{i} - 4\boldsymbol{k} \triangleq \boldsymbol{v}$. Now we need to find the magnitude of $\boldsymbol{n}$-directional component of $\boldsymbol{v}$. When the angle between $\boldsymbol{n}$ and $\boldsymbol{v}$ is $\theta$ the magnitude of $\boldsymbol{n}$-directional component of $\boldsymbol{v}$ can be written as $|\boldsymbol{v}|\cos\theta$. As $\boldsymbol{n}\cdot\boldsymbol{v} = |\boldsymbol{n}||\boldsymbol{v}|\cos\theta$, we can obtain the magnitude as

$$|\boldsymbol{v}|\cos\theta = |\boldsymbol{v}|\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}||\boldsymbol{v}|} = \frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}|}$$

The magnitude of $\boldsymbol{n}$ is $|\boldsymbol{n}| = \sqrt{3^2 + (-4)^2 + 5^2} = \sqrt{9 + 16 + 25} = 5\sqrt{2}$. Therefore

$$\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}|} = \frac{3\cdot(2) + (-4)\cdot(0) + 5\cdot(-4)}{\sqrt{3^2 + 4^2 + 5^2}} = \frac{6 - 20}{\sqrt{50}} = \frac{-14}{5\sqrt{2}}$$

c) calculate the approximate change of $f(x, y)$ in moving from the position $(x, y, z) = (1, -1, 1)$ to a new position of $(x, y) = (1.01, -0.99, 0.99)$ In case of three dimensional problems, the change of $f$ is

$$df = \frac{d\{f\}}{dx}dx + \frac{d\{f\}}{dy}dy + \frac{d\{f\}}{dz}dz$$

Now we need to find the change of $f$ at $(x, y, z) = (1, -1, 1)$ and $(dx, dy, dz) = (1.01 - 1, -0.99 - (-1), 0.99 - 1) = (0.01, 0.01, -0.01)$.

$$df = \left.\frac{d\{f\}}{dx}\right|_{(x,y,z)=(1,-1,1)} dx|_{dx=0.01} + \left.\frac{d\{f\}}{dy}\right|_{(x,y,z)=(1,-1,1)} dy|_{dy=0.01} + \left.\frac{d\{f\}}{dz}\right|_{(x,y,z)=(1,-1,1)} dz|_{dz=-0.01}$$
$$= 2\cdot(0.01) + 0 + (-4)\cdot(-0.01) = 0.02 + 0.04 = 0.06$$

In fact $f(1, -1, 1) = 1$ and $f(1.01, -0.99, 0.99) = 1.060102$. Thus the difference is $0.060102$.

d) calculate the percentage error in calculating $f(x, y, z)$ in around the position $(x, y) = (1, -1, 1)$ and there are errors of -1% in $x$ and +0.1% in $y$ and -0.1 % in $z$ In case of three dimensional problems, the percentage error in calculating $f$ is

$$\frac{df}{f} = \frac{\frac{d\{f\}}{dx}dx}{f} + \frac{\frac{d\{f\}}{dy}dy}{f} + \frac{\frac{d\{f\}}{dz}dz}{f}$$

Now we need to find the percentage error in calculating $f$ at $(x, y) = (1, -1, 1)$ and $(\frac{dx}{x}, \frac{dy}{y}) = (-0.01, 0.001, -0.001)$.

$$\frac{df}{f} = \left.\frac{\frac{d\{f\}}{dx}}{f}\cdot x\right|_{(x,y,z)=(1,-1,1)} \cdot \left.\frac{dx}{x}\right|_{\frac{dx}{x}=-0.01} + \left.\frac{\frac{d\{f\}}{dy}}{f}\cdot y\right|_{(x,y,z)=(1,-1,1)} \cdot \left.\frac{dy}{y}\right|_{\frac{dy}{y}=0.001}$$
$$+ \left.\frac{\frac{d\{f\}}{dz}}{f}\cdot z\right|_{(x,y,z)=(1,-1,1)} \cdot \left.\frac{dz}{z}\right|_{\frac{dz}{z}=-0.001}$$
$$= \left.\frac{3x^2 - 2xz^3 - y^3}{x^3 - y^2 + z - x^2z^3 + yz^2 - xy^3 + 1}\right|_{x=1,\ y=-1,\ z=1} \cdot (-0.01)$$

$$+ \frac{-2y + z^2 - 3xy^2}{x^3 - y^2 + z - x^2z^3 + yz^2 - xy^3 + 1}\Bigg|_{\substack{x=1\\ y=-1\\ z=1}} \cdot (0.001)$$

$$+ \frac{1 - 3x^2z^2 + 2yz}{x^3 - y^2 + z - x^2z^3 + yz^2 - xy^3 + 1}\Bigg|_{\substack{x=1\\ y=-1\\ z=1}} \cdot (-0.001)$$

$$= \frac{2}{1} \cdot (-0.01) + \frac{0}{1} \cdot (0.001) + \frac{-4}{1} \cdot (-0.001) = -0.02 + 0.004 = -0.016$$

Thus the percent error is -1.6 % whilst the real error is $\frac{0.984109 - 1}{1} = -0.0158907$ because $x^3 - y^2 + z - x^2z^3 + yz^2 - xy^3 + 1\Big|_{\substack{x = 1\cdot(1-0.01)\\ y = -1\cdot(1+0.001)\\ z = 1\cdot(1-0.001)}} = 0.984109$ and $x^3 - y^2 + z - x^2z^3 + yz^2 - xy^3 + 1\Big|_{\substack{x=1\\ y=-1}} = 1$

37) For the function of $f(x,y) = \sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{y}\right)$

a) find the gradient $\nabla f$ of the function $f(x,y)$ at the point $\left(\frac{3}{\pi}, \frac{6}{\pi}\right)$ The definition of $\nabla f$ in 3D is

$$\nabla f = \frac{d\{f\}}{dx}\boldsymbol{i} + \frac{d\{f\}}{dy}\boldsymbol{j} + \frac{d\{f\}}{dz}\boldsymbol{k}.$$

Now we handle two dimensional case at the point $\left(\frac{3}{\pi}, \frac{6}{\pi}\right)$ First, we need $\frac{d\{f\}}{dx}$ and $\frac{d\{f\}}{dy}$.

$$\frac{d\{f\}}{dx} = \frac{d\left\{\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{y}\right)\right\}}{dx} = \frac{d\left\{\sin\left(\frac{1}{x}\right)\right\}}{dx}\cos\left(\frac{1}{y}\right) = \cos\left(\frac{1}{y}\right)\frac{d\{\sin(t)\}}{dx}(\because t \triangleq \frac{1}{x})$$
$$= \cos\left(\frac{1}{y}\right)\frac{d\{t\}}{dx}\frac{d\{\sin(t)\}}{dt} = \cos\left(\frac{1}{y}\right)\frac{d\{x^{-1}\}}{dx}\cos(t) = -x^{-2}\cos\left(\frac{1}{y}\right)\cos\left(\frac{1}{x}\right)$$

and

$$\frac{d\{f\}}{dy} = \frac{d\left\{\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{y}\right)\right\}}{dy} = \sin\left(\frac{1}{x}\right)\frac{d\left\{\cos\left(\frac{1}{y}\right)\right\}}{dy} = \sin\left(\frac{1}{x}\right)\frac{d\{\cos(t)\}}{dy}(\because t \triangleq \frac{1}{y})$$
$$= \sin\left(\frac{1}{x}\right)\frac{d\{t\}}{dy}\frac{d\{\cos(t)\}}{dt} = \sin\left(\frac{1}{x}\right)\frac{d\{y^{-1}\}}{dy}(-\sin(t)) = y^{-2}\sin\left(\frac{1}{x}\right)\sin\left(\frac{1}{y}\right)$$

$$\nabla f = \left.\frac{d\{f\}}{dx}\right|_{(x,y)=\left(\frac{3}{\pi},\frac{6}{\pi}\right)} \boldsymbol{i} + \left.\frac{d\{f\}}{dy}\right|_{(x,y)=\left(\frac{3}{\pi},\frac{6}{\pi}\right)} \boldsymbol{j}$$

$$= -x^{-2}\cos\left(\frac{1}{y}\right)\cos\left(\frac{1}{x}\right)\Big|_{(x,y)=\left(\frac{3}{\pi},\frac{6}{\pi}\right)} \boldsymbol{i} + y^{-2}\sin\left(\frac{1}{x}\right)\sin\left(\frac{1}{y}\right)\Big|_{(x,y)=\left(\frac{3}{\pi},\frac{6}{\pi}\right)} \boldsymbol{j}$$

$$= -\left(\frac{\pi}{3}\right)^2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2}\boldsymbol{i} + \left(\frac{\pi}{6}\right)^2 \frac{\sqrt{3}}{2} \cdot \frac{1}{2}\boldsymbol{j} = -\frac{\sqrt{3}\pi^2}{36}\boldsymbol{i} + \frac{\sqrt{3}\pi^2}{144}\boldsymbol{j}$$

So at the point $(x, y) = \left(\frac{3}{\pi}, \frac{6}{\pi}\right)$, the gradient is $(-\frac{\sqrt{3}\pi^2}{36}, \frac{\sqrt{3}\pi^2}{144})$.

b) find the directional derivative of $f$ at the point $(x, y) = \left(\frac{3}{\pi}, \frac{6}{\pi}\right)$ in the direction of the vector $\boldsymbol{n} = 72\boldsymbol{i}+144\boldsymbol{j}$. At the point $(x, y) = \left(\frac{3}{\pi}, \frac{6}{\pi}\right)$ the gradient is $-\frac{\sqrt{3}\pi^2}{36}\boldsymbol{i} + \frac{\sqrt{3}\pi^2}{144}\boldsymbol{j} \triangleq \boldsymbol{v}$. Now we need to find the magnitude of $\boldsymbol{n}$-directional component of $\boldsymbol{v}$. When the angle between $\boldsymbol{n}$ and $\boldsymbol{v}$ is $\theta$ the magnitude of $\boldsymbol{n}$-directional component of $\boldsymbol{v}$ can be written as $|\boldsymbol{v}|\cos\theta$. As $\boldsymbol{n}\cdot\boldsymbol{v} = |\boldsymbol{n}||\boldsymbol{v}|\cos\theta$, we can obtain the magnitude as

$$|\boldsymbol{v}|\cos\theta = |\boldsymbol{v}|\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}||\boldsymbol{v}|} = \frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}|}$$

The magnitude of $\boldsymbol{n}$ is $|\boldsymbol{n}| = \sqrt{72^2 + 144^2} = \sqrt{25920} = 8 \cdot 9\sqrt{5} = 72\sqrt{5}$. Therefore

$$\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}|} = \frac{72 \cdot (-\frac{\sqrt{3}\pi^2}{36}) + 144 \cdot \frac{\sqrt{3}\pi^2}{144}}{72\sqrt{5}} = \frac{-\sqrt{3}\pi^2}{72\sqrt{5}}$$

c) calculate the approximate change of $f(x, y)$ in moving from the position $(x, y) = \left(\frac{3}{\pi}, \frac{6}{\pi}\right)$ to a new position of $(x, y) = \left(\frac{4}{\pi}, \frac{5}{\pi}\right)$ In case of three dimensional problems, the change of $f$ is

$$df = \frac{d\{f\}}{dx}dx + \frac{d\{f\}}{dy}dy + \frac{d\{f\}}{dz}dz$$

Now we need to find the change of $f$ at $(x, y) = \left(\frac{3}{\pi}, \frac{6}{\pi}\right)$ and $(dx, dy) = \left(\frac{4}{\pi} - \frac{3}{\pi}, \frac{5}{\pi} - \frac{6}{\pi}\right) = \left(\frac{1}{\pi}, -\frac{1}{\pi}\right)$.

$$df = \left.\frac{d\{f\}}{dx}\right|_{(x,y)=\left(\frac{3}{\pi},\frac{6}{\pi}\right)} dx|_{dx=\frac{1}{\pi}} + \left.\frac{d\{f\}}{dy}\right|_{(x,y)=\left(\frac{3}{\pi},\frac{6}{\pi}\right)} dy|_{dy=-\frac{1}{\pi}}$$

$$= -x^{-2}\cos\left(\frac{1}{y}\right)\cos\left(\frac{1}{x}\right)\Big|_{(x,y)=\left(\frac{3}{\pi},\frac{6}{\pi}\right)} dx|_{dx=\frac{1}{\pi}} + y^{-2}\sin\left(\frac{1}{x}\right)\sin\left(\frac{1}{y}\right)\Big|_{(x,y)=\left(\frac{3}{\pi},\frac{6}{\pi}\right)} dy|_{dy=-\frac{1}{\pi}}$$

$$= -\frac{\sqrt{3}\pi^2}{36} \cdot \left(\frac{1}{\pi}\right) + \frac{\sqrt{3}\pi^2}{144} \cdot \left(-\frac{1}{\pi}\right) = -\frac{\sqrt{3}\pi}{36} - \frac{\sqrt{3}\pi}{144} = -\frac{5\sqrt{3}\pi}{144} = -0.188937$$

In fact $f\left(\frac{3}{\pi}, \frac{6}{\pi}\right) = 0.75$ and $f\left(\frac{4}{\pi}, \frac{5}{\pi}\right) = 0.572$. Thus the difference is $-0.1779385972$.

d) calculate the percentage error in calculating $f(x, y)$ around the position $(x, y) = \left(\frac{6}{\pi}, \frac{3}{\pi}\right)$, and there are errors of $0.1\%$ in $x$ and $1\%$ in $y$. In case of three dimensional problems, the percentage error in calculating $f$ is

$$\frac{df}{f} = \frac{\frac{d\{f\}}{dx}dx}{f} + \frac{\frac{d\{f\}}{dy}dy}{f} + \frac{\frac{d\{f\}}{dz}dz}{f}$$

Now we need to find the percentage error in calculating $f$ at $(x, y) = \left(\frac{6}{\pi}, \frac{3}{\pi}\right)$, and $(\frac{dx}{x}, \frac{dy}{y}) =$

$(0.001, 0.01)$.

$$\frac{df}{f} = \left.\frac{\frac{d\{f\}}{dx}}{f}\cdot x\right|_{(x,y)=\left(\frac{6}{\pi},\frac{3}{\pi}\right)} \cdot \left.\frac{dx}{x}\right|_{\frac{dx}{x}=0.001} + \left.\frac{\frac{d\{f\}}{dy}}{f}\cdot y\right|_{(x,y)=\left(\frac{6}{\pi},\frac{3}{\pi}\right)} \cdot \left.\frac{dy}{y}\right|_{\frac{dy}{y}=0.01}$$

$$= \left.\frac{-x^{-2}\cos\left(\frac{1}{y}\right)\cos\left(\frac{1}{x}\right)\cdot x}{\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{y}\right)}\right|_{(x,y)=\left(\frac{6}{\pi},\frac{3}{\pi}\right)} \cdot (0.001) + \left.\frac{y^{-2}\sin\left(\frac{1}{x}\right)\sin\left(\frac{1}{y}\right)\cdot y}{\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{y}\right)}\right|_{(x,y)=\left(\frac{6}{\pi},\frac{3}{\pi}\right)} \cdot (0.01)$$

$$= \frac{-\left(\frac{\pi}{6}\right)\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot(0.001) + \left(\frac{\pi}{3}\right)\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot(0.01)}{\frac{1}{4}} = 0.0172311$$

Thus the percent error is 1.72 % whilst the real error is $\frac{\sin\left(\frac{\pi}{6.006}\right)\cos\left(\frac{\pi}{3.03}\right) - \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right)} = 0.016984$

38) For the function of $f(x,y,z) = \mathfrak{e}^{\frac{x}{y-z}}$

a) find the gradient $\nabla f$ of the function $f(x,y,z)$ at the point $(x,y,z) = (1,2,1)$ The definition of $\nabla f$ in 3D is

$$\nabla f = \frac{d\{f\}}{dx}\boldsymbol{i} + \frac{d\{f\}}{dy}\boldsymbol{j} + \frac{d\{f\}}{dz}\boldsymbol{k}.$$

Now we handle three dimensional case at the point $(1,2,1)$ First, we need $\frac{d\{f\}}{dx}$, $\frac{d\{f\}}{dy}$ and $\frac{d\{f\}}{dz}$.

$$\frac{d\{f\}}{dx} = \frac{d\left\{\mathfrak{e}^{\frac{x}{y-z}}\right\}}{dx} = \frac{d\{\mathfrak{e}^t\}}{dx}(\because t \triangleq \frac{x}{y-z}) = \frac{d\{t\}}{dx}\frac{d\{\mathfrak{e}^t\}}{dt} = \frac{d\left\{\frac{x}{y-z}\right\}}{dx}\frac{d\{\mathfrak{e}^t\}}{dt} = \frac{1}{y-z}\frac{d\{x\}}{dx}\mathfrak{e}^t = \frac{1}{y-z}\mathfrak{e}^{\frac{x}{y-z}}$$

and

$$\frac{d\{f\}}{dy} = \frac{d\left\{\mathfrak{e}^{\frac{x}{y-z}}\right\}}{dy} = \frac{d\{\mathfrak{e}^t\}}{dy}(\because t \triangleq \frac{x}{y-z})$$

$$= \frac{d\{t\}}{dy}\frac{d\{\mathfrak{e}^t\}}{dt} = \frac{d\left\{\frac{x}{y-z}\right\}}{dy}\mathfrak{e}^t = x\frac{d\{(y-z)^{-1}\}}{dy}\mathfrak{e}^{\frac{x}{y-z}} = x\frac{d\{s^{-1}\}}{dy}\mathfrak{e}^{\frac{x}{y-z}}(\because s \triangleq y-z)$$

$$= x\frac{d\{s\}}{dy}\frac{d\{s^{-1}\}}{ds}\mathfrak{e}^{\frac{x}{y-z}} = x\frac{d\{y-z\}}{dy}(-s^{-2})\mathfrak{e}^{\frac{x}{y-z}} = -x(y-z)^{-2}\mathfrak{e}^{\frac{x}{y-z}}$$

and

$$\frac{d\{f\}}{dz} = \frac{d\left\{\mathfrak{e}^{\frac{x}{y-z}}\right\}}{dz} = \frac{d\{\mathfrak{e}^t\}}{dz}(\because t \triangleq \frac{x}{y-z})$$

$$= \frac{d\{t\}}{dz}\frac{d\{\mathfrak{e}^t\}}{dt} = \frac{d\left\{\frac{x}{y-z}\right\}}{dz}\mathfrak{e}^t = x\frac{d\{(y-z)^{-1}\}}{dz}\mathfrak{e}^{\frac{x}{y-z}} = x\frac{d\{s^{-1}\}}{dz}\mathfrak{e}^{\frac{x}{y-z}}(\because s \triangleq y-z)$$

$$= x\frac{d\{s\}}{dz}\frac{d\{s^{-1}\}}{ds}\mathfrak{e}^{\frac{x}{y-z}} = x\frac{d\{y-z\}}{dz}(-s^{-2})\mathfrak{e}^{\frac{x}{y-z}} = x(y-z)^{-2}\mathfrak{e}^{\frac{x}{y-z}}$$

Therefore

$$\nabla f = \left.\frac{d\{f\}}{dx}\right|_{(x,y,z)=(1,2,1)} \boldsymbol{i} + \left.\frac{d\{f\}}{dy}\right|_{(x,y,z)=(1,2,1)} \boldsymbol{j} + \left.\frac{d\{f\}}{dz}\right|_{(x,y,z)=(1,2,1)} \boldsymbol{k}$$

$$= \left.\frac{1}{y-z}\mathfrak{e}^{\frac{x}{y-z}}\right|_{(x,y,z)=(1,2,1)} \boldsymbol{i} + \left.-x(y-z)^{-2}\mathfrak{e}^{\frac{x}{y-z}}\right|_{(x,y,z)=(1,2,1)} \boldsymbol{j} + \left.x(y-z)^{-2}\mathfrak{e}^{\frac{x}{y-z}}\right|_{(x,y,z)=(1,2,1)} \boldsymbol{k}$$

$$= \mathfrak{e}\boldsymbol{i} - \mathfrak{e}\boldsymbol{j} + \mathfrak{e}\boldsymbol{k}$$

So at the point $(x,y,z) = (1,2,1)$, the gradient is $(\mathfrak{e}, -\mathfrak{e}, \mathfrak{e})$.

b) find the directional derivative of $f$ at the point $(x,y,z) = (1,2,1)$ in the direction of the vector $\boldsymbol{n} = -\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$. At the point $(x,y,z) = (1,2,1)$ the gradient is $\mathfrak{e}\boldsymbol{i} - \mathfrak{e}\boldsymbol{j} + \mathfrak{e}\boldsymbol{k} \triangleq \boldsymbol{v}$. Now we need to find the magnitude of $\boldsymbol{n}$-directional component of $\boldsymbol{v}$. When the angle between $\boldsymbol{n}$ and $\boldsymbol{v}$ is $\theta$ the magnitude of $\boldsymbol{n}$-directional component of $\boldsymbol{v}$ can be written as $|\boldsymbol{v}|\cos\theta$. As $\boldsymbol{n}\cdot\boldsymbol{v} = |\boldsymbol{n}||\boldsymbol{v}|\cos\theta$, we can obtain the magnitude as

$$|\boldsymbol{v}|\cos\theta = |\boldsymbol{v}|\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}||\boldsymbol{v}|} = \frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}|}$$

The magnitude of $\boldsymbol{n}$ is $|\boldsymbol{n}| = \sqrt{(-1)^2 + (-1)^2 + 1^2} = \sqrt{3}$. Therefore

$$\frac{\boldsymbol{n}\cdot\boldsymbol{v}}{|\boldsymbol{n}|} = \frac{(-1)\cdot\mathfrak{e} + (-1)\cdot(-\mathfrak{e}) + 1\cdot(\mathfrak{e})}{\sqrt{3}} = \frac{-\mathfrak{e}+\mathfrak{e}+\mathfrak{e}}{\sqrt{3}} = \frac{\mathfrak{e}}{\sqrt{3}}$$

c) calculate the approximate change of $f(x,y,z)$ in moving from the position $(x,y,z) = (1,2,1)$ to a new position of $(x,y,z) = (0.99, 2.025, 0.975)$ In case of three dimensional problems, the change of $f$ is

$$df = \frac{d\{f\}}{dx}dx + \frac{d\{f\}}{dy}dy + \frac{d\{f\}}{dz}dz$$

Now we need to find the change of $f$ at $(x,y,z) = (1,2,1)$ and $(dx, dy, dz) = (0.99-1, 2.025-2, 0.975-1) = (-0.01, 0.025, -0.025)$.

$$df = \left.\frac{d\{f\}}{dx}\right|_{(x,y,z)=(1,2,1)} dx|_{dx=-0.01} + \left.\frac{d\{f\}}{dy}\right|_{(x,y,z)=(1,2,1)} dy|_{dy=0.025} + \left.\frac{d\{f\}}{dz}\right|_{(x,y,z)=(1,2,1)} dz|_{dz=-0.025}$$

$$= \mathfrak{e}\cdot(-0.01) - \mathfrak{e}\cdot(0.025) + \mathfrak{e}\cdot(-0.025) = -0.06\mathfrak{e} = -0.163097$$

In fact $f(1,2,1) = 2.71828$ and $f(0.99, 2.025, 0.075) = 2.56731$. Thus the difference is $-0.150976$.

d) calculate the percentage error in calculating $f(x,y,z)$ around the position $(x,y,z) = (1,2,1)$, and there are errors of -1 % in $x$ and 1.25% in $y$ and -2.5% in $z$. In case of three dimensional problems, the percentage error in calculating $f$ is

$$\frac{df}{f} = \frac{\frac{d\{f\}}{dx}dx}{f} + \frac{\frac{d\{f\}}{dy}dy}{f} + \frac{\frac{d\{f\}}{dz}dz}{f}$$

Now we need to find the percentage error in calculating $f$ at $(x,y) = (1,2,1)$, and $(\frac{dx}{x}, \frac{dy}{y}, \frac{dz}{z}) = (-0.01, 0.0125, -0.025)$.

$$\frac{df}{f} = \left.\frac{\frac{d\{f\}}{dx}}{f}\cdot x\right|_{(x,y,z)=(1,2,1)} \cdot \left.\frac{dx}{x}\right|_{\frac{dx}{x}=-0.01} + \left.\frac{\frac{d\{f\}}{dy}}{f}\cdot y\right|_{(x,y,z)=(1,2,1)} \cdot \left.\frac{dy}{y}\right|_{\frac{dy}{y}=0.0125}$$

$$+ \left.\frac{\frac{d\{f\}}{dz}}{f}\cdot z\right|_{(x,y,z)=(1,2,1)} \cdot \left.\frac{dz}{z}\right|_{\frac{dz}{z}=-0.025}$$

$$= \frac{\mathfrak{e}\cdot 1\cdot(-0.01) - \mathfrak{e}\cdot 2\cdot(0.0125) + \mathfrak{e}\cdot 1\cdot(-0.025)}{\mathfrak{e}} = -0.06$$

Thus the percent error is -6 % whilst the real error is $\frac{2.56731 - 2.71828}{2.71828} = -0.0555388$.

39) For the function of $f(x, y, z) = \frac{1}{x^2 - yz}$

a) find the gradient $\nabla f$ of the function $f(x, y, z)$ at the point $(x, y, z) = (-2, 1, 3)$ The definition of $\nabla f$ in 3D is

$$\nabla f = \frac{d\{f\}}{dx}\boldsymbol{i} + \frac{d\{f\}}{dy}\boldsymbol{j} + \frac{d\{f\}}{dz}\boldsymbol{k}.$$

Now we handle three dimensional case at the point $(-2, 1, 3)$ First, we need $\frac{d\{f\}}{dx}$, $\frac{d\{f\}}{dy}$ and $\frac{d\{f\}}{dz}$.

$$\frac{d\{f\}}{dx} = \frac{d\{(x^2 - yz)^{-1}\}}{dx} = \frac{d\{t^{-1}\}}{dx}(\because t \triangleq (x^2 - yz))$$
$$= \frac{d\{t\}}{dx}\frac{d\{t^{-1}\}}{dt} = \frac{d\{x^2 - yz\}}{dx} \cdot (-t^{-2}) = 2x \cdot (-t^{-2}) = -2x(x^2 - yz)^{-2}$$

and

$$\frac{d\{f\}}{dy} = \frac{d\{(x^2 - yz)^{-1}\}}{dy} = \frac{d\{t^{-1}\}}{dy}(\because t \triangleq (x^2 - yz))$$
$$= \frac{d\{t\}}{dy}\frac{d\{t^{-1}\}}{dt} = \frac{d\{x^2 - yz\}}{dy} \cdot (-t^{-2}) = (-z) \cdot (-t^{-2}) = z(x^2 - yz)^{-2}$$

and

$$\frac{d\{f\}}{dz} = \frac{d\{(x^2 - yz)^{-1}\}}{dz} = \frac{d\{t^{-1}\}}{dz}(\because t \triangleq (x^2 - yz))$$
$$= \frac{d\{t\}}{dz}\frac{d\{t^{-1}\}}{dt} = \frac{d\{x^2 - yz\}}{dz} \cdot (-t^{-2}) = (-y) \cdot (-t^{-2}) = y(x^2 - yz)^{-2}$$

Therefore

$$\nabla f = \left.\frac{d\{f\}}{dx}\right|_{(x,y,z)=(-2,1,3)}\boldsymbol{i} + \left.\frac{d\{f\}}{dy}\right|_{(x,y,z)=(-2,1,3)}\boldsymbol{j} + \left.\frac{d\{f\}}{dz}\right|_{(x,y,z)=(-2,1,3)}\boldsymbol{k}$$
$$= \left.-2x(x^2 - yz)^{-2}\right|_{(x,y,z)=(-2,1,3)}\boldsymbol{i} + \left.z(x^2 - yz)^{-2}\right|_{(x,y,z)=(-2,1,3)}\boldsymbol{j} + \left.y(x^2 - yz)^{-2}\right|_{(x,y,z)=(-2,1,3)}\boldsymbol{k}$$
$$= 4\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$$

So at the point $(x, y, z) = (-2, 1, 3)$, the gradient is $(4, 3, 1)$.

b) find the directional derivative of $f$ at the point $(x, y, z) = (-2, 1, 3)$ in the direction of the vector $\boldsymbol{n} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$. At the point $(x, y, z) = (-2, 1, 3)$ the gradient is $4\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k} \triangleq \boldsymbol{v}$. Now we need to find the magnitude of $\boldsymbol{n}$-directional component of $\boldsymbol{v}$. When the angle between $\boldsymbol{n}$ and $\boldsymbol{v}$ is $\theta$ the magnitude of $\boldsymbol{n}$-directional component of $\boldsymbol{v}$ can be written as $|\boldsymbol{v}|\cos\theta$. As $\boldsymbol{n} \cdot \boldsymbol{v} = |\boldsymbol{n}||\boldsymbol{v}|\cos\theta$, we can obtain the magnitude as

$$|\boldsymbol{v}|\cos\theta = |\boldsymbol{v}|\frac{\boldsymbol{n} \cdot \boldsymbol{v}}{|\boldsymbol{n}||\boldsymbol{v}|} = \frac{\boldsymbol{n} \cdot \boldsymbol{v}}{|\boldsymbol{n}|}$$

The magnitude of $\boldsymbol{n}$ is $|\boldsymbol{n}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$. Therefore

$$\frac{\boldsymbol{n} \cdot \boldsymbol{v}}{|\boldsymbol{n}|} = \frac{1 \cdot 4 + (-2) \cdot 3 + 3 \cdot 1}{\sqrt{14}} = \frac{4 - 6 + 3}{\sqrt{14}} = \frac{1}{\sqrt{14}}$$

c) calculate the approximate change of $f(x, y, z)$ in moving from the position $(x, y, z) = (-2, 1, 3)$ to a new position of $(x, y, z) = (-1.99, 0.99, 2.995)$ In case of three dimensional problems, the change of $f$ is

$$df = \frac{d\{f\}}{dx}dx + \frac{d\{f\}}{dy}dy + \frac{d\{f\}}{dz}dz$$

Now we need to find the change of $f$ at $(x, y, z) = (-2, 1, 3)$ and $(dx, dy, dz) = (-1.99 - (-2), 0.99 - 1, 2.995 - 3) = (0.01, -0.01, -0.005)$.

$$df = \left.\frac{d\{f\}}{dx}\right|_{(x,y,z)=(-2,1,3)} dx|_{dx=0.01} + \left.\frac{d\{f\}}{dy}\right|_{(x,y,z)=(-2,1,3)} dy|_{dy=-0.01} + \left.\frac{d\{f\}}{dz}\right|_{(x,y,z)=(-2,1,3)} dz|_{dz=-0.005}$$
$$= 4 \cdot 0.01 + 3 \cdot (-0.01) - 0.005 = 0.005$$

In fact $f(-2, 1, 3) = 1$ and $f(-1.99, 0.99, 2.995) = 1.00497$. Thus the difference is $0.00497$.

d) calculate the percentage error in calculating $f(x, y, z)$ around the position $(x, y, z) = (-2, 1, 3)$, and there are errors of 0.5 % in $x$ and -1% in $y$ and -0.2% in $z$. In case of three dimensional problems, the percentage error in calculating $f$ is

$$\frac{df}{f} = \frac{\frac{d\{f\}}{dx}dx}{f} + \frac{\frac{d\{f\}}{dy}dy}{f} + \frac{\frac{d\{f\}}{dz}dz}{f}$$

Now we need to find the percentage error in calculating $f$ at $(x, y) = (-2, 1, 3)$, and $(\frac{dx}{x}, \frac{dy}{y}, \frac{dz}{z}) = (0.005, -0.01, -0.002)$.

$$\frac{df}{f} = \left.\frac{\frac{d\{f\}}{dx}}{f} \cdot x\right|_{(x,y,z)=(-2,1,3)} \cdot \left.\frac{dx}{x}\right|_{\frac{dx}{x}=0.005} + \left.\frac{\frac{d\{f\}}{dy}}{f} \cdot y\right|_{(x,y,z)=(-2,1,3)} \cdot \left.\frac{dy}{y}\right|_{\frac{dy}{y}=-0.01}$$
$$+ \left.\frac{\frac{d\{f\}}{dz}}{f} \cdot z\right|_{(x,y,z)=(-2,1,3)} \cdot \left.\frac{dz}{z}\right|_{\frac{dz}{z}=-0.002}$$
$$= \frac{4 \cdot (-2) \cdot (0.005) + 3 \cdot (-0.01) + 3 \cdot (-0.002)}{1} = -0.076$$

Thus the percent error is -7.6 % whilst the real error is $\frac{\frac{1}{2.01^2 - 2.97 \cdot 0.998} - 1}{1} = -0.0706665$.

**DAY5**

40) The function $f(x,y) = xy(a-x-y)$ has stationary point(s) Give the location and the nature of stationary point(s).

At the stationary point, $\frac{d\{f(x,y)\}}{dx} = \frac{d\{f(x,y)\}}{dy} = 0$. Therefore we find $(x,y)$ which satisfies $\frac{d\{f(x,y)\}}{dx} = \frac{d\{f(x,y)\}}{dy} = 0$ as follows:

$$\frac{d\{f(x,y)\}}{dx} = ay - 2xy - y^2 = 0 \qquad ① \ ; \ \frac{d\{f(x,y)\}}{dy} = ax - 2xy - x^2 = 0 \qquad ②$$

①=② gives

$$ay - 2xy - y^2 = ax - 2xy - x^2 \ ; \ \therefore a(y-x) + x^2 - y^2 = 0$$
$$\therefore a(y-x) + (x-y)(x+y) = 0 \ ; \ \therefore (x-y)(x+y-a) = 0$$
$$\therefore y = x, y = a - x$$

When $y = x$ is substituted into ① we obtain

$$ax - 2x^2 - x^2 = ax - 3x^2 = x(a-3x) = 0$$

Thefore $(x,y) = (0,0), (\frac{a}{3}, \frac{a}{3})$ satisfy both ① and ②. When $y = a - x$ is substituted into ① we obtain

$$a(a-x) - 2x(a-x) - (a-x)^2 = a^2 - ax - 2ax + 2x^2 - a^2 - x^2 + 2xa$$
$$= -ax + x^2 = x(x-a) = 0 \qquad ③$$

Thefore $(x,y) = (0,a), (a,0)$ satisfy both ① and ②. Thus the stationary points are $(x,y) = (0,0), (\frac{a}{3}, \frac{a}{3}), (0,a), (a,0)$. Now we need $\frac{d^2f(x,y)}{dx^2}$,$\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$ to find out the nature of the stationary points.

$$\frac{d^2f(x,y)}{dx^2} = -2y \qquad ③ \ ; \ \frac{\partial^2 f(x,y)}{\partial x \partial y} = a - 2x - 2y \qquad ④ \ ; \ \frac{\partial^2 f(x,y)}{\partial y^2} = -2x \qquad ⑤$$

The discriminant $D$ is

$$D = \frac{d^2f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 \Bigg|_{(x,y)=(0,0)} = 4xy - (a-2x-2y)^2 = -a^2 < 0$$

$$D = \frac{d^2f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 \Bigg|_{(x,y)=(\frac{a}{3},\frac{a}{3})}$$
$$= \frac{4a^2}{3} - (a - \frac{4a}{3})^2 = \frac{4a^2}{3} - (-\frac{a}{3})^2 = \frac{4a^2}{3} - \frac{a^2}{9} = \frac{12a^2}{9} - \frac{a^2}{9} = \frac{11a^2}{9} > 0$$

$$D = \frac{d^2f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 \Bigg|_{(x,y)=(0,a)} = 4xy - (a-2x-2y)^2 = -a^2 < 0$$

$$D = \frac{d^2f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 \Bigg|_{(x,y)=(a,0)} = 4xy - (a-2x-2y)^2 = -a^2 < 0$$

Furthermore at $(x,y) = (\frac{a}{3}, \frac{a}{3})$

$$\frac{d^2f(x,y)}{dx^2} = \frac{-2a}{3}$$
$$> 0 \text{ for } a < 0$$
$$< 0 \text{ for } a > 0$$

Therefore $(x,y)=(\frac{a}{3},\frac{a}{3})$ is a local maximum when $a>0$ and a local minimum when $a<0$. $(x,y)=(0,0),(0,a),(a,0)$ are the saddle point.

41) The function $f(x,y)=x^3+y^3-x^2+xy-y^2$ has stationary point(s) Give the location and the nature of stationary point(s).

At the stationary point, $\frac{d\{f(x,y)\}}{dx}=\frac{d\{f(x,y)\}}{dy}=0$. Therefore we find $(x,y)$ which satisfies $\frac{d\{f(x,y)\}}{dx}=\frac{d\{f(x,y)\}}{dy}=0$ as follows:

$$\frac{d\{f(x,y)\}}{dx}=3x^2-2x+y=0 \qquad ① \;;\; \frac{d\{f(x,y)\}}{dy}=3y^2+x-2y=0 \qquad ②$$

①=② gives

$$3x^2-2x+y=3y^2+x-2y \;;\; \therefore 3x^2-3y^2-3x+3y=0 \;;\; \therefore (x-y)(x+y)-(x-y)=0$$
$$\therefore (x-y)(x+y-1)=0 \;;\; \therefore y=x, y=1-x$$

When $y=x$ is substituted into ① we obtain

$$3x^2-2x+x=3x^2-x=x(3x-1)=0 \;;\; x=0,\frac{1}{3}$$

Thefore $(x,y)=(0,0),(\frac{1}{3},\frac{1}{3})$ satisfy both ① and ②. When $y=1-x$ is substituted into ① we obtain

$$3x^2-2x+1-x=3x^2-3x+1=3(x-0.5)^2+0.25=0 \qquad ③$$

There is no real $x$ which satisfies into ③. Thus the stationary points are $(x,y)=(0,0),(\frac{1}{3},\frac{1}{3})$. Now we need $\frac{d^2f(x,y)}{dx^2}$,$\frac{\partial^2 f(x,y)}{\partial y\partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$ to find out the nature of the stationary points.

$$\frac{d^2f(x,y)}{dx^2}=6x-2 \;;\; \frac{\partial^2 f(x,y)}{\partial x\partial y}=1 \;;\; \frac{\partial^2 f(x,y)}{\partial y^2}=6y-2$$

The discriminant $D$ is

$$D=\frac{d^2f(x,y)}{dx^2}\cdot\frac{\partial^2 f(x,y)}{\partial y^2}-\left(\frac{\partial^2 f(x,y)}{\partial x\partial y}\right)^2\Bigg|_{(x,y)=(0,0)}=(-2)^2-1>0$$
$$D=\frac{d^2f(x,y)}{dx^2}\cdot\frac{\partial^2 f(x,y)}{\partial y^2}-\left(\frac{\partial^2 f(x,y)}{\partial x\partial y}\right)^2\Bigg|_{(x,y)=(\frac{1}{3},\frac{1}{3})}=(0)^2-1<0$$

Furthermore at $(x,y)=(0,0)$

$$\frac{d^2f(x,y)}{dx^2}=6\cdot 0-2<0$$

Therefore $(x,y)=(0,0)$ is a local maximum and $(x,y)=(\frac{1}{3},\frac{1}{3})$ is a saddle point.

42) The function $f(x,y)=x^2-xy+2y^2-x-2y$ has stationary point(s) Give the location and the nature of stationary point(s).

At the stationary point, $\frac{d\{f(x,y)\}}{dx}=\frac{d\{f(x,y)\}}{dy}=0$. Therefore we find $(x,y)$ which satisfies $\frac{d\{f(x,y)\}}{dx}=\frac{d\{f(x,y)\}}{dy}=0$ as follows:

$$\frac{d\{f(x,y)\}}{dx}=2x-y-1=0 \qquad ① \;;\; \frac{d\{f(x,y)\}}{dy}=-x+4y-2=0 \qquad ②$$

①+2×② gives $7y = 5$, *i.e.*, $y = 5/7$. When $y = 5/7$ is substituted into ② $x = 6/7$. Thus the stationary point is $(x, y) = (6/7, 5/7)$ Now we need $\dfrac{d^2 f(x,y)}{dx^2}$,$\dfrac{\partial^2 f(x,y)}{\partial y \partial x}$, $\dfrac{\partial^2 f(x,y)}{\partial y^2}$ to find out the nature of the stationary points.

$$\frac{d^2 f(x,y)}{dx^2} = 2 > 0 \qquad ③ \;;\quad \frac{\partial^2 f(x,y)}{\partial x \partial y} = -1 \qquad ④ \;;\quad \frac{\partial^2 f(x,y)}{\partial y^2} = 4 \qquad ⑤$$

The discriminant $D$ is

$$D = \frac{d^2 f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 \Bigg|_{(x,y)=(6/7,5/7)} = 2 \cdot 4 - (-1)^2 = 7 > 0$$

With ③ and $D > 0$, the stationary point $(x, y) = (6/7, 5/7)$ is a local minimum.

43) The function $f(x, y) = (x + y)\mathfrak{e}^{-xy}$ has stationary point(s) Give the location and the nature of stationary point(s).

At the stationary point, $\dfrac{d\{f(x,y)\}}{dx} = \dfrac{d\{f(x,y)\}}{dy} = 0$. Therefore we find $(x, y)$ which satisfies $\dfrac{d\{f(x,y)\}}{dx} = \dfrac{d\{f(x,y)\}}{dy} = 0$ as follows:

$$\frac{d\{f(x,y)\}}{dx} = \mathfrak{e}^{-xy} + (x+y)(-y)\mathfrak{e}^{-xy} = 0 \qquad ① \;;\quad \frac{d\{f(x,y)\}}{dy} = \mathfrak{e}^{-xy} + (x+y)(-x)\mathfrak{e}^{-xy} = 0 \qquad ②$$

① −② gives $y^2 = x^2$, *i.e.*, $y = \pm x$. When $y = x$ is substituted into ② $1 - 2x^2 = 0$ *i.e.*, $x = \pm\frac{1}{\sqrt{2}}$. When $y = -x$ is substituted into ② $1 = 0$ which is impossible. Therefore the stationary points are $(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Now we need $\dfrac{d^2 f(x,y)}{dx^2}$,$\dfrac{\partial^2 f(x,y)}{\partial y \partial x}$, $\dfrac{\partial^2 f(x,y)}{\partial y^2}$ to find out the nature of the stationary points.

$$\frac{d^2 f(x,y)}{dx^2} = \mathfrak{e}^{-xy}(-2y + xy^2 + y^3) \qquad ③ \;;\quad \frac{\partial^2 f(x,y)}{\partial x \partial y} = \mathfrak{e}^{-xy}(-2x - 2y + xy^2 + x^2 y) \qquad ④ \;;\quad \frac{\partial^2 f(x,y)}{\partial y^2} = \mathfrak{e}^{-x}$$

When $(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$$\frac{d^2 f(x,y)}{dx^2} = \mathfrak{e}^{-\frac{1}{2}}(-2\frac{1}{\sqrt{2}} + 2\frac{1}{2\sqrt{2}}) = \mathfrak{e}^{-\frac{1}{2}}(-2\frac{1}{\sqrt{2}} + 2\frac{1}{2\sqrt{2}}) = \mathfrak{e}^{-\frac{1}{2}}(-\frac{1}{\sqrt{2}}) \;;\quad \frac{\partial^2 f(x,y)}{\partial x \partial y} = \mathfrak{e}^{-xy}(-2x - 2y + xy^2 +$$

When $(x, y) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$$\frac{d^2 f(x,y)}{dx^2} = \mathfrak{e}^{-\frac{1}{2}}(\frac{1}{\sqrt{2}}) \;;\quad \frac{\partial^2 f(x,y)}{\partial x \partial y} = \mathfrak{e}^{-\frac{1}{2}}(\frac{3}{\sqrt{2}}) \;;\quad \frac{\partial^2 f(x,y)}{\partial y^2} = \mathfrak{e}^{-\frac{1}{2}}(\frac{1}{\sqrt{2}})$$

The discriminant $D$ is

$$D = \frac{d^2 f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 \Bigg|_{(x,y)=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})} = \frac{d^2 f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 \Bigg|_{(x,y)=(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})}$$

Thus the stationary points $(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ are saddle points.

44) The function $f(x, y) = \sin x + \sin y + \sin(x + y)$ where $0 < x < 2\pi, 0 < y < 2\pi$ has stationary point(s) Give the location and the nature of stationary point(s).

At the stationary point, $\dfrac{d\{f(x,y)\}}{dx} = \dfrac{d\{f(x,y)\}}{dy} = 0$. Therefore we find $(x,y)$ which satisfies $\dfrac{d\{f(x,y)\}}{dx} = \dfrac{d\{f(x,y)\}}{dy} = 0$ as follows:

$$\frac{d\{f(x,y)\}}{dx} = \cos x + \cos(x+y) = 0 \qquad ① \ ; \quad \frac{d\{f(x,y)\}}{dy} = \cos y + \cos(x+y) = 0 \qquad ②$$

①-② gives $\cos x - \cos y = 0$, *i.e.*, $\cos x = \cos y$ *i.e.*,

$$x = y \ ; \ x = 2\pi - y$$

When $x = 2\pi - y$ ① can be manipulated as

$$0 = \cos x + \cos(x+y) = \cos x + \cos 2\pi = \cos x + 1$$

$$\therefore \cos x = -1 \ ; \quad \therefore x = \pi$$

Therefore the stationary points are $(x,y) = (\pi,\pi)$.
When $x = y$, ① can be manipulated as

$$0 = \cos x + \cos(x+x) = \cos x + \cos 2x = \cos x + 2\cos^2 x - 1 = (2\cos x - 1)(\cos x + 1)$$

$$\therefore \cos x = -1, \frac{1}{2}$$

When $\cos x = -1$, we get $x = \pi$. When $\cos x = \frac{1}{2}$, we get $x = \frac{\pi}{3}, \frac{5\pi}{3}$. Therefore there are 3 stationary points at $(x,y) = (\pi,\pi), (\frac{\pi}{3},\frac{\pi}{3}), (\frac{5\pi}{3},\frac{5\pi}{3})$ Now we need $\dfrac{d^2 f(x,y)}{dx^2}, \dfrac{\partial^2 f(x,y)}{\partial y \partial x}, \dfrac{\partial^2 f(x,y)}{\partial y^2}$ to find out the nature of the stationary points.

$$\frac{d^2 f(x,y)}{dx^2} = -\sin x - \sin(x+y) \qquad ③ \ ; \quad \frac{\partial^2 f(x,y)}{\partial x \partial y} = -\sin(x+y) \qquad ④$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = -\sin y - sin(x+y) \qquad ⑤$$

The discriminant $D$ is

$$D = \frac{d^2 f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2$$

$$= (-\sin x - \sin(x+y))(-\sin y - sin(x+y)) - (-\sin(x+y))^2 = \sin x \sin y + \sin(x+y)(\sin x + \sin y)$$

When $(x,y) = (\pi,\pi)$, $D = 0$. In this case we need to investigate the values around $(x,y) = (\pi,\pi)$. When $x = \pi - \alpha$ where $0 < \alpha \ll \frac{\pi}{100}$, $f(x,x) > 0$. When $x = \pi + \alpha$ where $0 < \alpha \ll \frac{\pi}{100}$, $f(x,x) < 0$. Therefore $(x,y) = (\pi,\pi)$ is not local minimum nor local maximum.
When $(x,y) = (\frac{\pi}{3},\frac{\pi}{3})$

$$D = \sin x \sin y + \sin(x+y)(\sin x + \sin y)|_{(x,y)=(\frac{\pi}{3},\frac{\pi}{3})} = \left(\frac{\sqrt{3}}{2}\right)^2 + 2\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} = \frac{9}{4} > 0$$

$$\frac{d^2 f(x,y)}{dx^2} = -\sin x - \sin(x+y)|_{(x,y)=(\frac{\pi}{3},\frac{\pi}{3})} = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} < 0$$

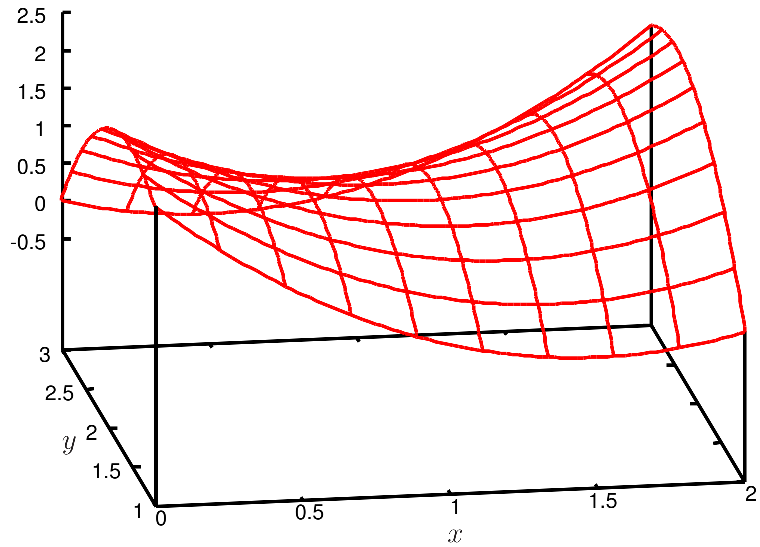
Therefore $(x,y) = (\frac{\pi}{3},\frac{\pi}{3})$ is a local maximum
When $(x,y) = (\frac{5\pi}{3},\frac{5\pi}{3})$

$$D = \sin x \sin y + \sin(x+y)(\sin x + \sin y)|_{(x,y)=(\frac{5\pi}{3},\frac{5\pi}{3})} = \left(\frac{-\sqrt{3}}{2}\right)^2 + 2(-\frac{\sqrt{3}}{2})(-\frac{\sqrt{3}}{2}) = \frac{9}{4} > 0$$

$$\frac{d^2 f(x,y)}{dx^2} = -\sin x - \sin(x+y)|_{(x,y)=(\frac{5\pi}{3},\frac{5\pi}{3})} = -(-\frac{\sqrt{3}}{2}) - (-\frac{\sqrt{3}}{2}) > 0$$

Therefore $(x,y)=(\frac{5\pi}{3},\frac{5\pi}{3})$ is a local minimum.

45) The function $f(x,y)=x^2+xy-y^2-4x+3y$ has stationary point(s) Give the location and the nature of stationary point(s).




At the stationary point, $\frac{d\{f(x,y)\}}{dx}=\frac{d\{f(x,y)\}}{dy}=0$. Therefore we find $(x,y)$ which satisfies $\frac{d\{f(x,y)\}}{dx}=\frac{d\{f(x,y)\}}{dy}=0$ as follows:

$$\frac{d\{f(x,y)\}}{dx}=2x+y-4=0 \qquad ① \quad ; \quad \frac{d\{f(x,y)\}}{dy}=x-2y+3=0 \qquad ②$$

①-②×2 gives $5y-10=0$, *i.e.*, $y=2$. When we put $y=2$ into ② we get $x=2y-3=4-3=1$. Therefore the stationary point is $(x,y)=(1,2)$. Now we need $\frac{d^2f(x,y)}{dx^2}$,$\frac{\partial^2f(x,y)}{\partial y\partial x}$, $\frac{\partial^2f(x,y)}{\partial y^2}$ to find out the nature of the stationary point.

$$\frac{d^2f(x,y)}{dx^2}=\frac{d\{2x+y-4\}}{dx}=2 \qquad ③ \quad ; \quad \frac{\partial^2f(x,y)}{\partial x\partial y}=\frac{d\{x-2y+3\}}{dx}=1 \qquad ④$$

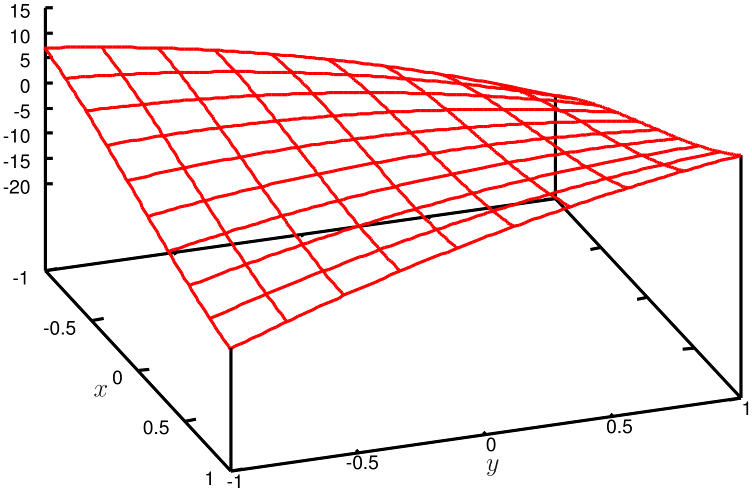
$$\frac{\partial^2f(x,y)}{\partial y^2}=\frac{d\{x-2y+3\}}{dy}=-2 \qquad ⑤$$

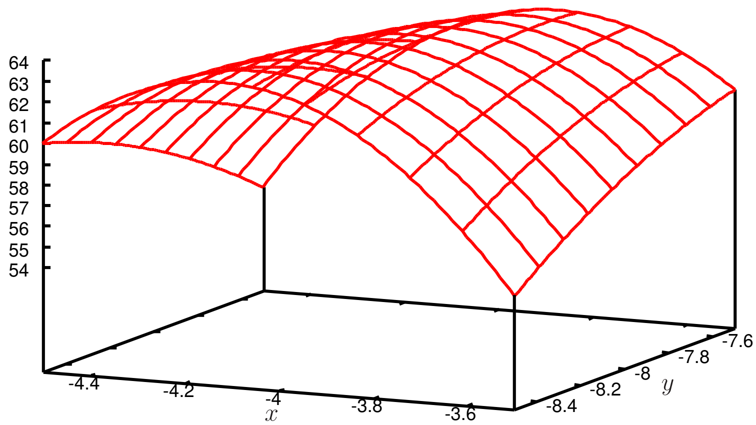
The value of the discriminant is

$$\frac{d^2f(x,y)}{dx^2}\cdot\frac{\partial^2f(x,y)}{\partial y^2}-\left(\frac{\partial^2f(x,y)}{\partial x\partial y}\right)^2=2\cdot(-2)-1^2=-4-1=-5<0$$

Therefore the the stationary point of $(x,y)=(1,2)$ corresponds to a saddle point.

46) The function $f(x,y)=12xy-3y^2+2x^3$ has stationary point(s). Give the location and the nature of stationary point(s).




At the stationary point, $\frac{d\{f(x,y)\}}{dx} = \frac{d\{f(x,y)\}}{dy} = 0$. Therefore we find $(x,y)$ which satisfies $\frac{d\{f(x,y)\}}{dx} = \frac{d\{f(x,y)\}}{dy} = 0$ as follows:

$$\frac{d\{f(x,y)\}}{dx} = 12y + 6x^2 = 0 \qquad ① \;\; ; \;\; \frac{d\{f(x,y)\}}{dy} = 12x - 6y = 0 \qquad ②$$

①+②×2 gives $24x + 6x^2 = 6x(x+4) = 0$, *i.e.*, $x = 0, -4$. When we put $x = 0, -4$ into ② we get $y = 0, -8$. Therefore the stationary point is $(x,y) = (0,0), (-4,-8)$. Now we need $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$ to find out the nature of the stationary point.

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\{12y + 6x^2\}}{dx} = 12x \qquad ③ \;\; ; \;\; \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{d\{12x - 6y\}}{dx} = 12 \qquad ④$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\{12x - 6y\}}{dy} = -6 \qquad ⑤$$

The value of the discriminant at $(x,y) = (0,0)$ is

$$\frac{d^2 f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 = 12 \cdot 0 \cdot (-6) - 12^2 < 0$$

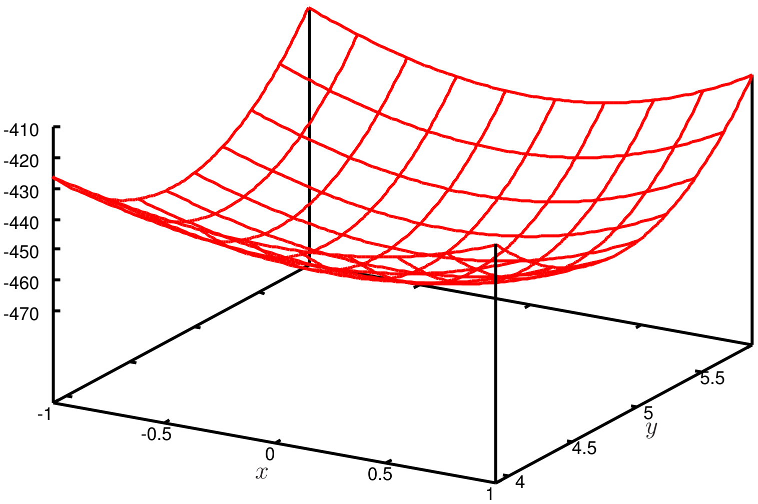
Therefore the the stationary point at $(x,y) = (0,0)$ corresponds to a saddle point. The value of the discriminant at $(x,y) = (-4,-8)$ is

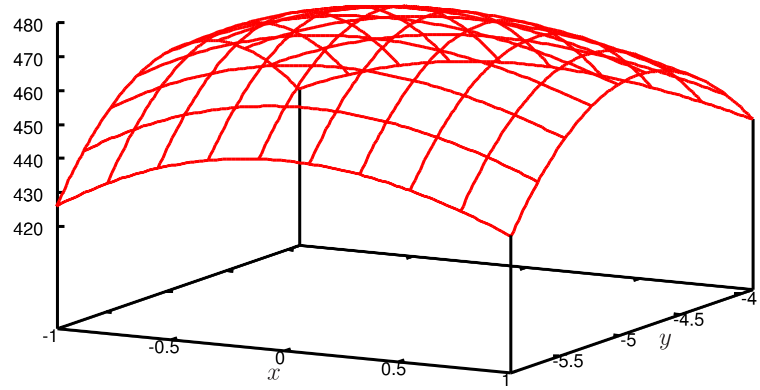
$$\frac{d^2 f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 = 12 \cdot (-4) \cdot (-6) - 12^2 > 0$$

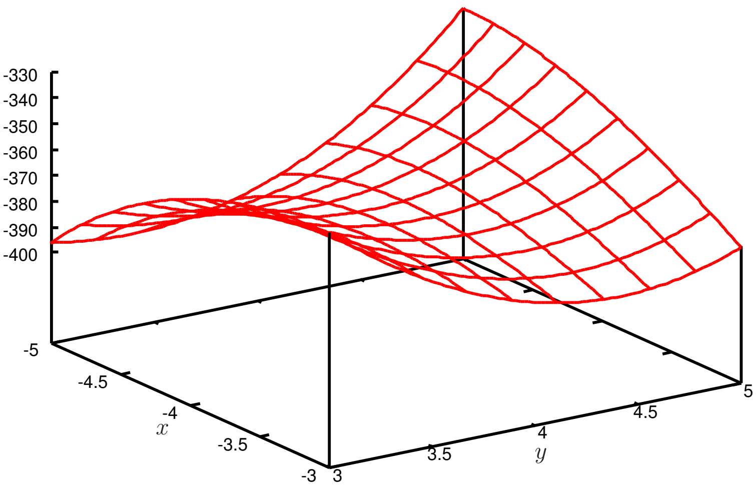
$$\left.\frac{d^2 f(x,y)}{dx^2}\right|_{x=-4} = \left.\frac{d\{12y + 6x^2\}}{dx}\right|_{x=-4} = 12 \cdot (-4) < 0$$

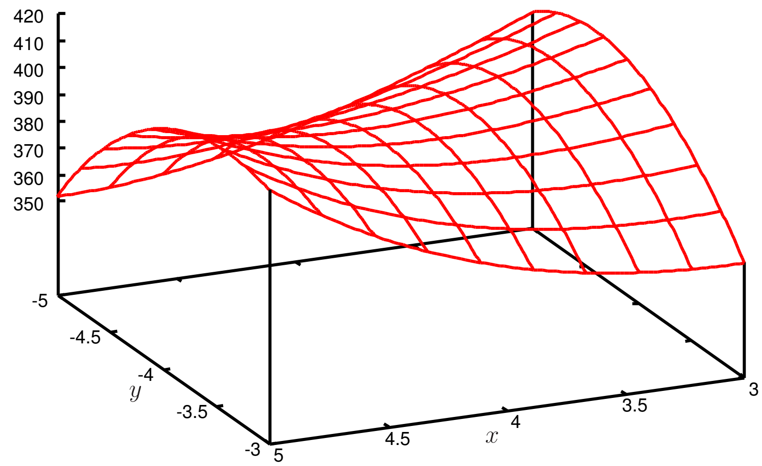
Therefore the the stationary point at $(x,y) = (-4,-8)$ corresponds to a local maximum point.

47) The function $f(x,y) = 2x^3 + 3x^2 y + 2y^3 - 144y + 7$ has stationary point(s). Give the location and the nature of stationary point(s).




At the stationary point, $\dfrac{d\{f(x,y)\}}{dx} = \dfrac{d\{f(x,y)\}}{dy} = 0$. Therefore we find $(x,y)$ which satisfies $\dfrac{d\{f(x,y)\}}{dx} = \dfrac{d\{f(x,y)\}}{dy} = 0$ as follows:

$$\frac{d\{f(x,y)\}}{dx} = 6x^2 + 6xy = 6x(x+y) = 0 \qquad ① \;\; ; \;\; \frac{d\{f(x,y)\}}{dy} = 3x^2 + 6y^2 - 144 = 0 \qquad ②$$

① gives $x = 0, -y$. When $x = 0$, ② gives $6y^2 = 144$ *i.e.*, $y = \pm 2\sqrt{6}$. When $x = -y$, ② gives $9y^2 = 144$ *i.e.*, $y = \pm 4$. Therefore the stationary point is $(x,y) = (0, \pm 2\sqrt{6}), (\mp 4, \pm 4)$. Now we need $\dfrac{d^2 f(x,y)}{dx^2}$, $\dfrac{\partial^2 f(x,y)}{\partial y \partial x}$, $\dfrac{\partial^2 f(x,y)}{\partial y^2}$ to find out the nature of the stationary point.

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\{6x^2 + 6xy\}}{dx} = 12x + 6y \qquad ③ \;\; ; \;\; \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{d\{3x^2 + 6y^2 - 144\}}{dx} = 6x \qquad ④$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\{3x^2 + 6y^2 - 144\}}{dy} = 12y \qquad ⑤$$

The value of the discriminant at $(x,y) = (0, \pm 2\sqrt{6})$ is

$$\frac{d^2 f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 = (12x + 6y) \cdot 12y - (6x)^2\Big|_{(x,y)=(0,2\sqrt{6})} = 72 \cdot (\pm 2\sqrt{6})^2 > 0$$

Therefore the stationary point at $(x,y)=(0,2\sqrt{6})$ corresponds to a local minimum point as $\frac{d^2f(x,y)}{dx^2}|_{(x,y)=(0,2\sqrt{6})}>0$ . The stationary point at $(x,y)=(0,-2\sqrt{6})$ corresponds to a local maximum point as $\frac{d^2f(x,y)}{dx^2}|_{(x,y)=(0,-2\sqrt{6})}<0$ . The value of the discriminant at $(x,y)=(\mp 4,\pm 4)$ is

$$\frac{d^2f(x,y)}{dx^2}\cdot\frac{\partial^2 f(x,y)}{\partial y^2}-\left(\frac{\partial^2 f(x,y)}{\partial x\partial y}\right)^2=(72-36-12^2)\cdot 4^2<0$$

Therefore the the stationary points at $(x,y)=(\mp 4,\pm 4)$ corresponds to saddle points.

48) Find the maximum value of $f(x,y)=s(s-x)(s-y)(x+y-s)$ when

$$0<x<s,$$
$$0<y<s,$$
$$s<x+y.$$

Hint Theorem 10a

The procedure to solve the problem is

a) find $x$ and $y$ which satisfy Equation (45)
b) confirm Theorem 10a

$$\frac{d\{f(x,y)\}}{dx}=\frac{d\{s(s-x)(s-y)(x+y-s)\}}{dx}$$

$$=s(s-y)\left\{\frac{d\{s-x\}}{dx}(x+y-s)+(s-x)\frac{d\{x+y-s\}}{dx}\right\}(\because s(s-y)\text{is constant})$$

$$=s(s-y)\{(-1)(x+y-s)+(s-x)\cdot 1\}=s(s-y)\{-x-y+s+s-x\}=s(s-y)\{2s-y-2x\}$$

$$\frac{d\{f(x,y)\}}{dy}=\frac{d\{s(s-x)(s-y)(x+y-s)\}}{dy}$$

$$=s(s-x)\left\{\frac{d\{s-y\}}{dy}(x+y-s)+(s-y)\frac{d\{x+y-s\}}{dy}\right\}(\because s(s-x)\text{is constant})$$

$$=s(s-x)\{-1\cdot(x+y-s)+(s-y)\cdot 1\}=s(s-x)\{-x-y+s+s-y\}=s(s-x)\{2s-x-2y\}$$

Since $s(s-x)$ and $s(s-y)\neq 0$, Equation (45) can be realised when

$$2s-y-2x=0 \quad ①$$
$$2s-x-2y=0 \quad ②$$

$2\times$ ①− ② gives

$$2s-3x=0,$$

*i.e.,*

$$x=\frac{2s}{3}$$

and when this is put into ① , we obtain

$$y=\frac{2s}{3}.$$

Therefore, if $f(x,y)$ has a local maximum or local minimum, that happens when

$$(x,y)=(\frac{2s}{3},\frac{2s}{3}).$$

Furthermore, we need to find the nature of this point. For that, we must find the second derivatives $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^2 f(x,y)}{\partial x \partial y}$ and $\frac{\partial^2 f(x,y)}{\partial y \partial x}$

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\{s(s-y)(2s-y-2x)\}}{dx}$$
$$= s(s-y)\frac{d\{2s-y-2x\}}{dx}\left(\because s(s-y) \text{ is constant}\right)$$
$$= s(s-y)(-2) < 0$$
$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\{s(s-x)(2s-x-2y)\}}{dy}$$
$$= s(s-x)\frac{d\{(2s-x-2y)\}}{dy}\left(\because s(s-x) \text{ is constant}\right) = -2s(s-x)$$
$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dy}$$
$$= \frac{d\{s(s-y)(2s-y-2x)\}}{dy} = s\left\{\frac{d\{s-y\}}{dy}(2s-y-2x) + (s-y)\frac{d\{2s-y-2x\}}{dy}\right\}$$
$$= s\left\{-(2s-y-2x) + (s-y)(-1)\right\} = -s\{2s-y-2x+s-y\} = -s\{3s-2y-2x\}$$
$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dx} = \frac{d\{s(s-x)(2s-x-2y)\}}{dx}$$
$$= s\left\{\frac{d\{(s-x)\}}{dx}(2s-x-2y) + (s-x)\frac{d\{(2s-x-2y)\}}{dx}\right\}$$
$$= s\{-(2s-x-2y) + (s-x)(-1)\}$$
$$= -s\{2s-x-2y+s-x\} = -s\{3s-2x-2y\}$$

$$\frac{d^2 f(x,y)}{dx^2}\frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial y \partial x}\right)^2\Bigg|_{\substack{x=\frac{2s}{3}\\ y=\frac{2s}{3}}} = s(s-y)(-2)\cdot(-2)s(s-x) - \{-s(3s-2x-2y)\}^2\Big|_{\substack{x=\frac{2s}{3}\\ y=\frac{2s}{3}}}$$
$$= 4s^2(s-y)(s-x) - s^2(3s-2x-2y)^2\Big|_{\substack{x=\frac{2s}{3}\\ y=\frac{2s}{3}}} = 4s^2(s-2s/3)^2 - s^2(3s-8s/3)^2 = 4s^2(s/3)^2 - s^2(s/3)^2$$
$$\left(\because s - \frac{2s}{3} = 1\cdot s - \frac{2s}{3} = \frac{3s}{3} - \frac{2s}{3} = \frac{s}{3}3s - 8\frac{s}{3} = \frac{9s}{3} - \frac{8s}{3} = \frac{s}{3}\right) = 3s^2(s/3)^2 > 0$$

Thus,

$$f(x,y) = s(s-x)(s-y)(x+y-s)$$

is has a local maximum when $(x,y) = (\frac{2s}{3}, \frac{2s}{3})$ and the maximum value is $f(\frac{2s}{3}, \frac{2s}{3}) = s\frac{2s}{3}\frac{2s}{3}\frac{s}{3} = \frac{2^2 s^4}{3^3}$.

49) What is $\frac{4x}{y} + \frac{2x}{3y}$.

$$\frac{4x}{y} + \frac{2x}{3y} = \frac{4x\times 3}{y\times 3} + \frac{2x}{3y} = \frac{12x}{3y} + \frac{2x}{3y} = \frac{12x+2x}{3y} = \frac{14x}{3y}$$

50) Simplify $\dfrac{s^6+(s2^5)^2}{s^4}$.

$$\frac{s^6+(s2^5)^2}{s^4}=\frac{s^6+s^2 2^{10}}{s^4}=\frac{s^6}{s^4}+\frac{s^2 2^{10}}{s^4}=s^2+\frac{2^{10}}{s^2}$$

51) If $a=9$, is $18a^2-4a>0$ true or false?

$$18a^2-4a>0\ ;\quad \therefore 18\times(9)^2-(4\times 9)>0\ ;\quad \therefore 18\times 81-36>0$$
$$\therefore 1458-36>0\ ;\quad \therefore 1422>0 \qquad \text{True}$$

52) Find $\dfrac{d^2 f(x,y)}{dx^2}+\dfrac{\partial^2 f(x,y)}{\partial y^2}$ when $f(x,y)=\ln(x^2+y^2)$

We use Equation (41) in order to find the second order differentiation. $u \triangleq x^2+y^2$

$$\frac{d\{f(x,y)\}}{dx}=\frac{d\{\ln(x^2+y^2)\}}{dx}=\frac{d\{\ln(u)\}}{dx}=\frac{d\{u\}}{dx}\frac{\partial \ln(u)}{\partial u}=\frac{d\{x^2+y^2\}}{dx}\frac{\partial\{\ln(u)\}}{\partial u}=2x\frac{1}{u}=\frac{2x}{x^2+y^2}$$

$$\therefore \frac{d^2 f(x,y)}{dx^2}=\frac{d\left\{\dfrac{2x}{x^2+y^2}\right\}}{dx}=\frac{\dfrac{\partial 2x}{\partial x}(x^2+y^2)-2x\dfrac{\partial\{x^2+y^2\}}{\partial x}}{(x^2+y^2)^2}$$
$$=\frac{2(x^2+y^2)-2x(2x)}{(x^2+y^2)^2}=\frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2}=\frac{-2x^2+2y^2}{(x^2+y^2)^2}$$

$$\frac{d\{f(x,y)\}}{dy}=\frac{d\{\ln(x^2+y^2)\}}{dy}=\frac{d\{\ln(u)\}}{dy}=\frac{d\{u\}}{dy}\frac{\partial\{\ln(u)\}}{\partial u}=\frac{d\{x^2+y^2\}}{dy}\frac{\partial\{\ln(u)\}}{\partial u}=2y\frac{1}{u}=\frac{2y}{x^2+y^2}$$

$$\therefore \frac{\partial^2 f(x,y)}{\partial y^2}=\frac{d\left\{\dfrac{2y}{x^2+y^2}\right\}}{dy}=\frac{\dfrac{d\{2y\}}{dy}(x^2+y^2)-2y\dfrac{d\{x^2+y^2\}}{dy}}{(x^2+y^2)^2}=\frac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2}$$
$$=\frac{2x^2+2y^2-4y^2}{(x^2+y^2)^2}=\frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{d^2 f(x,y)}{dx^2}+\frac{\partial^2 f(x,y)}{\partial y^2}=\frac{-2x^2+2y^2}{(x^2+y^2)^2}+\frac{2x^2-2y^2}{(x^2+y^2)^2}=0$$

53) Find $\dfrac{d^2y}{dx^2}$ and express $\dfrac{d^2y}{dx^2}$ using $y$, $\dfrac{d\{y\}}{dx}$ and $x$ (*i.e.,* produce a differential equation) when $y=a\sin(x+b)$.

First we find $\dfrac{d\{y\}}{dx}$

$$\frac{d\{y\}}{dx}=\frac{d\{a\sin(x+b)\}}{dx}=\frac{d\{u\}}{dx}\frac{\partial\{a\sin(u)\}}{\partial u}[\because u\triangleq x+b]=\frac{d\{x+b\}}{dx}(a\cos(u))=a\cos(x+b)$$

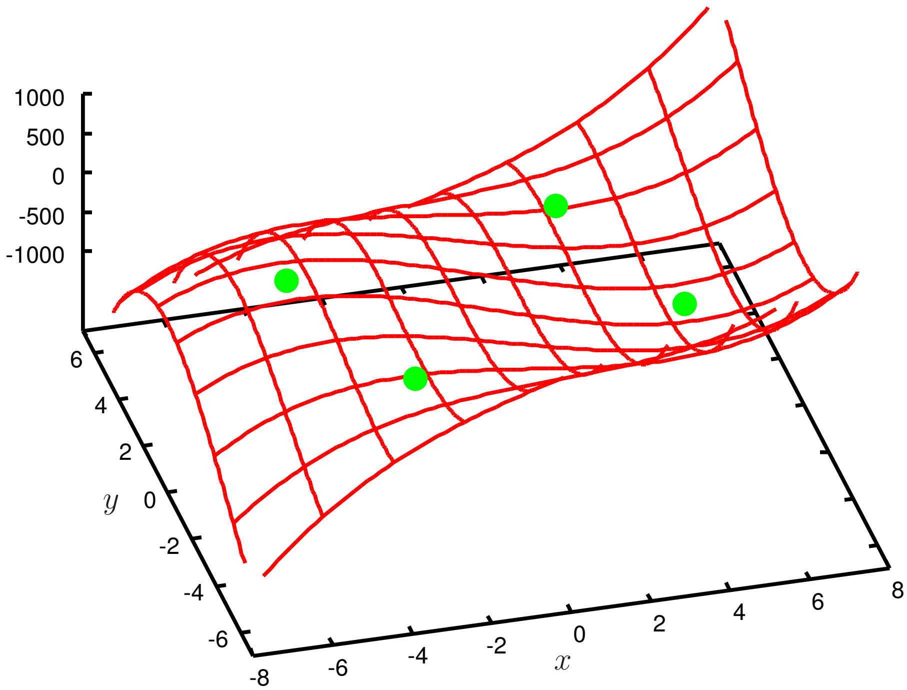
Then we find $\dfrac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2}=\frac{d\{a\cos(x+b)\}}{dx}=\frac{d\{u\}}{dx}\frac{\partial\{a\cos(u)\}}{\partial u}=\frac{d\{x+b\}}{dx}(-a\sin(u))=-a\sin(x+b)=-y$$

Thus the differential equation is $\dfrac{d^2y}{dx^2}+y=0$.

**DAY6**

54) The function $f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x$ has stationary point(s). Give the location and the nature of stationary point(s).




At the stationary point, $\dfrac{d\{f(x,y)\}}{dx} = \dfrac{d\{f(x,y)\}}{dy} = 0$. Therefore we find $(x,y)$ which satisfies $\dfrac{d\{f(x,y)\}}{dx} = \dfrac{d\{f(x,y)\}}{dy} = 0$ as follows:

$$\frac{d\{f(x,y)\}}{dx} = 6x^2 + 6y^2 - 150 = 0 \qquad ① \;\; ; \quad \frac{d\{f(x,y)\}}{dy} = 12xy - 9y^2 = 0 \qquad ②$$

② gives $y = 0, \frac{4x}{3}$. When $y = 0$, ① gives $6x^2 = 150$ *i.e.*, $x = \pm 5$. When $y = \frac{4x}{3}$, ① gives $6x^2 + \frac{32x^2}{3} - 150 = 0$ *i.e.*, $x = \pm 3$. In this case $y = \frac{4x}{3} = \pm 4$. Therefore the stationary point is $(x,y) = (\pm 5, 0), (\pm 3, \pm 4)$. Now we need $\dfrac{d^2 f(x,y)}{dx^2}, \dfrac{\partial^2 f(x,y)}{\partial y \partial x}, \dfrac{\partial^2 f(x,y)}{\partial y^2}$ to find out the nature of the stationary point.

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\{6x^2 + 6y^2 - 150\}}{dx} = 12x \qquad ③ \;\; ; \quad \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{d\{12xy - 9y^2\}}{dx} = 12y \qquad ④$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\{12xy - 9y^2\}}{dy} = 12x - 18y \qquad ⑤$$

The value of the discriminant at $(x,y) = (\pm 5, 0)$ is

$$\frac{d^2 f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 = 12x \cdot (12x - 18y) - (12y)^2$$
$$= 72(2x^2 - 3xy - 2y^2)|_{(x,y)=(\pm 5,0)} = 72 \cdot 2 \cdot (\pm 5)^2 > 0$$

Therefore the stationary point at $(x,y) = (5,0)$ corresponds to a local minimum point as $\dfrac{d^2 f(x,y)}{dx^2}|_{(x,y)=(5,0)} > 0$ . Therefore the stationary point at $(x,y) = (-5,0)$ corresponds to a local maximum point as $\frac{d^2f(x,y)}{dx^2}|_{(x,y)=(-5,0)} < 0$ . The stationary point at $(x,y) = (\pm3,\pm4)$ corresponds to a saddle point because the value of the discriminant at $(x,y) = (\pm3,\pm4)$ is

$$\frac{d^2f(x,y)}{dx^2}\cdot\frac{\partial^2f(x,y)}{\partial y^2} - \left(\frac{\partial^2f(x,y)}{\partial x\partial y}\right)^2 = 72(2x^2-3xy-2y^2)|_{(x,y)=(\pm3,\pm4)} = 72(-50) < 0$$

55) The function $f(x,y) = \mathfrak{e}^{x^2}\sin(y)$ for $-1 < x < 1$ and $-\pi < y < \pi$ has stationary point(s). Give the location and the nature of stationary point(s).

At the stationary point, $\frac{d\{f(x,y)\}}{dx} = \frac{d\{f(x,y)\}}{dy} = 0$. Therefore we find $(x,y)$ which satisfies $\frac{d\{f(x,y)\}}{dx} = \frac{d\{f(x,y)\}}{dy} = 0$ as follows:

$$\frac{d\{f(x,y)\}}{dx} = 2x\mathfrak{e}^{x^2}\sin y = 0 \qquad ① \;;\; \frac{d\{f(x,y)\}}{dy} = \mathfrak{e}^{x^2}\cos y = 0 \qquad ②$$

② gives $y = \pm\frac{\pi}{2}$. ① gives $x = 0$. When $(x,y) = (0,\pm\frac{\pi}{2})$ $\frac{d\{f(x,y)\}}{dx} = \frac{d\{f(x,y)\}}{dy} = 0$. Therefore the stationary point is $(x,y) = (0,\pm\frac{\pi}{2})$ Now we need $\frac{d^2f(x,y)}{dx^2}$,$\frac{\partial^2f(x,y)}{\partial y\partial x}$, $\frac{\partial^2f(x,y)}{\partial y^2}$ to find out the nature of the stationary point.

$$\frac{d^2f(x,y)}{dx^2} = 2\mathfrak{e}^{x^2}\sin y + (2x)^2\mathfrak{e}^{x^2}\sin y \qquad ③ \;;\; \frac{\partial^2f(x,y)}{\partial x\partial y} = 2x\mathfrak{e}^{x^2}\cos y \qquad ④$$

$$\frac{\partial^2f(x,y)}{\partial y^2} = -\mathfrak{e}^{x^2}\sin y \qquad ⑤$$

The value of the discriminant at $(x,y) = (0,\pm\frac{\pi}{2})$ is

$$\frac{d^2f(x,y)}{dx^2}\cdot\frac{\partial^2f(x,y)}{\partial y^2} - \left(\frac{\partial^2f(x,y)}{\partial x\partial y}\right)^2 = (\pm2)\cdot(\mp1) - 0^2 < 0$$

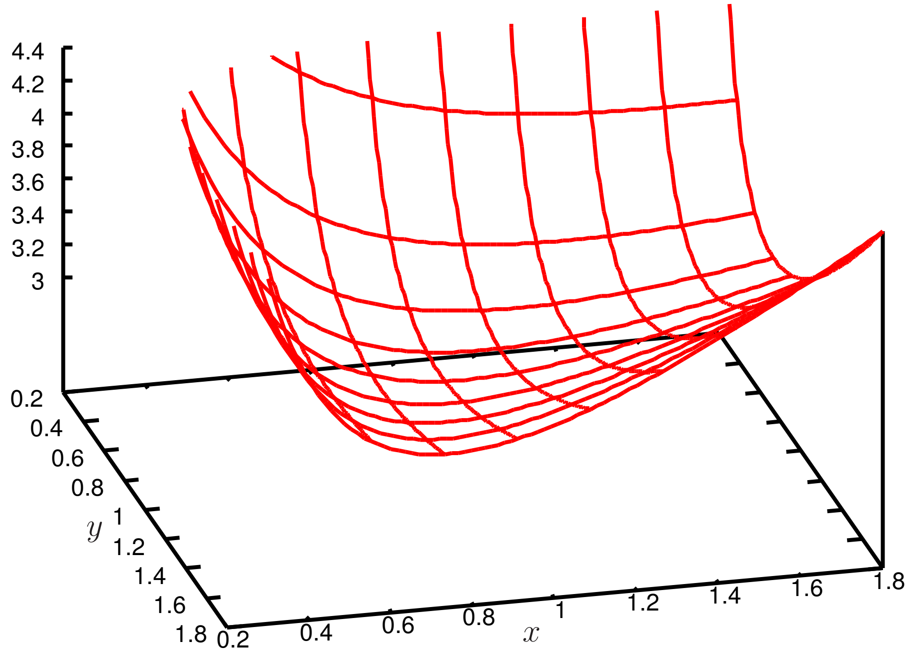
Therefore the stationary points at $(x,y) = (0,\pm\frac{\pi}{2})$ correspond to saddle points

56) The function $f(x,y) = \frac{1}{x} + \frac{1}{y} + xy$ has stationary point(s) when $x > 0$ and $y > 0$. Give the location and the nature of stationary point(s).

At the stationary point, $\frac{d\{f(x,y)\}}{dx} = \frac{d\{f(x,y)\}}{dy} = 0$. Therefore we find $(x,y)$ which satisfies $\frac{d\{f(x,y)\}}{dx} = \frac{d\{f(x,y)\}}{dy} = 0$ as follows:

$$\frac{d\{f(x,y)\}}{dx} = -x^{-2} + y = 0 \qquad ① \;;\; \frac{d\{f(x,y)\}}{dy} = -y^{-2} + x = 0 \qquad ②$$

① gives $y = x^{-2}$. When we put $y = x^{-2}$ into ② we get $-x^4 + x = x(-x^3+1) = 0$. As $x > 0$, we obtain $-x^3 + 1 = 0$ *i.e.*,$x = 1$. Then when we put $x = 1$ into $y = x^{-2}$, we obtain $y = 1$. Therefore the stationary point is $(x,y) = (1,1)$




Now we need $\frac{d^2f(x,y)}{dx^2}, \frac{\partial^2 f(x,y)}{\partial y \partial x}, \frac{\partial^2 f(x,y)}{\partial y^2}$ to find out the nature of the stationary point.

$$\frac{d^2f(x,y)}{dx^2} = \frac{d\{-x^{-2}+y\}}{dx} = 2x^{-3} \qquad ③ \ ; \ \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{d\{-y^{-2}+x\}}{dx} = 1 \qquad ④$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\{-y^{-2}+x\}}{dy} = 2y^{-3} \qquad ⑤$$

The value of the discriminant at $(x,y) = (1,1)$ is

$$\frac{d^2f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 = 2x^{-3} \cdot (2y^{-3}) - 1^2|_{(x,y)=(1,1)} = 4 - 1 > 0$$

Therefore the stationary point at $(x,y) = (1,1)$ corresponds to a local minimum point as $\frac{d^2f(x,y)}{dx^2}|_{(x,y)=(1,1)} > 0$ .

57) Find a local maximum and/or a local minimum of $f(x,y) = x^3 - 3axy + y^3$ ( $a > 0$) where $x$ and $y$ are real.

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{x^3 - 3axy + y^3\}}{dx} = 3x^2 - 3ay$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{x^3 - 3axy + y^3\}}{dy} = -3ax + 3y^2$$

$$\frac{d^2f(x,y)}{dx^2} = \frac{d\{3x^2 - 3ay\}}{dx} = 6x \ ; \ \frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\{-3ax + 3y^2\}}{dy} = 6y \ ; \ \frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{d\{3x^2 - 3ay\}}{dy} = -3a$$

$$\frac{d^2f(x,y)}{dx^2}\frac{\partial^2 f(x,y)}{\partial y^2} - (\frac{\partial^2 f(x,y)}{\partial y \partial x})^2 = (6x) \cdot (6y) - (-3a)^2 = 36xy - 9a^2$$

When $f(x,y)$ has a local maximum or minimum, both $\frac{d\{f(x,y)\}}{dx}$ and $\frac{d\{f(x,y)\}}{dy}$ equal zero. Therefore

$$\frac{d\{f(x,y)\}}{dx} = 3x^2 - 3ay = 0 \ ; \ \frac{d\{f(x,y)\}}{dy} = -3ax + 3y^2 = 0$$

$$\therefore -3ax + 3y^2 = 3x^2 - 3ay = 0 \ ; \ \therefore -ax + y^2 - x^2 + ay = 0$$

$$\therefore (x+y)(y-x) + a(y-x) = 0 \ ; \ \therefore (y-x)(x+y+a) = 0$$

Therefore $x+y+a=0$ or $y-x=0$. When $x+y+a=0$, we can manipulate $3x^2-3ay=0$ as

$$3x^2-3ay=0 \;\; ; \quad \therefore 3x^2-3a(-a-x)=0 \;\; ; \quad \therefore 3x^2+3ax+3a^2=0$$

$$\therefore x^2+ax+a^2=0 \;\; ; \quad \therefore x=\frac{-a\pm\sqrt{3}a\jmath}{2}$$

Since $x$ has to be real and $a\neq 0$, $x\neq\dfrac{-a\pm\sqrt{3}a\jmath}{2}$. Thus $x+y+a\neq 0$. When $y-x=0$,

$$3x^2-3ay=0 \;\; ; \quad \therefore 3x^2-3ax=0 \;\; ; \quad \therefore 3x(x-a)=0 \;\; ; \quad \therefore x=0,a$$

Thus the points $(x,y)=(0,0)$ and $(x,y)=(a,a)$ satisfy $\dfrac{d\{f(x,y)\}}{dx}=0$ and $\dfrac{d\{f(x,y)\}}{dy}=0$. Therefore, $(x,y)=(0,0)$ and $(x,y)=(a,a)$ are the possible local minimum or local maximum. When $(x,y)=(0,0)$ then

$$\frac{d^2f(x,y)}{dx^2}\frac{\partial^2 f(x,y)}{\partial y^2}-\left(\frac{\partial^2 f(x,y)}{\partial y\partial x}\right)^2\Bigg|_{\substack{x=0\\y=0}}=-9a^2<0.$$

Therefore following Theory 10c, the point $(x,y)=(0,0)$ is a saddle point, not local minimum or local maximum. When $(x,y)=(a,a)$

$$\frac{d^2f(x,y)}{dx^2}\frac{\partial^2 f(x,y)}{\partial y^2}-\left(\frac{\partial^2 f(x,y)}{\partial y\partial x}\right)^2\Bigg|_{\substack{x=a\\y=a}}=36\,(a)^2-9a^2=27a^2>0.$$

Furthermore $\dfrac{d^2f(a,a)}{dx^2}=6x\big|_{x=a}=6a>0$. Therefore, $f(x,y)$ has a local minimum at $(x,y)=(a,a)$ and $f(a,a)=a^3-3a^3+a^3=-a^3$.

58) Evaluate the following expression $-5^2-[7-3(8-2^3)]^2$.

$$-5^2-[7-3(8-2^3)]^2=-25-[7-3(8-8)]^2=-25-[7-3(0)]^2=-25-[7]^2=-25-49=-74$$

59) Simplify the following expression $(10x^6)^3(x+2y)^0$.

$$(10x^6)^3(x+2y)^0=(10x^6)^3\cdot 1=10^3x^{6\times 3}\cdot 1=10^3x^{18}$$

60) Solve the following equation $5+3(x-1)=3x-2(x-3)$.

$$5+3(x-1)=3x-2(x-3) \;\; ; \quad \therefore 5+3x-3=3x-2x+6$$

$$\therefore 3x-3x+2x=6-5+3 \;\; ; \quad \therefore 2x=4 \;\; ; \quad \therefore x=2$$

61) Find $\dfrac{d\{h(x,y)\}}{dx}$, $\dfrac{d\{h(x,y)\}}{dy}$, $\dfrac{d^2h(x,y)}{dx^2}$, $\dfrac{\partial^2 h(x,y)}{\partial y\partial x}$, $\dfrac{\partial^2 h(x,y)}{\partial y^2}$, $\dfrac{d^2h(x,y)}{dx^2}+\dfrac{\partial^2 h(x,y)}{\partial y^2}$ when $h(x,y)=a+bx+cy+dx^2+gxy+fy^2$

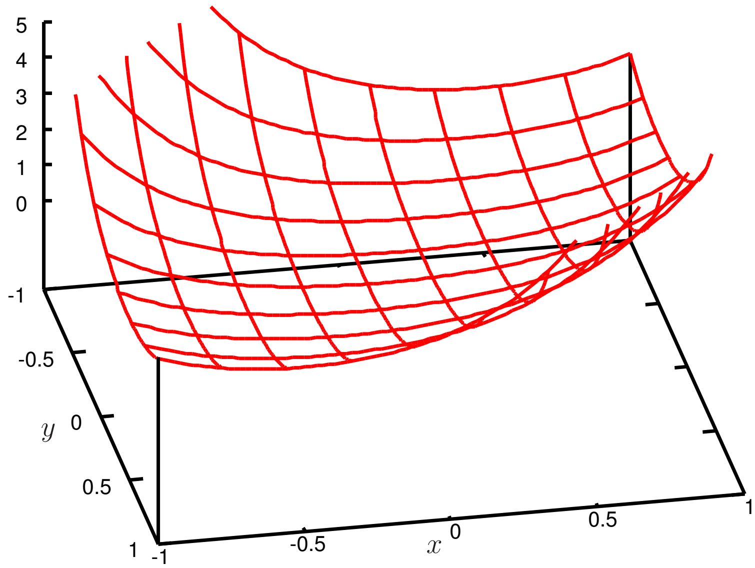
$$\frac{d\{h(x,y)\}}{dx} = \frac{d\{bx + dx^2 + gxy\}}{dx} = b + 2dx + gy$$
$$\frac{d\{h(x,y)\}}{dy} = \frac{d\{cy + gxy + fy^2\}}{dy} = c + gx + 2fy$$
$$\frac{\partial^2 h(x,y)}{\partial y \partial x} = \frac{d\left\{\frac{d\{h(x,y)\}}{dx}\right\}}{dy} = \frac{d\{b + 2dx + gy\}}{dy} = g$$
$$\frac{d^2 h(x,y)}{dx^2} = \frac{d\left\{\frac{d\{h(x,y)\}}{dx}\right\}}{dx} = \frac{d\{b + 2dx + gy\}}{dx} = 2d$$
$$\frac{\partial^2 h(x,y)}{\partial y^2} = \frac{d\left\{\frac{d\{h(x,y)\}}{dy}\right\}}{dy} = \frac{d\{c + gx + 2fy\}}{dy} = 2f \;\; ; \;\; \frac{d^2 h(x,y)}{dx^2} + \frac{\partial^2 h(x,y)}{\partial y^2} = 2d + 2f$$

62) Find $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{d^2 f(x,y)}{dx^2} + \frac{\partial^2 f(x,y)}{\partial y^2}$
when $f(x,y) = \mathfrak{e}^{ax}(\cos by + \sin by)$

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{\mathfrak{e}^{ax}(\cos by + \sin by)\}}{dx} = \frac{d\{\mathfrak{e}^{ax}\}}{dx}(\cos by + \sin by) + \mathfrak{e}^{ax}\frac{d\{(\cos by + \sin by)\}}{dx}$$
$$= \frac{d\{u\}}{dx}\frac{\partial \mathfrak{e}^u}{\partial u}(\cos by + \sin by) + \mathfrak{e}^{ax} \cdot 0 [\because u \triangleq ax]$$
$$= \frac{d\{ax\}}{dx}\mathfrak{e}^u(\cos by + \sin by) = a\mathfrak{e}^{ax}(\cos by + \sin by)$$
$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{\mathfrak{e}^{ax}(\cos by + \sin by)\}}{dy}$$
$$= \frac{d\{\mathfrak{e}^{ax}\}}{dy}(\cos by + \sin by) + \mathfrak{e}^{ax}\frac{d\{\cos by + \sin by\}}{dy}$$
$$= 0 \cdot (\cos by + \sin by) + \mathfrak{e}^{ax}\frac{d\{v\}}{dy}\frac{\partial\{\cos v + \sin v\}}{\partial v}[\because v \triangleq by]$$
$$= \mathfrak{e}^{ax}\frac{d\{by\}}{dy}(-\sin v + \cos v) = \mathfrak{e}^{ax} \cdot b \cdot (-\sin by + \cos by)$$
$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dy} = \frac{d\{a\mathfrak{e}^{ax}(\cos by + \sin by)\}}{dy}$$
$$= a\mathfrak{e}^{ax}\frac{d\{\cos by + \sin by\}}{dy} = a\mathfrak{e}^{ax}(-b \sin by + b \cos by)$$
$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\{a\mathfrak{e}^{ax}(\cos by + \sin by)\}}{dx}$$
$$= a(\cos by + \sin by)\frac{d\{\mathfrak{e}^{ax}\}}{dx} = a(\cos by + \sin by)(a\mathfrak{e}^{ax}) = a^2(\cos by + \sin by)\mathfrak{e}^{ax}$$
$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\{\mathfrak{e}^{ax} \cdot b \cdot (-\sin by + \cos by)\}}{dy} = \mathfrak{e}^{ax} \cdot b \cdot \frac{d\{-\sin by + \cos by\}}{dy}$$
$$= \mathfrak{e}^{ax} \cdot b \cdot (-b \cos by - b \sin by) = -b^2(\cos by + \sin by)\mathfrak{e}^{ax}$$

$$\therefore \frac{d^2 f(x,y)}{dx^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = a^2(\cos by + \sin by)\mathfrak{e}^{ax} + (-b^2(\cos by + \sin by)\mathfrak{e}^{ax}) = (a^2 - b^2)(\cos by + \sin by)\mathfrak{e}^{ax}$$

63) Find the co-ordinates of the point of inflection of the function $f(x,y) = \mathfrak{e}^{x^2+xy+y^2}$ and find the nature of the curve at this point.
Hint: Key point 10a




Firstly we need to find the co-ordinates at which the function turns. Let's find $\frac{d\{f(x,y)\}}{dx}$ and $\frac{d\{f(x,y)\}}{dy}$.
Let $u = x^2 + xy + y^2$ Now using the chain rule.

$$\frac{d\{f(x,y)\}}{dx} = \frac{d\{u\}}{dx} \cdot \frac{\partial\{f(x,y)\}}{\partial u}$$

Now we need $\frac{d\{u\}}{dx}$ and $\frac{\partial\{f(x,y)\}}{\partial u}$:

$$\frac{d\{u\}}{dx} = 2x + y \ \ ; \ \ \frac{\partial\{f(x,y)\}}{\partial u} = \mathfrak{e}^u = \mathfrak{e}^{x^2+xy+y^2}$$

Therefore

$$\frac{d\{f(x,y)\}}{dx} = (2x+y)\mathfrak{e}^{x^2+xy+y^2}$$

In the same way we find $\frac{d\{f(x,y)\}}{dy}$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{u\}}{dy} \cdot \frac{\partial\{f(x,y)\}}{\partial u}$$

Now we need $\frac{d\{u\}}{dy}$ and $\frac{\partial\{f(x,y)\}}{\partial u}$

$$\frac{d\{u\}}{dy} = 2y + x \ \ ; \ \ \frac{\partial\{f(x,y)\}}{\partial u} = \mathfrak{e}^u = \mathfrak{e}^{x^2+xy+y^2}$$

Therefore

$$\frac{d\{f(x,y)\}}{dy} = (2y+x)\mathfrak{e}^{x^2+xy+y^2}$$

At the point of inflection, both $\frac{d\{f(x,y)\}}{dx}$ and $\frac{d\{f(x,y)\}}{dy}$ equal 0.

$$\frac{d\{f(x,y)\}}{dx} = (2x+y)\mathfrak{e}^{x^2+xy+y^2} = 0 \;\; ; \;\; \frac{d\{f(x,y)\}}{dy} = (2y+x)\mathfrak{e}^{x^2+xy+y^2} = 0$$

$$\therefore 2x+y=0, \quad 2y+x=0 \quad (\because \mathfrak{e}^{x^2+xy+y^2} \neq 0) \;\; ; \;\; \therefore 2x+y = 2y+x \;\; ; \;\; \therefore x = y \;\; ;$$
$$\therefore 2x+x=0 \;\; ; \;\; \therefore x=y=0$$

Therefore the turning point is at $(x,y)=(0,0)$. Now we need to find the nature of this point. For that, we need $\frac{d^2f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$ and $\frac{\partial^2 f(x,y)}{\partial y^2}$. Using the product rule we can find these.

$$\frac{d^2f(x,y)}{dx^2} = ((2x+y)^2+2)\mathfrak{e}^{x^2+xy+y^2} \;\; ; \;\; \frac{\partial^2 f(x,y)}{\partial y^2} = ((2y+x)^2+2)\mathfrak{e}^{x^2+xy+y^2}$$
$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = ((2y+x)(2x+y)+1)\mathfrak{e}^{x^2+xy+y^2}$$

Now substitute in $x=0, y=0$ and using Theory 10a,10b,10c and 10d, we work out the nature of the point $(x,y)=(0,0)$.

$$\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=0\\y=0}} = \left.((2x+y)^2+2)\mathfrak{e}^{x^2+xy+y^2}\right|_{\substack{x=0\\y=0}} = 2$$
$$\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=0\\y=0}} = \left.((2y+x)^2+2)\mathfrak{e}^{x^2+xy+y^2}\right|_{\substack{x=0\\y=0}} = 2$$
$$\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=0\\y=0}} = \left.((2y+x)(2x+y)+1)\mathfrak{e}^{x^2+xy+y^2}\right|_{\substack{x=0\\y=0}} = 1$$
$$\frac{d^2f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left[\frac{\partial^2 f(x,y)}{\partial y \partial x}\right]^2 = 2\cdot 2 - 1^2 = 4-1 = 3 > 0$$

Therefore the point $(x,y)=(0,0)$ is the minimum point of inflection because $\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=0\\y=0}} > 0$ and $\left.\frac{d^2f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left[\frac{\partial^2 f(x,y)}{\partial y \partial x}\right]^2\right|_{\substack{x=0\\y=0}} > 0$.

**DAY7**

64) An electric potential $\phi$ is given by

$$\phi(x, y, z) = xy \sin z + x^2 y + y^2 z + z^2 x$$

Find the directional derivative of the electric potential $\phi$ at the point $P(1, -1, \pi)$ in the direction of the vector $\boldsymbol{n} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$.

- First approach The gradient $\nabla\phi$ is

$$\begin{aligned}\nabla\phi &= \frac{d\{\phi\}}{dx}\boldsymbol{i} + \frac{d\{\phi\}}{dy}\boldsymbol{j} + \frac{d\{\phi\}}{dy}\boldsymbol{k} \\ &= (y \sin z + 2xy + z^2)\boldsymbol{i} + (x \sin z + x^2 + 2yz)\boldsymbol{j} + (xy \cos z + y^2 + 2zx)\boldsymbol{k}\end{aligned}$$

The magnitude of $\boldsymbol{n}$ is $|\boldsymbol{n}| = \sqrt{1+1+1} = \sqrt{3}$. Therefore the unit vector of $\boldsymbol{n}$ is $\frac{\boldsymbol{n}}{|\boldsymbol{n}|} = \frac{\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k}}{\sqrt{3}}$. So the directional derivative at $P(1, -1, \pi)$ is

$$\begin{aligned}\nabla\phi \cdot \frac{\boldsymbol{n}}{|\boldsymbol{n}|}\bigg|_{(x,y,z)=(1,-1,\pi)} &= \frac{y \sin z + 2xy + z^2 - (x \sin z + x^2 + 2yz) - (xy \cos z + y^2 + 2zx)}{\sqrt{3}}\bigg|_{(x,y,z)=(1,-1,\pi)} \\ &= \frac{-2 + \pi^2 - 1 + 2\pi - 2 - 2\pi}{\sqrt{3}} = \frac{-5 + \pi^2}{\sqrt{3}}\end{aligned}$$

- Second approach The gradient $\nabla\phi$ is

$$\begin{aligned}\nabla\phi &= \frac{d\{\phi\}}{dx}\boldsymbol{i} + \frac{d\{\phi\}}{dy}\boldsymbol{j} + \frac{d\{\phi\}}{dy}\boldsymbol{k} \\ &= (y \sin z + 2xy + z^2)\boldsymbol{i} + (x \sin z + x^2 + 2yz)\boldsymbol{j} + (xy \cos z + y^2 + 2zx)\boldsymbol{k}\end{aligned}$$

At $P(1, -1, \pi)$, the gradient is $(-2 + \pi^2)\boldsymbol{i} + (1 - 2\pi)\boldsymbol{j} + (2 + 2\pi)\boldsymbol{k} \triangleq \boldsymbol{v}$. Now we need to find the magnitude of $\boldsymbol{n}$-directional component of $\boldsymbol{v}$. When the angle between $\boldsymbol{n}$ and $\boldsymbol{v}$ is $\theta$, the magnitude of $\boldsymbol{n}$-directional component of $\boldsymbol{v}$ can be written as $|\boldsymbol{v}| \cos\theta$. As $\boldsymbol{n} \cdot \boldsymbol{v} = |\boldsymbol{n}||\boldsymbol{v}| \cos\theta$, we can obtain the magnitude as

$$|\boldsymbol{v}| \cos\theta = |\boldsymbol{v}| \frac{\boldsymbol{n} \cdot \boldsymbol{v}}{|\boldsymbol{n}||\boldsymbol{v}|} = \frac{\boldsymbol{n} \cdot \boldsymbol{v}}{|\boldsymbol{n}|}$$

The magnitude of $\boldsymbol{n}$ is $|\boldsymbol{n}| = \sqrt{1+1+1} = \sqrt{3}$. Therefore

$$\frac{\boldsymbol{n} \cdot \boldsymbol{v}}{|\boldsymbol{n}|} = \frac{-2 + \pi^2 - (1 - 2\pi) - (2 + 2\pi)}{\sqrt{3}} = \frac{-2 + \pi^2 - 1 + 2\pi - 2 - 2\pi}{\sqrt{3}} = \frac{-5 + \pi^2}{\sqrt{3}}$$

65) A total resistance $Z$ given by the formula

$$\frac{1}{Z} = \jmath\omega L + \frac{1}{\jmath\omega C} + \frac{1}{R}$$

Find the derivative

$$\frac{dZ}{dC}.$$

Using the chain rule, we take the partial derivative of both sides with respect to $C$. Note that $L$, $C$ and $R$ are all independent of each other because $Z$ can take any value. Thus $\frac{dL}{dC} = \frac{dR}{dC} = 0$. On the other hand $Z$ changes depending on $C$. Therefore $Z$ is the function of $C$ and $\frac{dZ}{dC}$ does exist.

$$\frac{1}{Z} = \jmath\omega L + \frac{1}{\jmath\omega C} + \frac{1}{R}$$

$$\therefore \frac{d}{dC}\frac{1}{Z} = \frac{d\left(\jmath\omega L + \frac{1}{\jmath\omega C} + \frac{1}{R}\right)}{dC}$$

$$\therefore \frac{dZ}{dC}\frac{d}{dZ}\frac{1}{Z} = \frac{d\left(\jmath\omega L\right)}{dC} + \frac{d\left(\frac{1}{\jmath\omega C}\right)}{dC} + \frac{d\left(\frac{1}{R}\right)}{dC}$$

$$\therefore \frac{dZ}{dC}(-\frac{1}{z^2}) = \frac{d\left(\frac{1}{\jmath\omega C}\right)}{dC} = \jmath\omega\frac{d\left(\frac{1}{C}\right)}{dC} = \jmath\omega(-\frac{1}{C^2})$$

$$\therefore \frac{dZ}{dC} = \jmath\omega(\frac{z^2}{C^2})$$

66) Let $P = P(x, y)$, and $x = \mathfrak{e}^t$ and $y = \mathfrak{e}^{-t}$. Find the total derivative $\frac{dP}{dt}$ in terms of partial derivatives $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$. Hence find the second total derivative $\frac{d^2P}{dt^2}$ in terms of partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial^2 P}{\partial x^2}$, $\frac{\partial^2 P}{\partial y^2}$, and $\frac{\partial^2 P}{\partial x \partial y}$. You may assume that the two mixed partial derivatives are equal. From the chain rule, we can say that :

$$\begin{aligned} \frac{d\{P\}}{dt} &= \frac{d\{P\}}{dx}\cdot\frac{d\{x\}}{dt} + \frac{d\{P\}}{dy}\cdot\frac{d\{y\}}{dt} \\ &= \frac{d\{P\}}{dx}\cdot\frac{d\{\mathfrak{e}^t\}}{dt} + \frac{d\{P\}}{dy}\cdot\frac{d\{\mathfrak{e}^{-t}\}}{dt} \\ &= \frac{d\{P\}}{dx}\cdot\mathfrak{e}^t + \frac{d\{P\}}{dy}\cdot(-\mathfrak{e}^{-t}) \\ &= \frac{d\{P\}}{dx}\cdot x - \frac{d\{P\}}{dy}\cdot y \qquad ③ \end{aligned}$$

Now we differentiate again with respect to $t$ as

$$\frac{\partial^2 P}{\partial t^2} = \frac{d\left\{\frac{d\{P\}}{dt}\right\}}{dt} = \frac{d\left\{x\frac{d\{P\}}{dx} - y\frac{d\{P\}}{dy}\right\}}{dt}$$

$$= \frac{d\{x\}}{dt}\frac{d\{P\}}{dx} + x\frac{d\left\{\frac{d\{P\}}{dx}\right\}}{dt} - \frac{d\{y\}}{dt}\frac{d\{P\}}{dy} - y\frac{d\left\{\frac{d\{P\}}{dy}\right\}}{dt}$$

$$= \frac{d\{x\}}{dt}\frac{d\{P\}}{dx} + x(\frac{d\{x\}}{dt}\frac{d\left\{\frac{d\{P\}}{dx}\right\}}{dx} + \frac{d\{y\}}{dt}\frac{d\left\{\frac{d\{P\}}{dx}\right\}}{dy})$$

$$-\frac{d\{y\}}{dt}\frac{d\{P\}}{dy} - y(\frac{d\{x\}}{dt}\frac{d\left\{\frac{d\{P\}}{dy}\right\}}{dx} + \frac{d\{y\}}{dt}\frac{d\left\{\frac{d\{P\}}{dy}\right\}}{dy})$$

$$= \frac{d\{\mathfrak{e}^t\}}{dt}\frac{d\{P\}}{dx} + x(\frac{d\{\mathfrak{e}^t\}}{dt}\frac{d\left\{\frac{d\{P\}}{dx}\right\}}{dx} + \frac{d\{\mathfrak{e}^{-t}\}}{dt}\frac{d\left\{\frac{d\{P\}}{dx}\right\}}{dy})$$

$$
\begin{aligned}
&-\frac{d\left\{\mathfrak{e}^{-t}\right\}}{dt}\frac{d\left\{P\right\}}{dy}-y(\frac{d\left\{\mathfrak{e}^{t}\right\}}{dt}\frac{d\left\{\frac{d\left\{P\right\}}{dy}\right\}}{dx}+\frac{d\left\{\mathfrak{e}^{-t}\right\}}{dt}\frac{d\left\{\frac{d\left\{P\right\}}{dy}\right\}}{dy})\\
&=\mathfrak{e}^{t}\frac{d\left\{P\right\}}{dx}+x(\mathfrak{e}^{t}\frac{d\left\{\frac{d\left\{P\right\}}{dx}\right\}}{dx}-\mathfrak{e}^{-t}\frac{d\left\{\frac{d\left\{P\right\}}{dx}\right\}}{dy})\\
&+\mathfrak{e}^{-t}\frac{d\left\{P\right\}}{dy}-y(\mathfrak{e}^{t}\frac{d\left\{\frac{d\left\{P\right\}}{dy}\right\}}{dx}-\mathfrak{e}^{-t}\frac{d\left\{\frac{d\left\{P\right\}}{dy}\right\}}{dy})\\
&=x\frac{d\left\{P\right\}}{dx}+x(x\frac{d^2P}{dx^2}-y\frac{\partial^2P}{\partial x\partial y})\\
&+y\frac{d\left\{P\right\}}{dy}-y(x\frac{\partial^2P}{\partial y\partial x}-y\frac{\partial^2P}{\partial y^2})\\
&=x\frac{d\left\{P\right\}}{dx}+x^2\frac{d^2P}{dx^2}-xy\frac{\partial^2P}{\partial x\partial y}+y\frac{d\left\{P\right\}}{dy}-yx\frac{\partial^2P}{\partial y\partial x}+y^2\frac{\partial^2P}{\partial y^2}\\
&=x\frac{d\left\{P\right\}}{dx}+x^2\frac{d^2P}{dx^2}-2xy\frac{\partial^2P}{\partial y\partial x}+y\frac{d\left\{P\right\}}{dy}+y^2\frac{\partial^2P}{\partial y^2}(\because\frac{\partial^2P}{\partial y\partial x}=\frac{\partial^2P}{\partial x\partial y})
\end{aligned}
$$

**DAY8**

67) Locate all stationary points for the function $f(x,y) = 2x^3 + 3x^2y + 2y^3 - 144y + 7$. How many stationary points are there ?

At the stationary point, $\frac{d\{f(x,y)\}}{dx} = \frac{d\{f(x,y)\}}{dy} = 0$. Therefore we find $(x,y)$ which satisfies $\frac{d\{f(x,y)\}}{dx} = \frac{d\{f(x,y)\}}{dy} = 0$ as follows:

$$\frac{d\{f(x,y)\}}{dx} = 6x^2 + 6xy = 6x(x+y) = 0 \qquad ① \; ; \quad \frac{d\{f(x,y)\}}{dy} = 3x^2 + 6y^2 - 144 = 0 \qquad ②$$

① gives $x = 0, -y$. When $x = 0$, ② gives $6y^2 = 144$ *i.e.*, $y = \pm 2\sqrt{6}$. When $x = -y$, ② gives $9y^2 = 144$ *i.e.*, $y = \pm 4$. Therefore the stationary points are 4 points of $(x,y) = (0, \pm 2\sqrt{6}), (\mp 4, \pm 4)$.

68) A point $(x,y) = (-4,-8)$ is one of the stationary points of the function $f(x,y) = 12xy - 3y^2 + 2x^3$. Find the nature of this stationary point.

We need $\frac{d^2 f(x,y)}{dx^2}, \frac{\partial^2 f(x,y)}{\partial y \partial x}, \frac{\partial^2 f(x,y)}{\partial y^2}$ to find out the nature of the stationary point.

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\{12y + 6x^2\}}{dx} = 12x \qquad ③ \; ; \quad \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{d\{12x - 6y\}}{dx} = 12 \qquad ④$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\{12x - 6y\}}{dy} = -6 \qquad ⑤$$

The value of the discriminant at $(x,y) = (-4,-8)$ is

$$\frac{d^2 f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2 = 12 \cdot (-4) \cdot (-6) - 12^2 > 0$$

$$\left.\frac{d^2 f(x,y)}{dx^2}\right|_{x=-4} = \left.\frac{d\{12y + 6x^2\}}{dx}\right|_{x=-4} = 12 \cdot (-4) < 0$$

Therefore the the stationary point at $(x,y) = (-4,-8)$ corresponds to a local maximum point.

69) Explain why, for the function $f(x,y) = (x+y)\mathfrak{e}^{-xy}$, the stationary point at $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$ is a saddle point despite both $\frac{d^2 f(x,y)}{dx^2}$ and $\frac{\partial^2 f(x,y)}{\partial y^2}$ being negative.

We need $\frac{d^2 f(x,y)}{dx^2}, \frac{\partial^2 f(x,y)}{\partial y \partial x}, \frac{\partial^2 f(x,y)}{\partial y^2}$ to find out the nature of the stationary points.

$$\frac{d^2 f(x,y)}{dx^2} = \mathfrak{e}^{-xy}(-2y + xy^2 + y^3) \qquad ③ \; ; \quad \frac{\partial^2 f(x,y)}{\partial x \partial y} = \mathfrak{e}^{-xy}(-2x - 2y + xy^2 + x^2 y) \qquad ④$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \mathfrak{e}^{-xy}(-2x + yx^2 + x^3) \qquad ⑤$$

When $(x,y) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$$\frac{d^2 f(x,y)}{dx^2} = \mathfrak{e}^{-\frac{1}{2}}(\frac{1}{\sqrt{2}}) \; ; \quad \frac{\partial^2 f(x,y)}{\partial x \partial y} = \mathfrak{e}^{-\frac{1}{2}}(\frac{3}{\sqrt{2}}) \; ; \quad \frac{\partial^2 f(x,y)}{\partial y^2} = \mathfrak{e}^{-\frac{1}{2}}(\frac{1}{\sqrt{2}})$$

The discriminant $D$ is

$$D = \left.\frac{d^2 f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2\right|_{(x,y)=(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}$$

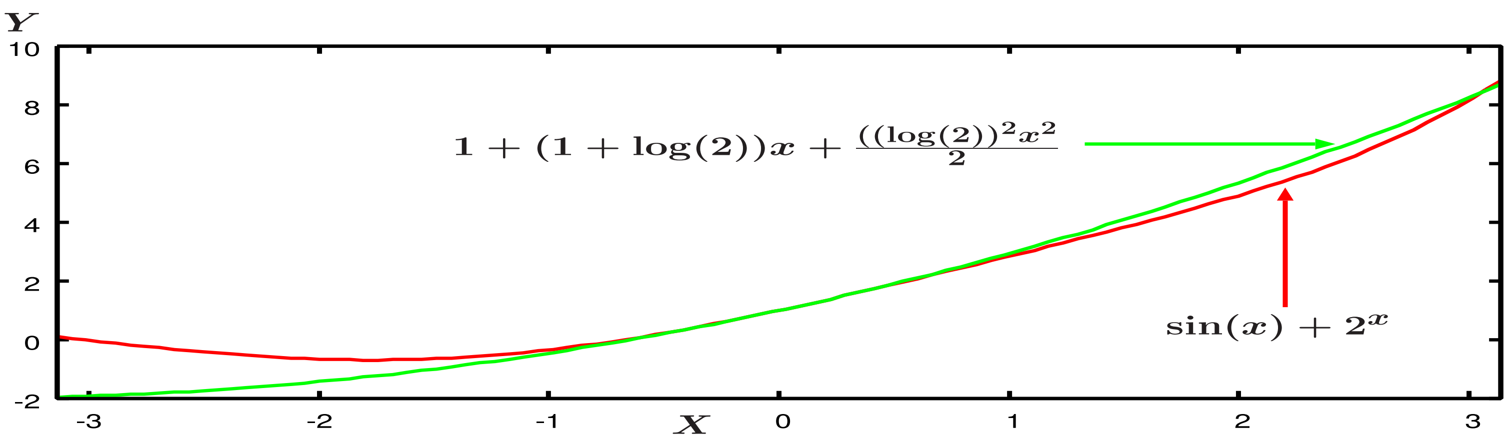
$$= \left.\frac{d^2 f(x,y)}{dx^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2 f(x,y)}{\partial x \partial y}\right)^2\right|_{(x,y)=(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})} = \mathfrak{e}^{-\frac{1}{2}-\frac{1}{2}}(\frac{1}{2} - \frac{9}{2}) < 0$$

Thus the stationary point $(x,y) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ is a saddle point.

1) **DAY1**
2) For the function

$$f(x) = \sin x + 2^x$$

a) find the Taylor polynomial of degree three for about the point $x = 0$ by calculating the appropriate derivatives.




Taylor series is written as

$$f(x) = f(a) + (x-a)\left.\frac{\partial f}{\partial x}\right|_{x=a} + \frac{(x-a)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=a} + 100\frac{h(x-a)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=a} + \cdots + \frac{(x-a)^n}{n!}\left.\frac{\partial^n f}{\partial x^n}\right|_{x=a}$$

First we need to find out $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, and $\frac{\partial^3 f}{\partial x^3}$.

$$f(x) = \sin x + 2^x \;\; ; \;\; \frac{d\{f(x)\}}{dx} = \frac{d\{\sin x + 2^x\}}{dx} = \cos x + 2^x \ln 2$$

$$\frac{d^2 f(x)}{dx^2} = \frac{d\left\{\frac{d\{f(x)\}}{dx}\right\}}{dx} = \frac{d\{\cos x + 2^x \ln 2\}}{dx} = -\sin x + 2^x(\ln 2)^2$$

$$\frac{\partial^3 f(x)}{\partial x^3} = \frac{d\left\{\frac{d^2 f(x)}{dx^2}\right\}}{dx} = \frac{d\{-\sin x + 2^x(\ln 2)^2\}}{dx} = -\cos x + 2^x(\ln 2)^3$$

Then we need to find $\left.\frac{\partial f}{\partial x}\right|_{x=0}$, $\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=0}$, and $\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=0}$.

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.\cos x + 2^x \ln 2\right|_{x=0} = \cos 0 + 2^0 \ln 2 = 1 + \ln 2$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=0} = \left.-\sin x + 2^x(\ln 2)^2\right|_{x=0} = -\sin 0 + 2^0(\ln 2)^2 = (\ln 2)^2$$

$$\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=0} = -\cos 0 + 2^0(\ln 2)^3 = -1 + (\ln 2)^3$$

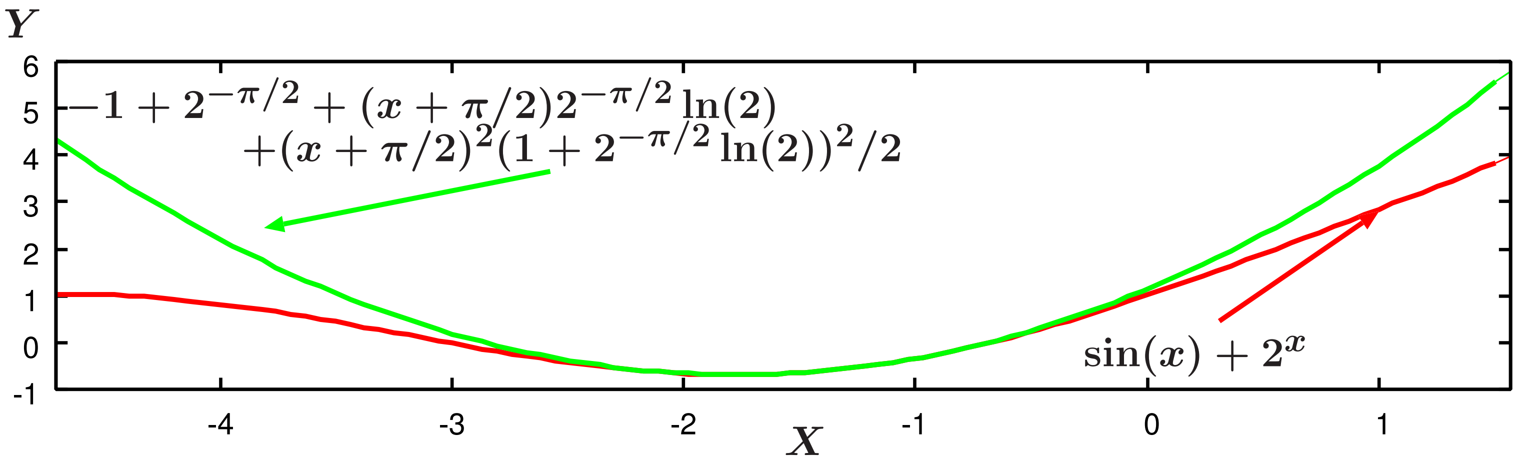
Now by simply substituting into the formula for the Taylor series expansion, we get

$$f(x) = f(0) + x\left.\frac{\partial f}{\partial x}\right|_{x=0} + \frac{x^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=0} + \frac{x^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=0}$$

$$\therefore f(x) = \sin 0 + 2^0 + x(1 + \ln 2) + \frac{x^2}{2!}(\ln 2)^2 + \frac{x^3}{3!}\left(-1 + (\ln 2)^3\right)$$

$$= 1 + (1 + \ln 2)x + \frac{(\ln 2)^2 x^2}{2} + \frac{(-1 + (\ln 2)^3)\, x^3}{6}$$

b) find the quadratic approximation $f(x)$ around $x = -\frac{\pi}{2}$ of the Taylor series expansion




In order to work out $f(x)$ around $x = -\frac{\pi}{2}$, we need to find out $a$ in Equation (82). we get $a = -\frac{\pi}{2}$. This means we are going to find out the Taylor series expansion around $x = -\frac{\pi}{2}$ We have already found $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, and $\frac{\partial^3 f}{\partial x^3}$. Since the quadratic approximation is required, we just need to find out $f(-\frac{\pi}{2})$ $\frac{\partial f}{\partial x}\Big|_{x=-\frac{\pi}{2}}$, and $\frac{\partial^2 f}{\partial x^2}\Big|_{x=-\frac{\pi}{2}}$.

$$f(-\frac{\pi}{2}) = \sin(-\frac{\pi}{2}) + 2^{-\frac{\pi}{2}} = -1 + 2^{-\frac{\pi}{2}}$$

$$\frac{d\{f(x)\}}{dx}\Big|_{x=-\frac{\pi}{2}} = \cos(-\frac{\pi}{2}) + 2^{-\frac{\pi}{2}}\ln 2 = 2^{-\frac{\pi}{2}}\ln 2$$

$$\frac{d^2 f(x)}{dx^2}\Big|_{x=-\frac{\pi}{2}} = -\sin(-\frac{\pi}{2}) + 2^{-\frac{\pi}{2}}(\ln 2)^2 = 1 + 2^{-\frac{\pi}{2}}(\ln 2)^2$$

Now by simply substituting these into the formula for the Taylor series expansion, we get

$$f(x) = f(-\frac{\pi}{2}) + (x + \frac{\pi}{2})\ \frac{\partial f}{\partial x}\Big|_{x=-\frac{\pi}{2}} + \frac{(x+\frac{\pi}{2})^2}{2!}\ \frac{\partial^2 f}{\partial x^2}\Big|_{x=-\frac{\pi}{2}}$$

$$\therefore f(x) = -1 + 2^{-\frac{\pi}{2}} + (x + \frac{\pi}{2})\cdot 2^{-\frac{\pi}{2}}\ln 2 + \frac{(x+\frac{\pi}{2})^2}{2!}(1 + 2^{-\frac{\pi}{2}}(\ln 2)^2)$$

$$= -1 + 2^{-\frac{\pi}{2}} + (x + \frac{\pi}{2})\cdot 2^{-\frac{\pi}{2}}\ln 2 + \frac{(x+\frac{\pi}{2})^2(1 + 2^{-\frac{\pi}{2}}(\ln 2)^2)}{2} \qquad ①$$

c) the value that the quadratic approximation gives for $f(-\frac{2\pi}{5})$ correct to 5 decimal places and the percentage error correct to 2 significant figures.
We have already worked out the second order approximation of $f(x)$ around $x = -\frac{\pi}{2}$. In order to find out $f(-\frac{2\pi}{5})$ using the approximation of $f(x)$ around $x = -\frac{\pi}{2}$, we need to know the exact value of $x - a$.

$$x - a = -\frac{2\pi}{5} + \frac{\pi}{2}$$
$$= \frac{\pi}{10}$$

Therefore we obtain $x - a = \frac{\pi}{10}$ which satisfies the condition of $|x - a| \ll 1$. By substituting $x = -\frac{\pi}{2}$ into ①, we get

$$f(-\frac{2\pi}{5}) = -1 + 2^{-\frac{\pi}{2}} + \frac{\pi}{10}\cdot 2^{-\frac{\pi}{2}}\ln 2 + \frac{(\frac{\pi}{10})^2(1 + 2^{-\frac{\pi}{2}}(\ln 2)^2)}{2} = -0.532515 \simeq -0.53252$$

correct to 5 decimal places. Since $f(-\frac{2\pi}{5}) = -0.53254$, the error is

$$\frac{-0.53252 + 0.53254}{-0.53254} = -3.75559 \cdot 10^{-5}.$$

The percentage error is

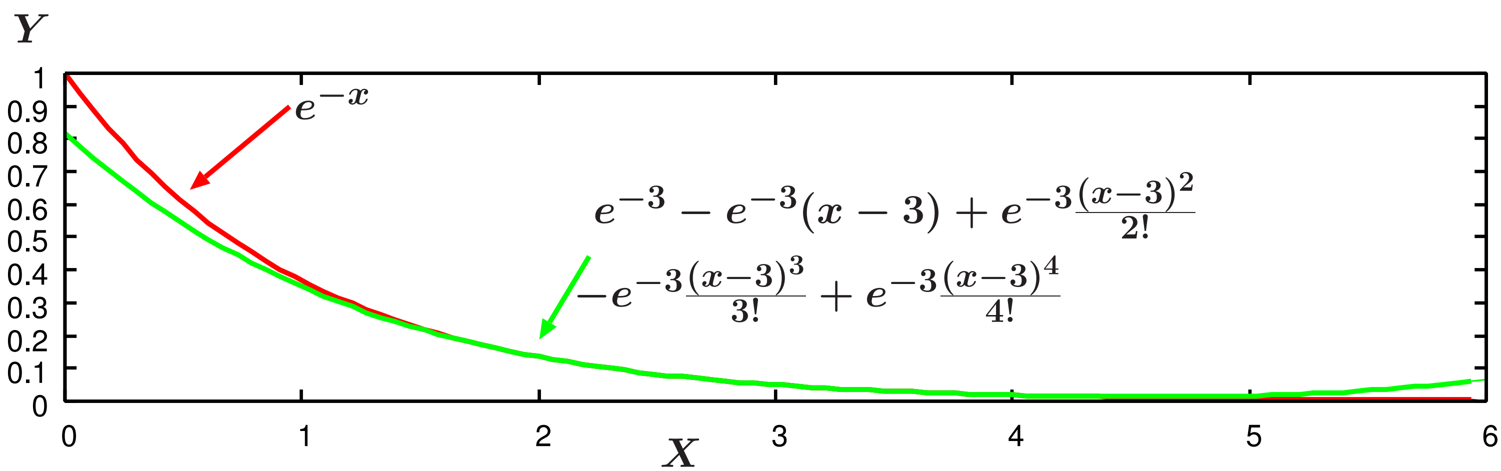
$$\frac{-0.53252 + 0.53254}{-0.53254} \times 100 = -3.75559 \cdot 10^{-3}\% \simeq -3.8 \cdot 10^{-3}\%$$

correct to 2 significant figures.

3) For the function

$$f(x) = \mathfrak{e}^{-x}$$

a) find the Taylor polynomial of degree four for about the point $x = 3$.




Taylor series is written as

$$f(x) = f(a) + (x-a)\left.\frac{\partial f}{\partial x}\right|_{x=a} + \frac{(x-a)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=a} + \frac{(x-a)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=a} + \frac{(x-a)^4}{4!}\left.\frac{\partial^4 f}{\partial x^4}\right|_{x=a}$$

Since we are going to find out the Taylor series around $x = 3$, we set $a$ as 3. First we need to find out $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^3 f}{\partial x^3}$, and $\frac{\partial^4 f}{\partial x^4}$.

$$f(x) = \mathfrak{e}^{-x} \quad ; \quad \therefore \frac{\partial f}{\partial x} = \frac{d\{\mathfrak{e}^{-x}\}}{dx} = -\mathfrak{e}^{-x} \quad ; \quad \frac{\partial^2 f}{\partial^2 x} = \frac{d\{-\mathfrak{e}^{-x}\}}{dx} = -(-\mathfrak{e}^{-x}) = \mathfrak{e}^{-x}$$

$$\frac{\partial^3 f}{\partial^3 x} = \frac{d\{\mathfrak{e}^{-x}\}}{dx} = -\mathfrak{e}^{-x} \quad ; \quad \frac{\partial^4 f}{\partial^4 x} = \frac{d\{-\mathfrak{e}^{-x}\}}{dx} = -(-\mathfrak{e}^{-x}) = \mathfrak{e}^{-x}$$

Second we need to find out $f(3)$, $\left.\frac{\partial f}{\partial x}\right|_{x=3}$, $\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=3}$, $\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=3}$, and $\left.\frac{\partial^4 f}{\partial x^4}\right|_{x=3}$.

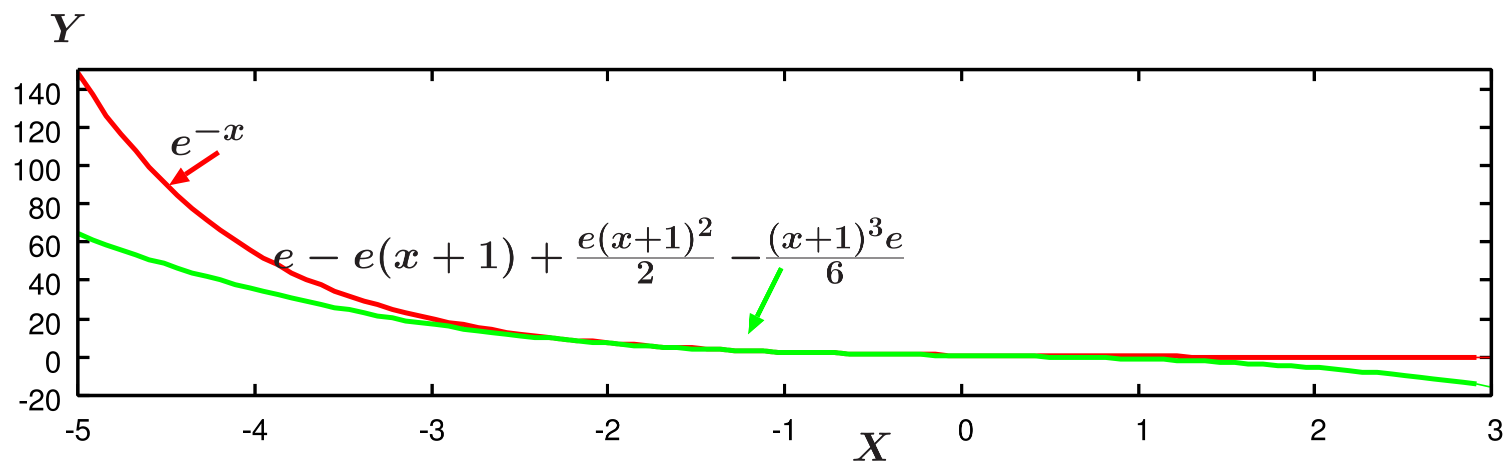
$$f(3) = \mathfrak{e}^{-3} \quad ; \quad \therefore \left.\frac{\partial f}{\partial x}\right|_{x=3} = -\mathfrak{e}^{-3} \quad ; \quad \left.\frac{\partial^2 f}{\partial x^2}\right|_{x=3} = \mathfrak{e}^{-3} \quad ; \quad \left.\frac{\partial^3 f}{\partial x^3}\right|_{x=3} = -\mathfrak{e}^{-3} \quad ; \quad \left.\frac{\partial^4 f}{\partial x^4}\right|_{x=3} = \mathfrak{e}^{-3}$$

Now simply substitute into the formula for the Taylor series expansion we get

$$f(x) = f(3) + (x-3)\left.\frac{\partial f}{\partial x}\right|_{x=3} + \frac{(x-3)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=3} + \frac{(x-3)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=3} + \frac{(x-3)^4}{4!}\left.\frac{\partial^4 f}{\partial x^4}\right|_{x=3}$$

$$= \mathfrak{e}^{-3} - \mathfrak{e}^{-3}(x-3) + \mathfrak{e}^{-3}\frac{(x-3)^2}{2!} - \mathfrak{e}^{-3}\frac{(x-3)^3}{3!} + \mathfrak{e}^{-3}\frac{(x-3)^4}{4!}$$

b) find the cubic approximation $f(x)$ around $x = -1$ of the Taylor series expansion




As we are working out the Taylor series expansion around $x = -1$ We have already found $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^3 f}{\partial x^3}$, and $\frac{\partial^4 f}{\partial x^4}$. Since the cubic approximation is required, we just need to find out $f(-1)$, $\frac{\partial f}{\partial x}\Big|_{x=-1}$, $\frac{\partial^2 f}{\partial x^2}\Big|_{x=-1}$, and $\frac{\partial^3 f}{\partial x^3}\Big|_{x=-1}$.

$$f(-1) = \mathfrak{e} \quad ; \quad ; \quad \frac{d\{f(x)\}}{dx}\bigg|_{x=-1} = -\mathfrak{e} \quad ; \quad \frac{d^2 f(x)}{dx^2}\bigg|_{x=-1} = \mathfrak{e} \quad ; \quad \frac{\partial^3 f(x)}{\partial x^3}\bigg|_{x=-1} = -\mathfrak{e}$$

Now by simply substituting these into the formula for the Taylor series expansion, we get

$$f(x) = f(-1) + (x+1)\frac{\partial f}{\partial x}\bigg|_{x=-1} + \frac{(x+1)^2}{2!}\frac{\partial^2 f}{\partial x^2}\bigg|_{x=-1} + \frac{(x+1)^3}{3!}\frac{\partial^3 f}{\partial x^3}\bigg|_{x=-1}$$

$$\therefore f(x) = \mathfrak{e} + (x+1)(-\mathfrak{e}) + \frac{(x+1)^2}{2!}\mathfrak{e} + \frac{(x+1)^3}{3!}(-\mathfrak{e})$$

$$= \mathfrak{e} - \mathfrak{e}(x+1) + \frac{(x+1)^2\mathfrak{e}}{2} - \frac{(x+1)^3\mathfrak{e}}{6} \qquad ①$$

c) Find the value that the quadratic approximation gives for $f(-0.9)$ correct to 4 decimal places and the percentage error correct to 2 significant figures.
We have already worked out the second order approximation of $f(x)$ around $x = -1$. In order to find out $f(-0.9)$ using the approximation of $f(x)$ around $x = -1$, we need to know the exact value of $x - a$.

$$x - a = -0.9 + 1 = 0.1$$

Therefore we obtain $x - a = 0.1$ which satisfies the condition of $|x - a| \ll 1$. By substituting $x = -0.9$ into up to 3rd terms of ①, we get

$$f(-0.9) = \mathfrak{e} - 0.1\mathfrak{e} + \frac{0.1^2\mathfrak{e}}{2} = 2.46005 \simeq 2.4601$$

correct to 4 decimal places. Since $f(-0.9) = 2.4596$, the percentage error is

$$\frac{2.4601 - 2.4596}{2.4596} \times 100 = 0.0203285\% \simeq 0.020\%$$

correct to 2 significant figures.

4) Let

$$f(x) = \cos(\mathfrak{e}^{2x})$$

a) find the Taylor polynomial of degree two for about the point $x = -1$ by calculating the appropriate derivatives.

Taylor series is written as

$$f(x) = f(a) + (x-a)\left.\frac{\partial f}{\partial x}\right|_{x=a} + \frac{(x-a)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=a} + \frac{(x-a)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=a} + \cdots + \frac{(x-a)^n}{n!}\left.\frac{\partial^n f}{\partial x^n}\right|_{x=a}$$

First we need to find out $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$.

$$f(x) = \cos(\mathfrak{e}^{2x}) \ ; \quad \frac{d\{f(x)\}}{dx} = \frac{d\{\cos(\mathfrak{e}^{2x})\}}{dx} = \frac{d\{\cos(A)\}}{dx} \because A \triangleq \mathfrak{e}^{2x}$$

$$= \frac{d\{A\}}{dx}\frac{d\{\cos(A)\}}{dA} = \frac{d\{\mathfrak{e}^{2x}\}}{dx}(-\sin(A)) = -2\mathfrak{e}^{2x}\sin(\mathfrak{e}^{2x})$$

$$\frac{d^2 f(x)}{dx^2} = \frac{d\left\{\frac{d\{f(x)\}}{dx}\right\}}{dx} = \frac{d\{-2\mathfrak{e}^{2x}\sin(\mathfrak{e}^{2x})\}}{dx} = \frac{d\{-2\mathfrak{e}^{2x}\}}{dx}\sin(\mathfrak{e}^{2x}) - 2\mathfrak{e}^{2x}\frac{d\{\sin(\mathfrak{e}^{2x})\}}{dx}$$

$$= -4\mathfrak{e}^{2x}\sin(\mathfrak{e}^{2x}) - 2\mathfrak{e}^{2x}\frac{d\{\sin(A)\}}{dx} = -4\mathfrak{e}^{2x}\sin(\mathfrak{e}^{2x}) - 2\mathfrak{e}^{2x}\frac{d\{A\}}{dx}\frac{d\{\sin(A)\}}{dA}$$

$$= -4\mathfrak{e}^{2x}\sin(\mathfrak{e}^{2x}) - 2\mathfrak{e}^{2x}\cdot(2\mathfrak{e}^{2x})\cos(A) = -4\mathfrak{e}^{2x}\sin(\mathfrak{e}^{2x}) - 4\mathfrak{e}^{4x}\cos(\mathfrak{e}^{2x}) = -4\mathfrak{e}^{2x}(\sin(\mathfrak{e}^{2x}) + \mathfrak{e}^{2x}\cos(\mathfrak{e}^{2x}))$$

Then we need to find $\left.\frac{\partial f}{\partial x}\right|_{x=-1}$, and $\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=-1}$.

$$\left.\frac{\partial f}{\partial x}\right|_{x=-1} = \left.-2\mathfrak{e}^{2x}\sin(\mathfrak{e}^{2x})\right|_{x=-1} = -2\mathfrak{e}^{-2}\sin(\mathfrak{e}^{-2})$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=-1} = \left.-4\mathfrak{e}^{2x}(\sin(\mathfrak{e}^{2x}) + \cos(\mathfrak{e}^{2x}))\right|_{x=-1} = -4\mathfrak{e}^{-2}(\sin(\mathfrak{e}^{-2}) + \mathfrak{e}^{-2}\cos(\mathfrak{e}^{-2}))$$

Now by simply substituting into the formula for the Taylor series expansion, we get

$$f(x) = f(-1) + (x+1)\left.\frac{\partial f}{\partial x}\right|_{x=-1} + \frac{(x+1)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=-1}$$

$$\therefore f(x) = \cos(\mathfrak{e}^{-2}) + (x+1)\cdot(-2\mathfrak{e}^{-2}\sin(\mathfrak{e}^{-2})) + \frac{(x+1)^2}{2!}\cdot(-4\mathfrak{e}^{-2}(\sin(\mathfrak{e}^{-2}) + \mathfrak{e}^{-2}\cos(\mathfrak{e}^{-2}))) \qquad ①$$

b) the value that the quadratic approximation gives for $f(-1.1)$ correct to 5 decimal places and the percentage error correct to 2 significant figures.
We have already worked out the second order approximation of $f(x)$ around $x = -1$. In order to find out $f(-1.1)$ using the approximation of $f(x)$, we need to know the exact value of $x - a$.

$$\begin{aligned} x - a &= -1.1 + 1 \\ &= -0.1 \end{aligned}$$

Thus we obtain $x-a = -0.1$ which satisfies the condition of $|x-a| \ll 1$. By substituting $x-a = -0.1$ into ①, we get

$$f(-1.1) = \cos(\mathfrak{e}^{-2}) - 0.1(-2\mathfrak{e}^{-2}\sin(\mathfrak{e}^{-2})) + \frac{0.01}{2}(-4\mathfrak{e}^{-2}(\sin(\mathfrak{e}^{-2}) + \cos(\mathfrak{e}^{-2}))) = 0.991461 \simeq 0.99146$$

correct to 5 decimal places. Since $f(-1.1) = 0.993868$, the error is

$$\frac{0.99146 - 0.993868}{0.993868} = -0.00242286$$

The percentage error is

$$\frac{0.99146 - 0.993868}{0.993868} \times 100 = -0.242286\% \simeq -0.24\%$$

correct to 2 significant figures.

5) Evaluate the expression of $10 \div 2 + 4 \cdot (-5) - 12$

$$10 \div 2 + 4 \cdot (-5) - 12 = 5 + (-20) - 12 = 5 - 20 - 12 = 5 - 32 = -27$$

6) Solve for $x$ for $\frac{2x-1}{-3} < -5$

$$\frac{2x-1}{-3} < -5 \ ; \quad \therefore 2x - 1 > -5 \cdot (-3) \ ; \quad \therefore 2x - 1 > 15 \ ; \quad \therefore 2x > 15 + 1 \ ; \quad \therefore x > 8$$

7) Solve for $x$ for $(2x+1)^2 = 25$

$$(2x+1)^2 = 25 \ ; \quad \therefore 2x + 1 = \pm 25^{0.5} \ ; \quad \therefore 2x + 1 = \pm 5 \ ; \quad \therefore 2x = \pm 5 - 1$$
$$\therefore 2x = 5 - 1, -5 - 1 \ ; \quad \therefore 2x = 4, -6 \ ; \quad \therefore x = 2, -3$$

8) Solve for $x$ for $|2x+1| + 3 \geq 7$

$$|2x+1| + 3 \geq 7 \ ; \quad \therefore |2x+1| \geq 7 - 3 \ ; \quad \therefore |2x+1| \geq 4 \ ; \quad \therefore 2x + 1 \geq 4, 2x + 1 \leq -4$$
$$\therefore 2x \geq 4 - 1, 2x \leq -4 - 1 \ ; \quad \therefore 2x \geq 3, 2x \leq -5 \ ; \quad \therefore x \geq 3/2, x \leq -5/2$$

9) Evaluate the following expression

$$|67 - 12(7-9)| - |23 - 43|$$

$$|67 - 12(7-9)| - |23 - 43| = |67 - 12(-2)| - |-20| = |67 + 24| - 20 = 91 - 20 = 71$$

10) Find $\frac{d\{y\}}{dx}$ of $y = \ln x$.

$$\frac{d\{y\}}{dx} = \frac{1}{x}$$

11) Find out $a_1$,$a_2$,$a_3$,$a_4$, and $a_5$ of the following arithmetic progressions.

a) $a_n = 1 + 4n$

b) $a_n = 1 + \frac{n}{2}$

c) $a_n = 1 + n$

a)

$$a_1 = 1 + 4 \cdot 1 = 5 \ ; \quad a_2 = 1 + 4 \cdot 2 = 9 \ ; \quad a_3 = 1 + 4 \cdot 3 = 13 \ ; \quad a_4 = 1 + 4 \cdot 4 = 17 \ ; \quad a_5 = 1 + 4 \cdot 5 = 21$$

The sequences is$\cdots$ $1, 5, 9, 13, 17, 21, \cdots$

b)

$$a_1 = 1 + \frac{1}{2} = 1.5 \ ; \quad a_2 = 1 + \frac{2}{2} = 2 \ ; \quad a_3 = 1 + \frac{3}{2} = 2.5 \ ; \quad a_4 = 1 + \frac{4}{2} = 3 \ ; \quad a_5 = 1 + \frac{5}{2} = 3.5$$

The sequences is$\cdots$ $1, 1.5, 2, 2.5, 3, 3.5 \cdots$

c) The first five terms of $a_{n+1} = 1 + n$ would be when $n = 0, 1, 2, 3, 4$. Therefore when

$$a_1 = 1 + 1 = 2 \ ; \quad a_2 = 1 + 2 = 3 \ ; \quad a_3 = 1 + 3 = 4 \ ; \quad a_4 = 1 + 4 = 5 \ ; \quad a_5 = 1 + 5 = 6$$

The sequences is$\cdots$ $1, 2, 3, 4, 5, 6, \cdots$

12) Find out $a_1$,$a_2$ and $a_3$ of the following geometric progressions.

a) $a_n = 4^n$

b) $a_n = 4 \cdot (-1)^n$

c) $a_n = 2\mathfrak{e}^n$

a)

$$a_1 = 4^1 = 4 \ ; \ a_2 = 4^2 = 16 \ ; \ a_3 = 4^3 = 64$$

The sequences is $\cdots\ 0.25, 1, 4, 16, 64, \cdots$

b)

$$a_1 = 4 \cdot (-1)^1 = -4 \ ; \ a_2 = 4 \cdot (-1)^2 = 4 \ ; \ a_3 = 4 \cdot (-1)^3 = -4$$

The sequences is$\cdots\ 4, -4, 4, -4, 4, \cdots$

c)

$$a_1 = 2\mathfrak{e}^1 = 2\mathfrak{e} \ ; \ a_2 = 2\mathfrak{e}^2 \ ; \ a_3 = 2\mathfrak{e}^3$$

13) Find the Taylor polynomial of degree three for $\sin(5x)$ about the point $x = 0$ by calculating the appropriate derivatives.

The Taylor series can be written as

$$f(x) = f(a) + (x-a)\left.\frac{\partial f}{\partial x}\right|_{x=a} + \frac{(x-a)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=a} + \frac{(x-a)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=a} + \cdots + \frac{(x-a)^n}{n!}\left.\frac{\partial^n f}{\partial x^n}\right|_{x=a}$$

where $a = 0$ because we need the approximation of $f$ about $x = 0$. First we need to find out $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, and $\frac{\partial^3 f}{\partial x^3}$

$$f(x) = \sin(5x) \ ; \ \frac{d\{f(x)\}}{dx} = \frac{d\{\sin(5x)\}}{dx} = 5\cos(5x)$$

$$\frac{d^2 f(x)}{dx^2} = \frac{d\left\{\frac{d\{f(x)\}}{dx}\right\}}{dx} = \frac{d\{5\cos(5x)\}}{dx} = -5^2\sin(5x) \ ; \ \frac{\partial^3 f(x)}{\partial x^3} = \frac{d\left\{\frac{d^2 f(x)}{dx^2}\right\}}{dx}$$
$$= \frac{d\{-5^2\sin(5x)\}}{dx} = -5^3\cos(5x)$$

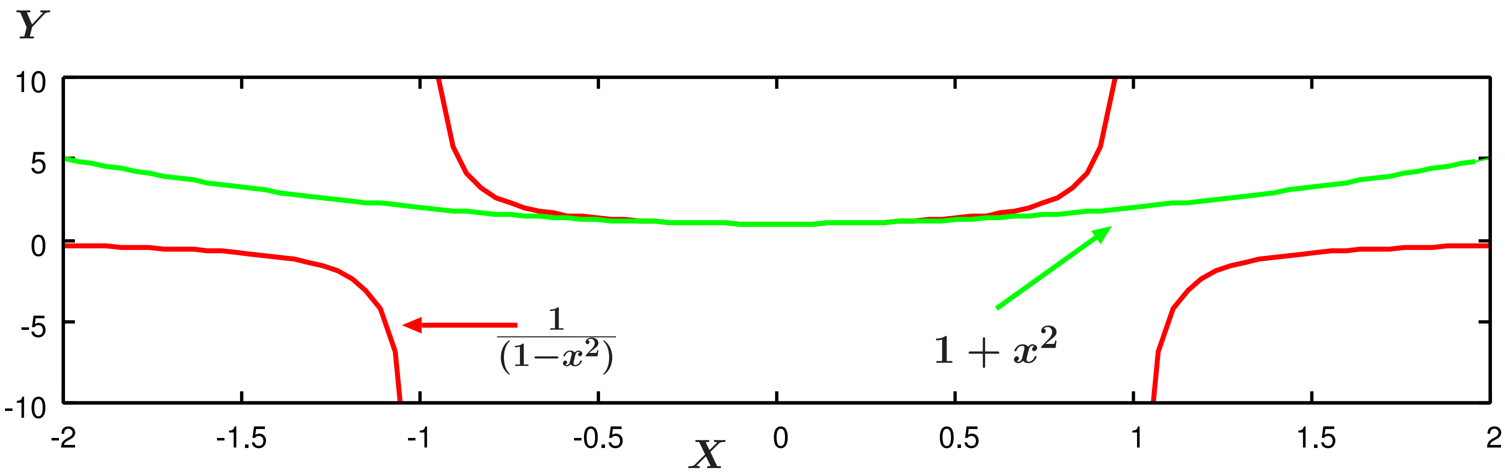
Then we need to find $f(0)$, $\left.\frac{\partial f}{\partial x}\right|_{x=0}$, $\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=0}$, and $\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=0}$.

$$f(0) = \sin(5 \cdot 0) = 0 \ ; \ \frac{d\{f(x)\}}{dx} = 5\cos(5x)|_{x=0} = 5 \ ;$$
$$\frac{d^2 f(x)}{dx^2} = -5^2\sin(5x)|_{x=0} = 0 \ ; \ \frac{\partial^3 f(x)}{\partial x^3} = -5^3\cos(5x)|_{x=0} = -5^3$$

By substituting these into Equation (82),

$$f(x) = f(0) + x\left.\frac{\partial f}{\partial x}\right|_{x=0} + \frac{x^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=0} + \frac{x^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=0}$$
$$= 0 + 5x + 0 \cdot \frac{x^2}{2!} - 5^3 \cdot \frac{x^3}{3!}$$
$$= 5x - 5^3\frac{x^3}{6}$$

14) Find the Taylor polynomial of degree two for $\frac{1}{1-x^2}$ about the point $x = 0$ by calculating the appropriate derivatives.




The Taylor series can be written as

$$f(x) = f(a) + (x-a)\left.\frac{\partial f}{\partial x}\right|_{x=a} + \frac{(x-a)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=a} + \frac{(x-a)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=a} + \cdots + \frac{(x-a)^n}{n!}\left.\frac{\partial^n f}{\partial x^n}\right|_{x=a}$$

where $a = 0$ because we need the approximation of $f$ about $x = 0$. First we need to find out $\frac{\partial f}{\partial x}$, and $\frac{\partial^2 f}{\partial x^2}$

$$f(x) = (1-x^2)^{-1} \;\; ; \;\; \frac{d\left\{f(x)\right\}}{dx} = \frac{d\left\{(1-x^2)^{-1}\right\}}{dx} = \frac{d\left\{(u)^{-1}\right\}}{dx} (\because u \triangleq 1-x^2)$$

$$= \frac{d\left\{u\right\}}{dx}\frac{\partial\left\{(u)^{-1}\right\}}{\partial u} = \frac{d\left\{1-x^2\right\}}{dx}\cdot(-(u)^{-2}) = -2x(-(1-x^2)^{-2}) = 2x(1-x^2)^{-2}$$

$$\frac{d^2 f(x)}{dx^2} = \frac{d\left\{\frac{d\left\{f(x)\right\}}{dx}\right\}}{dx} = \frac{d\left\{2x(1-x^2)^{-2}\right\}}{dx} = 2x\frac{d\left\{(1-x^2)^{-2}\right\}}{dx} + (1-x^2)^{-2}\frac{d\left\{2x\right\}}{dx}$$

$$= 2x\frac{d\left\{(1-x^2)^{-2}\right\}}{dx} + 2(1-x^2)^{-2} = 2x\frac{d\left\{u^{-2}\right\}}{dx} + 2(1-x^2)^{-2} (\because u \triangleq 1-x^2)$$

$$= 2x\frac{d\left\{u\right\}}{dx}\frac{\partial\left\{u^{-2}\right\}}{\partial u} + 2(1-x^2)^{-2} = 2x(-2x)(-2(1-x^2)^{-3}) + 2(1-x^2)^{-2} = 8x^2(1-x^2)^{-3} + 2(1-x^2)^{-2}$$

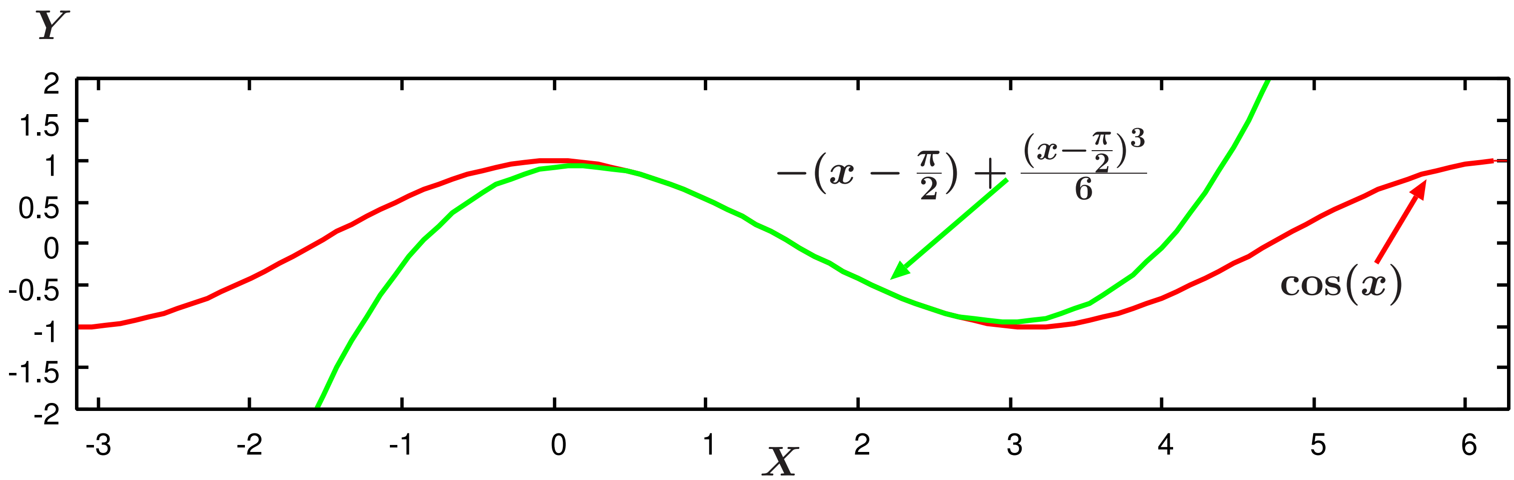
Then we need to find $f(0)$, $\frac{\partial f}{\partial x}\big|_{x=0}$, and $\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=0}$.

$$f(0) = (1-0^2)^{-1} = 1 \;\; ; \;\; \frac{d\left\{f(x)\right\}}{dx} = 2x(1-x^2)^{-2}|_{x=0} = 0 \;\; ; \;\; \frac{d^2 f(x)}{dx^2} = 8x^2(1-x^2)^{-3} + 2(1-x^2)^{-2}|_{x=0} = 2$$

By substituting these into Equation (82),

$$f(x) = f(0) + x\left.\frac{\partial f}{\partial x}\right|_{x=0} + \frac{x^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=0}$$
$$= 1 + 0\cdot x + 2\cdot\frac{x^2}{2!}$$
$$= 1 + x^2$$

15) Find the Taylor polynomial of degree three for $\cos(x)$ about the point $x = \frac{\pi}{2}$ by calculating the appropriate derivatives.




The Taylor series can be written as

$$f(x)=f(a)+(x-a)\left.\frac{\partial f}{\partial x}\right|_{x=a}+\frac{(x-a)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=a}+\frac{(x-a)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=a}+\cdots+\frac{(x-a)^n}{n!}\left.\frac{\partial^n f}{\partial x^n}\right|_{x=a}$$

where $a=\frac{\pi}{2}$ because we need the approximation of $f$ about $x=\frac{\pi}{2}$. First we need to find out $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, and $\frac{\partial^3 f}{\partial x^3}$

$$f(x)=\cos(x)\ \ ;\ \ \frac{d\left\{f(x)\right\}}{dx}=\frac{d\left\{\cos(x)\right\}}{dx}=-\sin(x)$$

$$\frac{d^2 f(x)}{dx^2}=\frac{d\left\{\frac{d\left\{f(x)\right\}}{dx}\right\}}{dx}=\frac{d\left\{-\sin(x)\right\}}{dx}=-\cos(x)\ \ ;\ \ \frac{\partial^3 f(x)}{\partial x^3}=\frac{d\left\{\frac{d^2 f(x)}{dx^2}\right\}}{dx}=\frac{d\left\{-\cos(x)\right\}}{dx}=\sin(x)$$

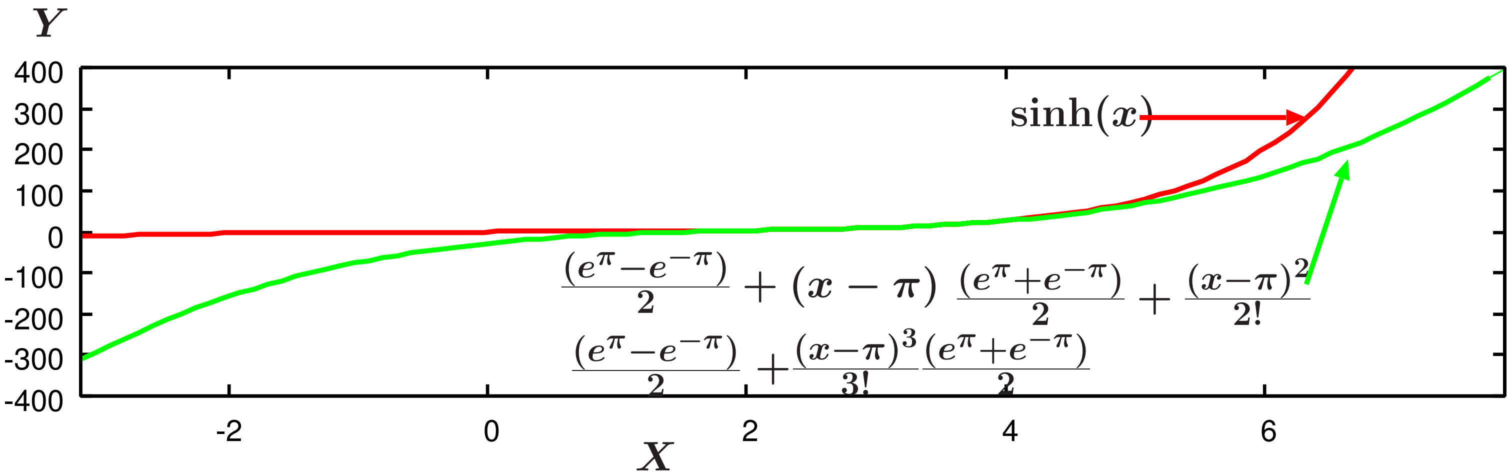
Then we need to find $f\left(\frac{\pi}{2}\right)$, $\left.\frac{\partial f}{\partial x}\right|_{x=\frac{\pi}{2}}$, $\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=\frac{\pi}{2}}$, and $\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=\frac{\pi}{2}}$.

$$f(x)=\cos(x)|_{x=\frac{\pi}{2}}=0\ \ ;\ \ \frac{d\left\{f(x)\right\}}{dx}=-\sin(x)|_{x=\frac{\pi}{2}}=-1\ \ ;$$
$$\frac{d^2 f(x)}{dx^2}=-\cos(x)|_{x=\frac{\pi}{2}}=0\ \ ;\ \ \frac{\partial^3 f(x)}{\partial x^3}=\sin(x)|_{x=\frac{\pi}{2}}=1$$

By substituting these into Equation (82),

$$\begin{aligned}f(x)&=f(\frac{\pi}{2})+(x-\frac{\pi}{2})\left.\frac{\partial f}{\partial x}\right|_{x=\frac{\pi}{2}}+\frac{(x-\frac{\pi}{2})^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=\frac{\pi}{2}}+\frac{(x-\frac{\pi}{2})^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=\frac{\pi}{2}}\\&=0+(x-\frac{\pi}{2})\cdot(-1)+\frac{(x-\frac{\pi}{2})^2}{2!}\cdot 0+\frac{(x-\frac{\pi}{2})^3}{3!}\cdot 1\\&=-(x-\frac{\pi}{2})+\frac{(x-\frac{\pi}{2})^3}{6}\end{aligned}$$

16) Find the Taylor polynomial of degree three for $\sinh(x)$ about the point $x=\pi$ by calculating the appropriate derivatives.




The Taylor series can be written as

$$f(x) = f(a) + (x-a)\left.\frac{\partial f}{\partial x}\right|_{x=a} + \frac{(x-a)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=a} + \frac{(x-a)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=a} + \cdots + \frac{(x-a)^n}{n!}\left.\frac{\partial^n f}{\partial x^n}\right|_{x=a}$$

where $a = \pi$ because we need the approximation of $f$ about $x = \pi$.
First we need to find out $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, and $\frac{\partial^3 f}{\partial x^3}$

$$f(x) = \sinh(x) = \frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2} \quad ; \quad \frac{d\{f(x)\}}{dx} = \frac{d\left\{\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2}\right\}}{dx} = \frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}$$

$$\frac{d^2 f(x)}{dx^2} = \frac{d\left\{\frac{d\{f(x)\}}{dx}\right\}}{dx} = \frac{d\left\{\frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}\right\}}{dx} = \frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2} \quad ; \quad \frac{\partial^3 f(x)}{\partial x^3} = \frac{d\left\{\frac{d^2 f(x)}{dx^2}\right\}}{dx} = \frac{d\left\{\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2}\right\}}{dx} = \frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}$$

Then we need to find $f(\pi)$, $\left.\frac{\partial f}{\partial x}\right|_{x=\pi}$, $\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=\pi}$, and $\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=\pi}$.

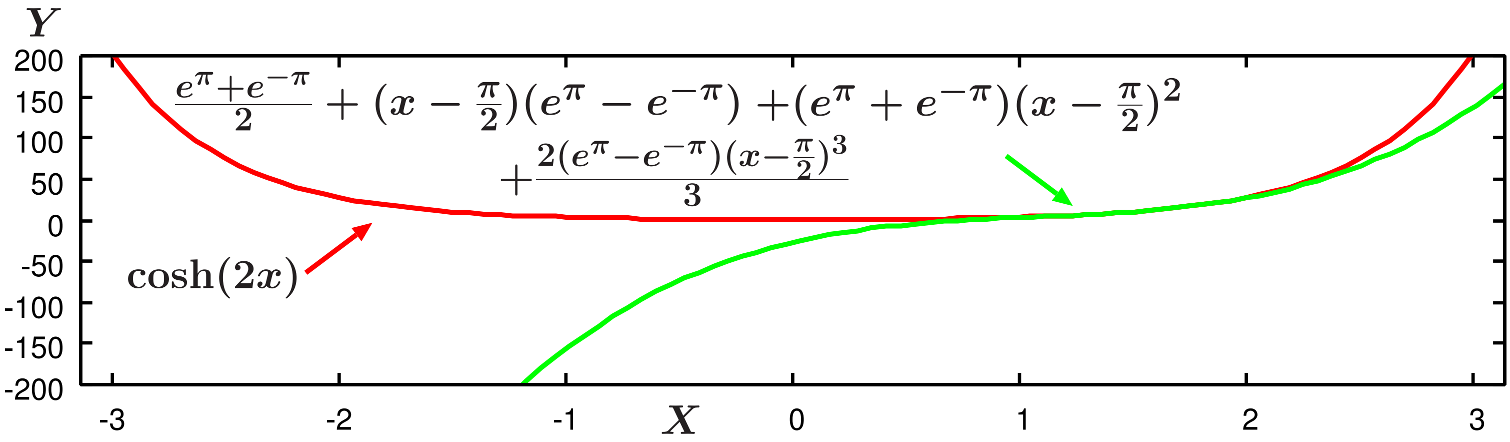
$$f(x) = \left.\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2}\right|_{x=\pi} = \frac{\mathfrak{e}^\pi - \mathfrak{e}^{-\pi}}{2} \quad ; \quad \frac{d\{f(x)\}}{dx} = \left.\frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}\right|_{x=\pi} = \frac{\mathfrak{e}^\pi + \mathfrak{e}^{-\pi}}{2}$$

$$\frac{d^2 f(x)}{dx^2} = \left.\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2}\right|_{x=\pi} = \frac{\mathfrak{e}^\pi - \mathfrak{e}^{-\pi}}{2} \quad ; \quad \frac{\partial^3 f(x)}{\partial x^3} = \left.\frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}\right|_{x=\pi} = \frac{\mathfrak{e}^\pi + \mathfrak{e}^{-\pi}}{2}$$

By substituting these into Equation (82),

$$f(x) = f(\pi) + (x-\pi)\left.\frac{\partial f}{\partial x}\right|_{x=\pi} + \frac{(x-\pi)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=\pi} + \frac{(x-\pi)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=\pi}$$
$$= \frac{\mathfrak{e}^\pi - \mathfrak{e}^{-\pi}}{2} + (x-\pi)\frac{\mathfrak{e}^\pi + \mathfrak{e}^{-\pi}}{2} + \frac{(x-\pi)^2}{2!}\frac{\mathfrak{e}^\pi - \mathfrak{e}^{-\pi}}{2} + \frac{(x-\pi)^3}{3!}\frac{\mathfrak{e}^\pi + \mathfrak{e}^{-\pi}}{2}$$

17) Find the Taylor polynomial of degree three for $\cosh(2x)$ about the point $x = \frac{\pi}{2}$ by calculating the appropriate derivatives.




The Taylor series can be written as

$$f(x) = f(a) + (x-a)\left.\frac{\partial f}{\partial x}\right|_{x=a} + \frac{(x-a)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=a} + \frac{(x-a)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=a} + \cdots + \frac{(x-a)^n}{n!}\left.\frac{\partial^n f}{\partial x^n}\right|_{x=a}$$

where $a = \frac{\pi}{2}$ because we need the approximation of $f$ about $x = \frac{\pi}{2}$.
First we need to find out $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, and $\frac{\partial^3 f}{\partial x^3}$

$$f(x) = \cosh(2x) = \frac{\mathfrak{e}^{2x}+\mathfrak{e}^{-2x}}{2} \quad ; \quad \frac{d\{f(x)\}}{dx} = \frac{d\left\{\frac{\mathfrak{e}^{2x}+\mathfrak{e}^{-2x}}{2}\right\}}{dx} = \frac{2\mathfrak{e}^{2x}-2\mathfrak{e}^{-2x}}{2} = \mathfrak{e}^{2x}-\mathfrak{e}^{-2x}$$

$$\frac{d^2 f(x)}{dx^2} = \frac{d\left\{\frac{d\{f(x)\}}{dx}\right\}}{dx} = \frac{d\{\mathfrak{e}^{2x}-\mathfrak{e}^{-2x}\}}{dx} = 2(\mathfrak{e}^{2x}+\mathfrak{e}^{-2x}) \quad ;$$

$$\frac{\partial^3 f(x)}{\partial x^3} = \frac{d\left\{\frac{d^2 f(x)}{dx^2}\right\}}{dx} = \frac{d\{2(\mathfrak{e}^{2x}+\mathfrak{e}^{-2x})\}}{dx} = 4(\mathfrak{e}^{2x}-\mathfrak{e}^{-2x})$$

Then we need to find $f\left(\frac{\pi}{2}\right)$, $\left.\frac{\partial f}{\partial x}\right|_{x=\frac{\pi}{2}}$, $\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=\frac{\pi}{2}}$, and $\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=\frac{\pi}{2}}$.

$$f(x) = \left.\frac{\mathfrak{e}^{2x}+\mathfrak{e}^{-2x}}{2}\right|_{x=\frac{\pi}{2}} = \frac{\mathfrak{e}^{\pi}+\mathfrak{e}^{-\pi}}{2} \quad ; \quad \frac{d\{f(x)\}}{dx} = \left.\mathfrak{e}^{2x}-\mathfrak{e}^{-2x}\right|_{x=\frac{\pi}{2}} = \mathfrak{e}^{\pi}-\mathfrak{e}^{-\pi}$$

$$\frac{d^2 f(x)}{dx^2} = \left.2(\mathfrak{e}^{2x}+\mathfrak{e}^{-2x})\right|_{x=\frac{\pi}{2}} = 2(\mathfrak{e}^{\pi}+\mathfrak{e}^{-\pi}) \quad ; \quad \frac{\partial^3 f(x)}{\partial x^3} = \left.4(\mathfrak{e}^{2x}-\mathfrak{e}^{-2x})\right|_{x=\frac{\pi}{2}} = 4(\mathfrak{e}^{\pi}-\mathfrak{e}^{-\pi})$$

By substituting these into Equation (82),

$$f(x) = f(\frac{\pi}{2}) + (x-\frac{\pi}{2})\left.\frac{\partial f}{\partial x}\right|_{x=\frac{\pi}{2}} + \frac{(x-\frac{\pi}{2})^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=\frac{\pi}{2}} + \frac{(x-\frac{\pi}{2})^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=\frac{\pi}{2}}$$

$$= \frac{\mathfrak{e}^{\pi}+\mathfrak{e}^{-\pi}}{2} + (x-\frac{\pi}{2})(\mathfrak{e}^{\pi}-\mathfrak{e}^{-\pi}) + \frac{(x-\frac{\pi}{2})^2}{2!}\cdot 2(\mathfrak{e}^{\pi}+\mathfrak{e}^{-\pi}) + \frac{(x-\frac{\pi}{2})^3}{3!}\cdot 4(\mathfrak{e}^{\pi}-\mathfrak{e}^{-\pi})$$

$$= \frac{\mathfrak{e}^{\pi}+\mathfrak{e}^{-\pi}}{2} + (x-\frac{\pi}{2})(\mathfrak{e}^{\pi}-\mathfrak{e}^{-\pi}) + (\mathfrak{e}^{\pi}+\mathfrak{e}^{-\pi})(x-\frac{\pi}{2})^2 + \frac{2(\mathfrak{e}^{\pi}-\mathfrak{e}^{-\pi})(x-\frac{\pi}{2})^3}{3}$$

**DAY2**

18) For the function

$$f(x,y) = \frac{1}{\sin(x)} + \frac{1}{\cos(y)}$$

find

a) the Taylor polynomial of degree two about the point $(x,y) = \left(\frac{\pi}{2}, -\frac{\pi}{3}\right)$

We are going to use Equation (83) with $(a,b) = \left(\frac{\pi}{2}, -\frac{\pi}{3}\right)$ because we need to find out the approximation about the point $(x,y) = \left(\frac{\pi}{2}, -\frac{\pi}{3}\right)$

| | | |
|---|---|---|
| $a$ | $\rightarrow$ | $\frac{\pi}{2}$ |
| $b$ | $\rightarrow$ | $-\frac{\pi}{3}$ |

Thus Equation (83) can be re-written as

$$f(x,y) = f\left(\frac{\pi}{2}, -\frac{\pi}{3}\right) + (x-\frac{\pi}{2})\frac{d\{f(x,y)\}}{dx}\bigg|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}} + (y+\frac{\pi}{3})\frac{d\{f(x,y)\}}{dy}\bigg|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}}$$
$$+\frac{1}{2!}\left[(x-\frac{\pi}{2})^2\frac{d^2f(x,y)}{dx^2}\bigg|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}} + 2(x-\frac{\pi}{2})(y+\frac{\pi}{3})\frac{\partial^2 f(x,y)}{\partial y\partial x}\bigg|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}} + (y+\frac{\pi}{3})^2\frac{\partial^2 f(x,y)}{\partial y^2}\bigg|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}}\right]$$
$$+\frac{1}{3!}\left[(x-\frac{\pi}{2})^3\frac{\partial^3 f(x,y)}{\partial x^3}\bigg|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}} + 3(x-\frac{\pi}{2})^2(y+\frac{\pi}{3})\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\bigg|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}}\right.$$
$$\left.+3(x-\frac{\pi}{2})(y+\frac{\pi}{3})^2\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\bigg|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}} + (y+\frac{\pi}{3})^3\frac{\partial^3 f(x,y)}{\partial y^3}\bigg|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}}\right] \quad ①$$

where $|x-\frac{\pi}{2}| \ll 1$ and $|y+\frac{\pi}{3}| \ll 1$. First, we need to find out $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$.

$$f(x,y) = \frac{1}{\sin(x)} + \frac{1}{\cos(y)} = (\sin(x))^{-1} + (\cos(y))^{-1}$$

$$\therefore \frac{d\{f(x,y)\}}{dx} = \frac{d\{(\sin(x))^{-1} + (\cos(y))^{-1}\}}{dx} = \frac{d\{(\sin(x))^{-1}\}}{dx} = -(\sin(x))^{-2}\cos(x)$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{(\sin(x))^{-1} + (\cos(y))^{-1}\}}{dy} = \frac{d\{(\cos(y))^{-1}\}}{dy} = -(\cos(y))^{-2}(-\sin(y)) = (\cos(y))^{-2}\sin(y)$$

$$\frac{d^2f(x,y)}{dx^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\{-(\sin(x))^{-2}\cos(x)\}}{dx}$$
$$= \frac{d\{-(\sin(x))^{-2}\}}{dx}\cos(x) - (\sin(x))^{-2}\frac{d\{\cos(x)\}}{dx}$$
$$= 2(\sin(x))^{-3}\cos^2(x) + (\sin(x))^{-2}\sin(x) = 2(\sin(x))^{-3}\cos^2(x) + (\sin(x))^{-1}$$

$$\frac{\partial^2 f(x,y)}{\partial y\partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dx} = \frac{d\{(\cos(y))^{-2}\sin(y)\}}{dx} = 0 \ ;$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{(\cos(y))^{-2}\sin(y)\}}{dy}$$

$$= \frac{d\{(\cos(y))^{-2}\}}{dy}\sin(y) + (\cos(y))^{-2}\frac{d\{\sin(y)\}}{dy}$$
$$= 2(\cos(y))^{-3}\sin^2(y) + (\cos(y))^{-2}\cos(y)$$
$$= 2(\cos(y))^{-3}\sin^2(y) + (\cos(y))^{-1}$$

Second, we need to find out $f\left(\frac{\pi}{2}, -\frac{\pi}{3}\right)$, $\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}}$, $\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}}$, $\left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}}$.

$$f\left(\frac{\pi}{2}, -\frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{2}\right)} + \frac{1}{\cos\left(-\frac{\pi}{3}\right)} = 1 + 2 = 3 \;\; ; \;\; \left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}} = -(\sin\left(\frac{\pi}{2}\right))^{-2}\cos\left(\frac{\pi}{2}\right) = 0$$

$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}} = (\cos\left(-\frac{\pi}{3}\right))^{-2}\sin\left(-\frac{\pi}{3}\right) = 2^2 \cdot \left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$

$$\left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}} = 2(\sin\left(\frac{\pi}{2}\right))^{-3}\cos^2\left(\frac{\pi}{2}\right) + (\sin\left(\frac{\pi}{2}\right))^{-1} = 2(\sin\left(\frac{\pi}{2}\right))^{-3}\cos^2\left(\frac{\pi}{2}\right) + (\sin\left(\frac{\pi}{2}\right))^{-1} = 1$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}} = 0$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=\frac{\pi}{2}\\ y=-\frac{\pi}{3}}} = 2(\cos\left(-\frac{\pi}{3}\right))^{-3}\sin^2\left(-\frac{\pi}{3}\right) + (\cos\left(-\frac{\pi}{3}\right))^{-1}$$
$$= 2(\cos\left(-\frac{\pi}{3}\right))^{-3}\sin^2\left(-\frac{\pi}{3}\right) + (\cos\left(-\frac{\pi}{3}\right))^{-1}$$
$$= 2^4 \cdot \left(-\frac{\sqrt{3}}{2}\right)^2 + 2$$
$$= 2^4 \cdot \left(-\frac{\sqrt{3}}{2}\right)^2 + 2 = 12 + 2 = 14$$

Finally by substituting these into ①, we get

$$f(x,y) = 3 + 0 \cdot (x - \frac{\pi}{2}) + 2\sqrt{3}(y + \frac{\pi}{3}) + \frac{1}{2!}\left[(x - \frac{\pi}{2})^2 + 2(x - \frac{\pi}{2})(y + \frac{\pi}{3}) \cdot 0 + 14 \cdot (y + \frac{\pi}{3})^2\right]$$
$$= 3 + 2\sqrt{3}(y + \frac{\pi}{3}) + \frac{(x - \frac{\pi}{2})^2}{2} + 7 \cdot (y + \frac{\pi}{3})^2$$

b) the quadratic approximation $f(x,y)$ of the Taylor series expansion around $(x,y) = (\frac{11}{6}\pi, 0)$
In order to work out $f(x,y)$ around $(x,y) = (\frac{11}{6}\pi, 0)$ , we need to find out $a$ and $b$ in Equation (83). we get $a = \frac{11}{6}\pi$ and $b = 0$. This means we are going to find out the Taylor series expansion around $(x,y) = (\frac{11}{6}\pi, 0)$ using Equation (83). We have already found $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$ previously. Since the quadratic approximation is required, we just need to find out $f\left(\frac{11}{6}\pi, 0\right)$, $\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=\frac{11}{6}\pi\\ y=0}}$ ,

$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=\frac{11}{6}\pi\\y=0}}, \left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=\frac{11}{6}\pi\\y=0}}, \left.\frac{\partial^2f(x,y)}{\partial y\partial x}\right|_{\substack{x=\frac{11}{6}\pi\\y=0}}, \text{ and } \left.\frac{\partial^2f(x,y)}{\partial y^2}\right|_{\substack{x=\frac{11}{6}\pi\\y=0}}.$$

$$f\left(\frac{11}{6}\pi,0\right)=\left(\sin\left(\frac{11}{6}\pi\right)\right)^{-1}+(\cos(0))^{-1}=-2+1=-1$$

$$\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=\frac{11}{6}\pi\\y=0}}=-\left(\sin\left(\frac{11}{6}\pi\right)\right)^{-2}\cos\left(\frac{11}{6}\pi\right)=-(-2)^2\cdot\frac{\sqrt{3}}{2}=-2\sqrt{3}$$

$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=\frac{11}{6}\pi\\y=0}}=(\cos(0))^{-2}\sin(0)=0$$

$$\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=\frac{11}{6}\pi\\y=0}}=2\left(\sin\left(\frac{11}{6}\pi\right)\right)^{-3}\cos^2\left(\frac{11}{6}\pi\right)+\left(\sin\left(\frac{11}{6}\pi\right)\right)^{-1}$$

$$=2(-2)^3\left(\frac{\sqrt{3}}{2}\right)^2-2=-16\cdot\frac{3}{4}-2=-12-2=-14$$

$$\left.\frac{\partial^2f(x,y)}{\partial y\partial x}\right|_{\substack{x=\frac{11}{6}\pi\\y=0}}=0\ ;\ \left.\frac{\partial^2f(x,y)}{\partial y^2}\right|_{\substack{x=\frac{11}{6}\pi\\y=0}}=2\left(\cos(0)\right)^{-3}\sin^2(0)+(\cos(0))^{-1}=1$$

Now we find $f(x,y)$ using $(a,b)=(\frac{11}{6}\pi,0)$ as follows:

$$f(x,y)=f(\frac{11}{6}\pi,0)+(x-\frac{11}{6}\pi)\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=\frac{11}{6}\pi\\y=0}}+y\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=\frac{11}{6}\pi\\y=0}}$$

$$+\frac{1}{2!}\left[(x-\frac{11}{6}\pi)^2\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=\frac{11}{6}\pi\\y=0}}+2(x-\frac{11}{6}\pi)y\left.\frac{\partial^2f(x,y)}{\partial y\partial x}\right|_{\substack{x=\frac{11}{6}\pi\\y=0}}+y^2\left.\frac{\partial^2f(x,y)}{\partial y^2}\right|_{\substack{x=\frac{11}{6}\pi\\y=0}}\right]$$

$$=-1-2\sqrt{3}(x-\frac{11}{6}\pi)+0\cdot y+\frac{1}{2!}\left[-14(x-\frac{11}{6}\pi)^2+2(x-\frac{11}{6}\pi)y\cdot 0+y^2\right]$$

$$=-1-2\sqrt{3}(x-\frac{11}{6}\pi)-7(x-\frac{11}{6}\pi)^2+y^2/2 \qquad ②$$

c) the value that the quadratic approximation gives for $f(\frac{11}{6}\pi,0.2)$ correct to 4 decimal places and the percentage error correct to 2 significant figures.
We have already worked out the approximation of $f(x,y)$ around $(x,y)=(\frac{11}{6}\pi,0)$. In order to find out $f(\frac{11}{6}\pi,0.2)$ using the approximation of $f(x,y)$ around $(x,y)=(\frac{11}{6}\pi,0)$ we need to know the exact value of $x-a$ and $y-b$.

$$x-a=\frac{11}{6}\pi-\frac{11}{6}\pi=0$$
$$y-b=0.2-0=0.2$$

Therefore we obtain $(x-a,y-b)=(0,0.2)$ which satisfy the condition of $|x-a|\ll 1$ and $|y-b|\ll 1$. By substituting $(x,y)=(\frac{11}{6}\pi,0.2)$ into the degree two of ②, we get

$$f(\frac{11}{6}\pi,0.2)\simeq -1-2\sqrt{3}\cdot 0-7\cdot 0+0.2^2/2=-1+0.02=-0.98$$

correct to 4 decimal places. Since $f(\frac{11}{6}\pi,0.2)=-0.979644\simeq -0.9796$ correct to 4 decimal places, the percentage error is

$$\frac{-0.98-(-0.9796)}{-0.9796}\times 100=0.040833\%\simeq 0.041\%$$

correct to 2 significant figures.

d) Find the value that the linear approximation gives for $f(\frac{16}{9}\pi, 0)$ correct to 4 decimal places and the percentage error correct to 2 significant figures. We have already worked out the approximation of $f(x,y)$ around $(x,y) = (\frac{11}{6}\pi, 0)$. In order to find out $f(\frac{16}{9}\pi, 0)$ using the approximation of $f(x,y)$ around $(x,y) = (\frac{11}{6}\pi, 0)$, we need to know the exact value of $x-a$ and $y-b$.

$$x - a = \frac{16}{9}\pi - \frac{11}{6}\pi = -\frac{\pi}{18}$$
$$y - b = 0 - 0 = 0$$

Therefore we obtain $(x-a, y-b) = \left(-\frac{\pi}{18}, 0\right)$ which satisfy the condition of $|x-a| \ll 1$ and $|y-b| \ll 1$. By substituting $(x,y) = \left(\frac{16}{9}\pi, 0\right)$ into the degree one of ②, we get

$$f(\frac{16}{9}\pi, 0) \simeq -1 - 2\sqrt{3} \cdot \left(-\frac{\pi}{18}\right) = -0.395401$$

correct to 4 decimal places. Since $f(\frac{16}{9}\pi, 0) = -0.5557$ correct to 4 decimal places, the percentage error is

$$\frac{-0.3954 - (-0.5557)}{-0.5557} \times 100 = -28.84\% \simeq -29\%$$

correct to 2 significant figures.

19) Evaluate the expression of

$$(\sqrt{3+6}+1)^2 \div 2 - 1$$

$$(\sqrt{3+6}+1)^2 \div 2 - 1 = (\sqrt{9}+1)^2 \div 2 - 1 = (3+1)^2 \div 2 - 1 = (4)^2 \div 2 - 1 = 16 \div 2 - 1 = 8 - 1 = 7$$

20) Find $\frac{d^2 f(x,y)}{dx^2}$ of $f(x,y) = x^3 - y^3 + 6yx^2 + 32yx - y + 2x + x^3$.

$$\frac{d\{f(x,y)\}}{dx} = 3x^2 + 12yx + 32y + 2 + 3x^2 \ ; \quad \therefore \frac{d^2 f(x,y)}{dx^2} = 6x + 12y + 6x = 12x + 12y$$

21) Find $\frac{\partial^2 f(x,y)}{\partial y \partial x}$ of $f(x,y) = \sin xy + \mathfrak{e}^x - \mathfrak{e}^{-y}$.

$$\frac{d\{f(x,y)\}}{dx} = y\cos xy + \mathfrak{e}^x \ ; \quad \therefore \frac{\partial^2 f(x,y)}{\partial y \partial x} = \cos xy - xy\sin xy$$

22) Find $\frac{\partial^2 f(x,y)}{\partial y^2}$ of $f(x,y) = \mathfrak{e}^y + \sin x + \cos xy$.

$$\frac{d\{f(x,y)\}}{dy} = \mathfrak{e}^y - x\sin xy \ ; \quad \therefore \frac{\partial^2 f(x,y)}{\partial y^2} = \mathfrak{e}^y - x^2\cos xy$$

23) Solve the equation of $3(x+1) - 8 = x + 3$

$$3(x+1) - 8 = x + 3 \therefore 3x + 3 - 8 = x + 3 \therefore 3x - x = 3 - 3 + 8 \therefore 2x = 8 \therefore x = 4$$

24) Find $\frac{d\{y\}}{dx}$ of $y = x^2$.

$$\frac{d\{y\}}{dx} = 2x$$

25) Find $\dfrac{d\{y\}}{dx}$ of $y = 3x + 2$.

$$\frac{d\{y\}}{dx} = 3$$

26) Find $\dfrac{d\{y\}}{dx}$ of $y = 12$.

$$\frac{d\{y\}}{dx} = 0$$

27) Find $\dfrac{d\{y\}}{dx}$ of $y = (x + 1)(x + 5)$.

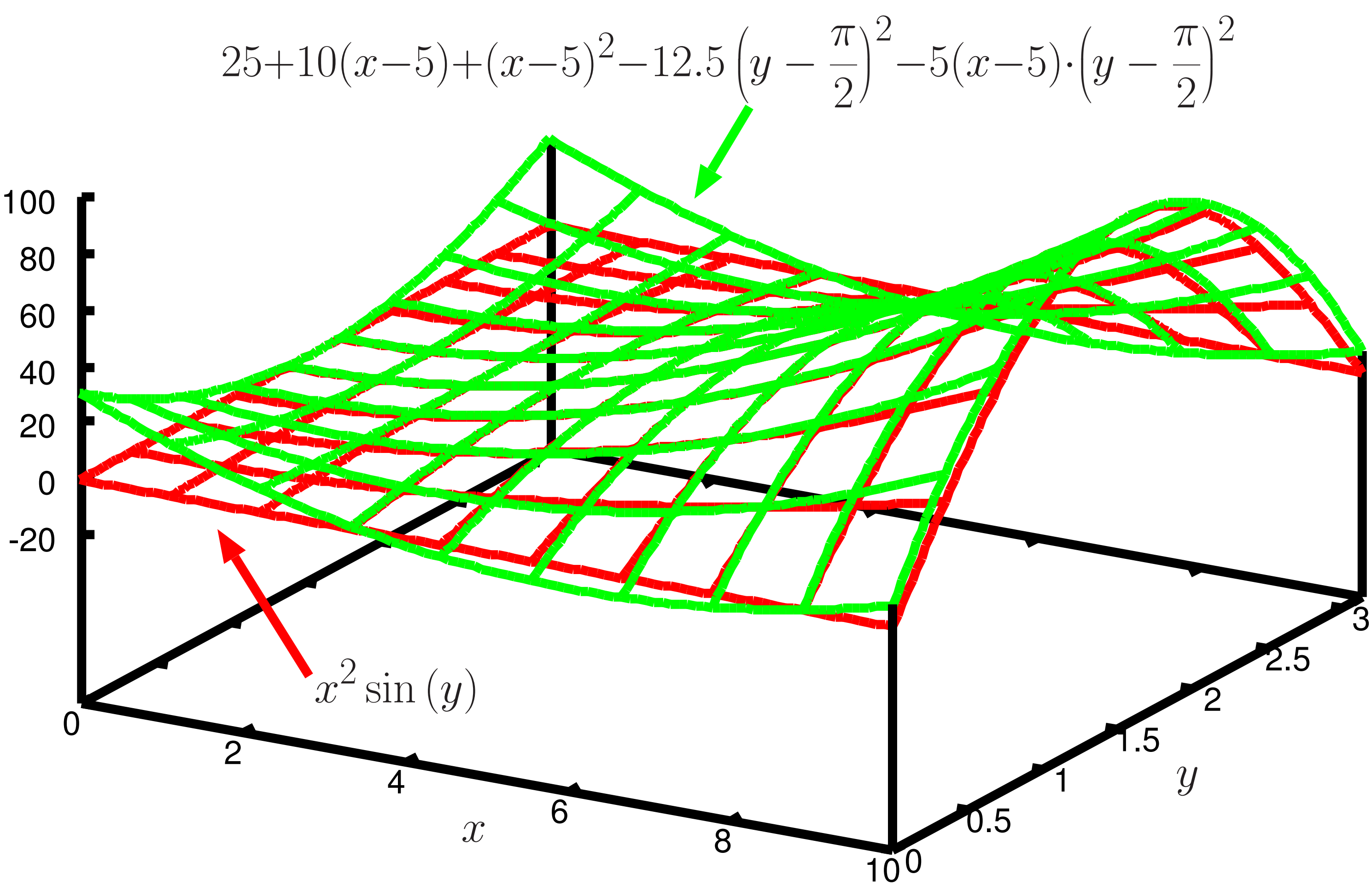
$$y = x^2 + 6x + 5 \ ; \quad \therefore \frac{d\{y\}}{dx} = 2x + 6$$

28) For the function

$$f(x, y) = x^2 \sin(y)$$

find

a) the Taylor polynomial of degree three about the point $(x, y) = \left(5, \frac{\pi}{2}\right)$




We are going to use Equation (83) with $(a, b) = \left(5, \frac{\pi}{2}\right)$ because we need to find out the approximation about the point $(x, y) = \left(5, \frac{\pi}{2}\right)$

| | | |
|---|---|---|
| $a$ | $\rightarrow$ | $5$ |
| $b$ | $\rightarrow$ | $\frac{\pi}{2}$ |

Thus Equation (83) can be re-written as

$$f(x,y) = f\left(5,\frac{\pi}{2}\right) + (x-5)\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=5\\y=\frac{\pi}{2}}} + (y-\frac{\pi}{2})\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=5\\y=\frac{\pi}{2}}}$$

$$+\frac{1}{2!}\left[(x-5)^2\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=5\\y=\frac{\pi}{2}}} + 2(x-5)(y-\frac{\pi}{2})\left.\frac{\partial^2 f(x,y)}{\partial y\partial x}\right|_{\substack{x=5\\y=\frac{\pi}{2}}} + (y-\frac{\pi}{2})^2\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=5\\y=\frac{\pi}{2}}}\right]$$

$$+\frac{1}{3!}\left[(x-5)^3\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=5\\y=\frac{\pi}{2}}} + 3(x-5)^2(y-\frac{\pi}{2})\left.\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\right|_{\substack{x=5\\y=\frac{\pi}{2}}}\right.$$

$$\left.+3(x-5)(y-\frac{\pi}{2})^2\left.\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\right|_{\substack{x=5\\y=\frac{\pi}{2}}} + (y-\frac{\pi}{2})^3\left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=5\\y=\frac{\pi}{2}}}\right] \qquad ①$$

where $|x-5| \ll 1$ and $|y-\frac{\pi}{2}| \ll 1$.

First, we need to find out $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y\partial x^2}$, $\frac{\partial^3 f(x,y)}{\partial y^2\partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$

$$f(x,y) = x^2\sin(y) \quad ; \quad \therefore \frac{d\{f(x,y)\}}{dx} = \frac{d\{x^2\sin(y)\}}{dx} = \frac{d\{x^2\}}{dx}\sin(y) = 2x\sin(y)$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{x^2\sin(y)\}}{dy} = x^2\frac{d\{\sin(y)\}}{dy} = x^2\cos(y)$$

$$\frac{d^2f(x,y)}{dx^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\{2x\sin(y)\}}{dx} = \frac{d\{2x\}}{dx}\sin(y) = 2\sin(y)$$

$$\frac{\partial^2 f(x,y)}{\partial y\partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dx} = \frac{d\{x^2\cos(y)\}}{dx} = \frac{d\{x^2\}}{dx}\cos(y) = 2x\cos(y)$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{x^2\cos(y)\}}{dy} = x^2\frac{d\{\cos(y)\}}{dy} = -x^2\sin(y)$$

$$\frac{\partial^3 f(x,y)}{\partial x^3} = \frac{d\left\{\frac{d^2f(x,y)}{dx^2}\right\}}{dx} = \frac{d\{2\sin(y)\}}{dx} = 0$$

$$\frac{\partial^3 f(x,y)}{\partial y\partial x^2} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y\partial x}\right\}}{dx} = \frac{d\{2x\cos(y)\}}{dx} = \frac{d\{2x\}}{dx}\cos(y) = 2\cos(y)$$

$$\frac{\partial^3 f(x,y)}{\partial y^2\partial x} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dx} = \frac{d\{-x^2\sin(y)\}}{dx} = \frac{d\{-x^2\}}{dx}\sin(y) = -2x\sin(y)$$

$$\frac{\partial^3 f(x,y)}{\partial y^3} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dy} = \frac{d\{-x^2\sin(y)\}}{dy} = -x^2\frac{d\{\sin(y)\}}{dy} = -x^2\cos(y)$$

Second, we need to find out $f\left(5, \frac{\pi}{2}\right)$, $\left.\frac{d\left\{f(x,y)\right\}}{dx}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}}$, $\left.\frac{d\left\{f(x,y)\right\}}{dy}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}}$, $\left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}}$,
$\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}}$, $\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}}$, $\left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}}$, $\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}}$, and $\left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}}$

$$f\left(5, \frac{\pi}{2}\right) = 5^2 \sin\left(\frac{\pi}{2}\right) = 25 \;\; ; \;\; \left.\frac{d\left\{f(x,y)\right\}}{dx}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}} = 2 \cdot 5 \sin\left(\frac{\pi}{2}\right) = 10$$

$$\left.\frac{d\left\{f(x,y)\right\}}{dy}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}} = 5^2 \cos\left(\frac{\pi}{2}\right) = 0 \;\; ; \;\; \left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}} = 2 \sin\left(\frac{\pi}{2}\right) = 2$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}} = 2 \cdot 5 \cos\left(\frac{\pi}{2}\right) = 0 \;\; ; \;\; \left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}} = -5^2 \sin\left(\frac{\pi}{2}\right) = -25$$

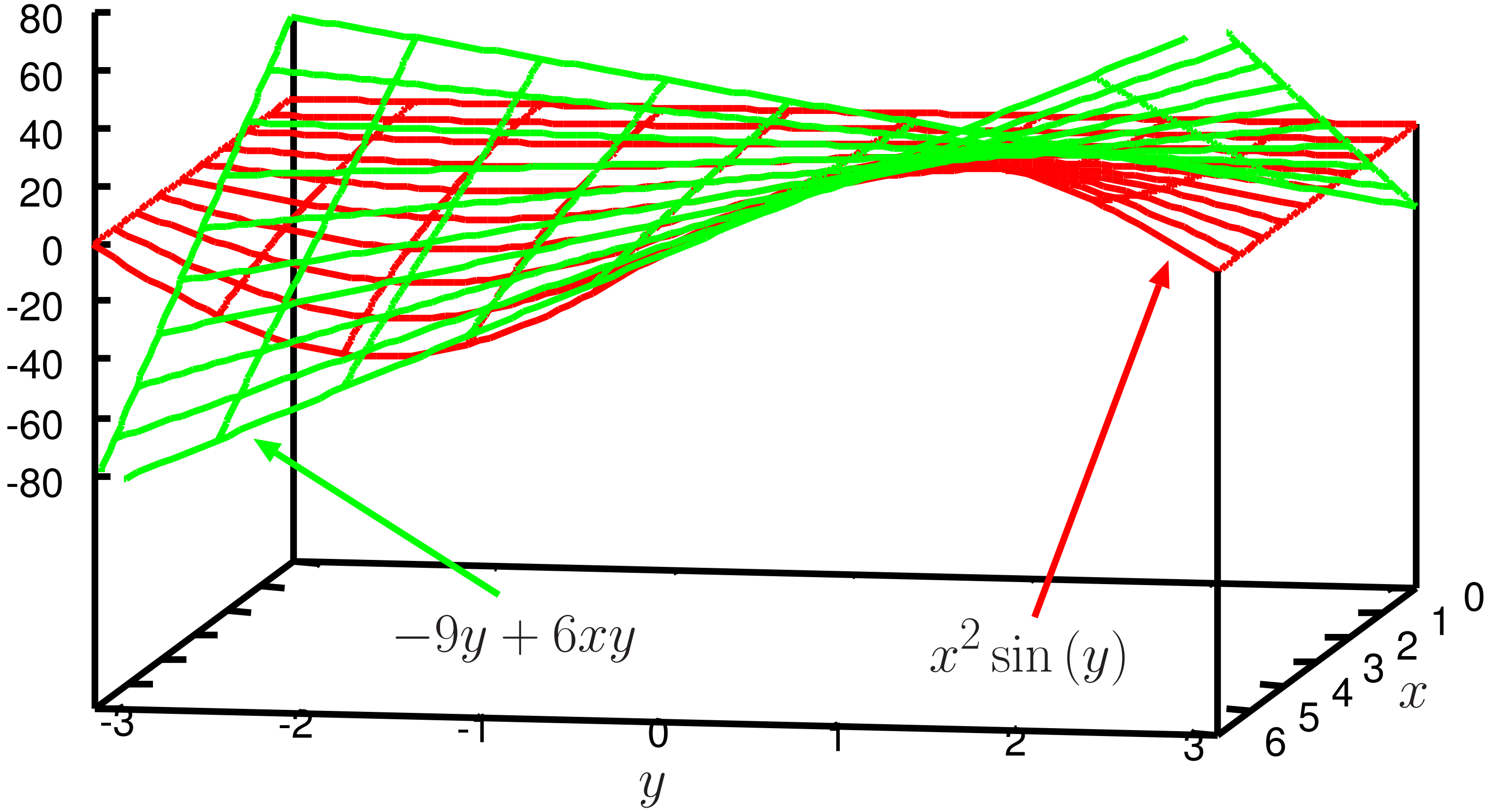
$$\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}} = 0 \;\; ; \;\; \left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}} = 2 \cos\left(\frac{\pi}{2}\right) = 0$$

$$\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}} = -2 \cdot 5 \sin\left(\frac{\pi}{2}\right) = -10 \;\; ; \;\; \left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=5 \\ y=\frac{\pi}{2}}} = -5^2 \cos\left(\frac{\pi}{2}\right) = 0$$

Finally by substituting these into ①, we get

$$\begin{aligned} f(x,y) &= 25 + 10(x-5) + 0 \cdot (y - \frac{\pi}{2}) + \frac{1}{2!}\left[2(x-5)^2 + 2(x-5)(y-\frac{\pi}{2}) \cdot 0 \; -25(y-\frac{\pi}{2})^2\right] \\ &+ \frac{1}{3!}\left[0 \cdot (x-5)^3 + 3(x-5)^2(y-\frac{\pi}{2}) \cdot 0 \; +3(x-5)(y-\frac{\pi}{2})^2 \cdot (-10) + (y-\frac{\pi}{2})^3 \cdot 0\right] \\ &= 25 + 10(x-5) + (x-5)^2 - 12.5(y-\frac{\pi}{2})^2 - 5(x-5)(y-\frac{\pi}{2})^2 \end{aligned}$$

b) the quadratic approximation $f(x,y)$ of the Taylor series expansion around $(x,y) = (3,0)$




In order to work out $f(x,y)$ , we need to find out $a$ and $b$ in Equation (83). Since

$$a = 3 \;\; ; \;\; b = 0$$

we get $a = 3$ and $b = 0$. This means we are going to find out the Taylor series expansion around $(x,y) = (3,0)$ using Equation (83). We have already found $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y \partial x^2}$, $\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$ previously. Since the quadratic approximation is required, we just need to find out $f(3,0)$, $\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=3\\y=0}}$, $\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=3\\y=0}}$, $\left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=3\\y=0}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=3\\y=0}}$, and $\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=3\\y=0}}$.

$$f(3,0) = 3^2 \sin(0) = 0 \;\; ; \;\; \left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=3\\y=0}} = 2 \cdot 3 \sin(0) = 0 \;\; ;$$
$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=3\\y=0}} = 3^2 \cos(0) = 9$$
$$\left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=3\\y=0}} = 2\sin(0) = 0 \;\; ; \;\; \left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=3\\y=0}} = 2 \cdot 3 \cos(0) = 6 \;\; ;$$
$$\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=3\\y=0}} = -3^2 \sin(0) = 0$$

Now we find $f(x,y)$ using $(a,b)=(3,0)$ as follows:

$$f(x,y)=f(3,0)+(x-3)\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=3\\y=0}}+y\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=3\\y=0}}$$

$$+\frac{1}{2!}\left[(x-3)^2\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=3\\y=0}}+2(x-3)y\left.\frac{\partial^2 f(x,y)}{\partial y\partial x}\right|_{\substack{x=3\\y=0}}+y^2\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=3\\y=0}}\right]$$

$$=0+(x-3)\cdot 0+9y+\frac{1}{2!}\left[0\cdot(x-3)^2+2(x-3)y\cdot 6+0\cdot y^2\right]=9y+6(x-3)y \qquad ②$$

c) the value that the linear approximation gives for $f(2.9,\pi/12)$ correct to 4 decimal places and the percentage error correct to 2 significant figures.
We have already worked out the approximation of $f(x,y)$ around $(x,y)=(3,0)$. In order to find out $f(2.9,\pi/12)$ using the approximation of $f(x,y)$, we need to know the exact value of $x-a$ and $y-b$.

$$x-a=2.9-3=-0.1 \;\; ; \;\; y-b=\pi/12$$

Therefore we obtain $(x-a,y-b)=(-0.1,\pi/12)$ which satisfy the condition of $|x-a|\ll 1$ and $|y-b|\ll 1$. By substituting $(x,y)=(2.9,\pi/12)$ into the degree one of ②, we get

$$f(2.9,\pi/12)=9\cdot\frac{\pi}{12}=\frac{3\pi}{4}\simeq 2.356$$

correct to 4 decimal places. Since $f(2.9,\pi/12)=2.17667\simeq 2.177$ correct to 4 decimal places, the percentage error is

$$\frac{2.356-2.177}{2.177}\times 100=8.22232\%\simeq 8.2\%$$

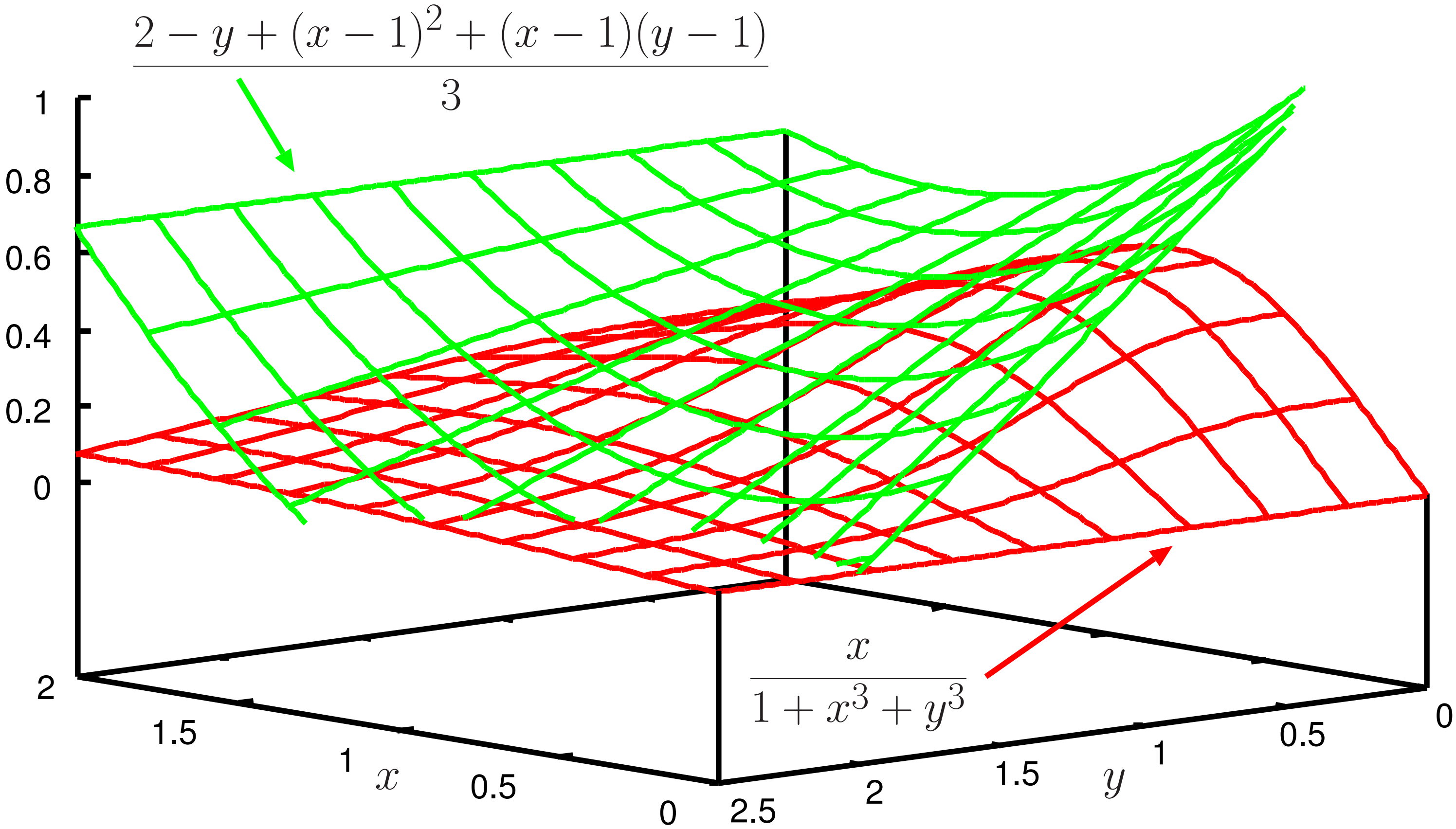
correct to 2 significant figures.

29) For the function

$$f(x,y)=\frac{x}{1+x^3+y^3}$$

find

a) the Taylor polynomial of degree two about the point $(x,y)=(1,1)$




We are going to use Equation (83) with $(a,b)=(1,1)$ because we need to find out the approximation about the point $(x,y)=(1,1)$

| | | |
|---|---|---|
| $a$ | $\to$ | $1$ |
| $b$ | $\to$ | $1$ |

Thus Equation (83) can be re-written as

$$f(x,y) = f(1,1) + (x-1)\frac{d\{f(x,y)\}}{dx}\bigg|_{\substack{x=1\\y=1}} + (y-1)\frac{d\{f(x,y)\}}{dy}\bigg|_{\substack{x=1\\y=1}}$$
$$+\frac{1}{2!}\left[(x-1)^2\frac{d^2f(x,y)}{dx^2}\bigg|_{\substack{x=1\\y=1}} + 2(x-1)(y-1)\frac{\partial^2 f(x,y)}{\partial y\partial x}\bigg|_{\substack{x=1\\y=1}} + (y-1)^2\frac{\partial^2 f(x,y)}{\partial y^2}\bigg|_{\substack{x=1\\y=1}}\right]$$
$$+\frac{1}{3!}\left[(x-1)^3\frac{\partial^3 f(x,y)}{\partial x^3}\bigg|_{\substack{x=1\\y=1}} + 3(x-1)^2(y-1)\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\bigg|_{\substack{x=1\\y=1}}\right.$$
$$\left.+3(x-1)(y-1)^2\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\bigg|_{\substack{x=1\\y=1}} + (y-1)^3\frac{\partial^3 f(x,y)}{\partial y^3}\bigg|_{\substack{x=1\\y=1}}\right] \quad ①$$

where $|x-1| \ll 1$ and $|y-1| \ll 1$.

First, we need to find out $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$.

$$f(x,y) = \frac{x}{1+x^3+y^3} = x(1+x^3+y^3)^{-1}$$

$$\therefore \frac{d\{f(x,y)\}}{dx} = \frac{d\{x(1+x^3+y^3)^{-1}\}}{dx} = \frac{d\{x\}}{dx}(1+x^3+y^3)^{-1} + x\frac{d\{(1+x^3+y^3)^{-1}\}}{dx}$$
$$= (1+x^3+y^3)^{-1} - x(1+x^3+y^3)^{-2}(3x^2) = (1+x^3+y^3)^{-1} - 3x^3(1+x^3+y^3)^{-2}$$
$$= (1+x^3+y^3)(1+x^3+y^3)^{-2} - 3x^3(1+x^3+y^3)^{-2} = (1-2x^3+y^3)(1+x^3+y^3)^{-2}$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{x(1+x^3+y^3)^{-1}\}}{dy} = \frac{d\{x\}}{dy}(1+x^3+y^3)^{-1} + x\frac{d\{(1+x^3+y^3)^{-1}\}}{dy}$$
$$= -x(1+x^3+y^3)^{-2}(3y^2) = -3xy^2(1+x^3+y^3)^{-2}$$

$$\frac{d^2f(x,y)}{dx^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\{(1-2x^3+y^3)(1+x^3+y^3)^{-2}\}}{dx}$$
$$= \frac{d\{(1-2x^3+y^3)\}}{dx}(1+x^3+y^3)^{-2} + (1-2x^3+y^3)\frac{d\{(1+x^3+y^3)^{-2}\}}{dx}$$
$$= -6x^2(1+x^3+y^3)^{-2} - 2(1-2x^3+y^3)(1+x^3+y^3)^{-3}(3x^2)$$

$$\frac{\partial^2 f(x,y)}{\partial y\partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dy} = \frac{d\{(1-2x^3+y^3)(1+x^3+y^3)^{-2}\}}{dy}$$
$$= \frac{d\{(1-2x^3+y^3)\}}{dy}(1+x^3+y^3)^{-2} + (1-2x^3+y^3)\frac{d\{(1+x^3+y^3)^{-2}\}}{dy}$$
$$= 3y^2(1+x^3+y^3)^{-2} - 2(1-2x^3+y^3)(1+x^3+y^3)^{-3}(3y^2)$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{-3xy^2(1+x^3+y^3)^{-2}\}}{dy}$$
$$= \frac{d\{-3xy^2\}}{dy}(1+x^3+y^3)^{-2} - 3xy^2\frac{d\{(1+x^3+y^3)^{-2}\}}{dy}$$
$$= -6xy(1+x^3+y^3)^{-2} + 6xy^2(1+x^3+y^3)^{-3}(3y^2)$$

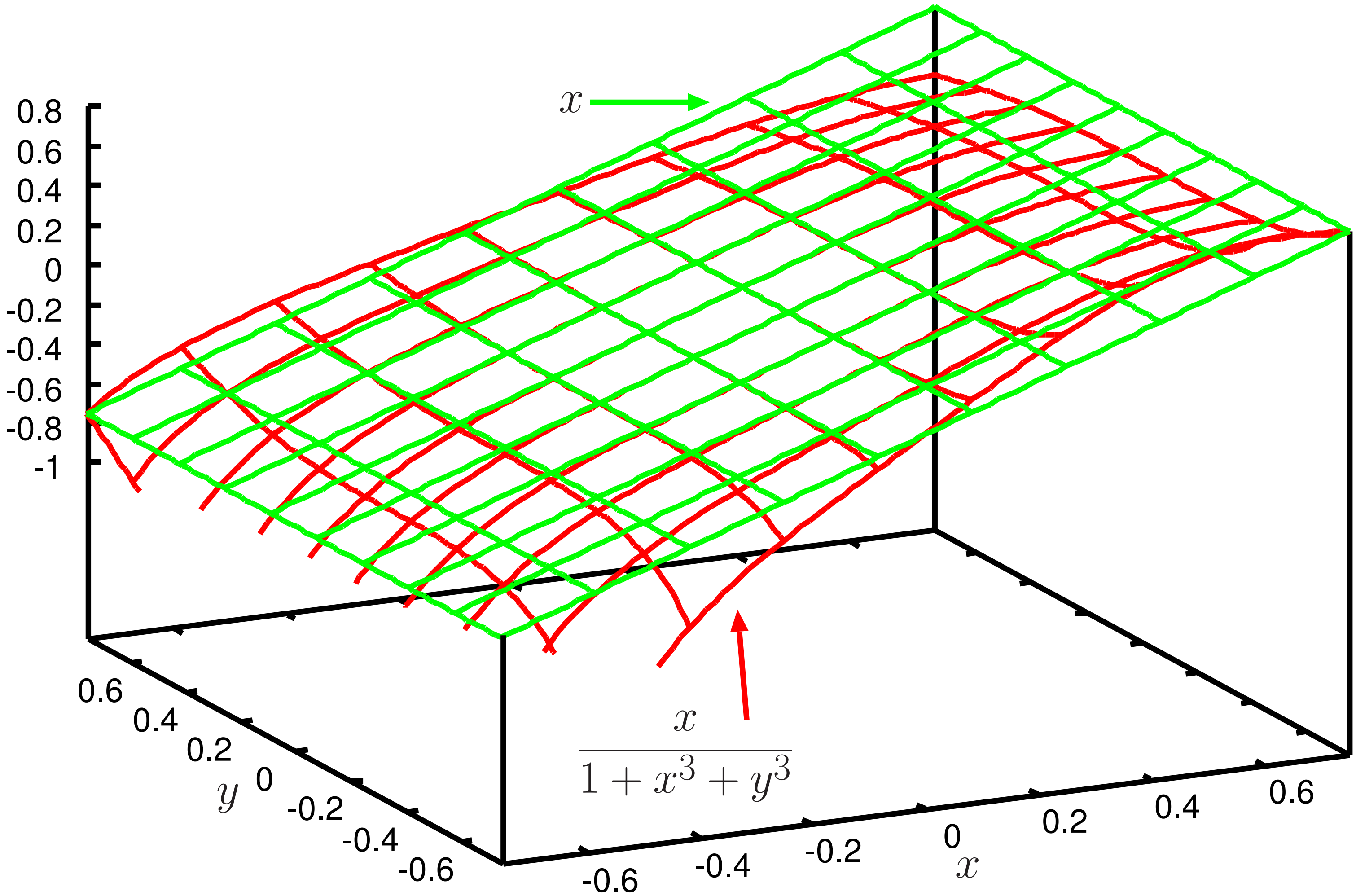
Second, we need to find out $f\,(1,1)$, $\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=1\\y=1}}$, $\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=1\\y=1}}$, $\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=1\\y=1}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y\partial x}\right|_{\substack{x=1\\y=1}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=1\\y=1}}$.

$$f\,(1,1) = \frac{1}{1+1^3+1^3} = \frac{1}{3} \;;\; \left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=1\\y=1}} = (1-2+1^3)(1+1^3+1^3)^{-2} = 0$$
$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=1\\y=1}} = -3(1+1^3+1^3)^{-2} = -3\cdot 3^{-2} = -3^{-1}$$
$$\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=1\\y=1}} = -6(1+1^3+1^3)^{-2} - 2(1-2+1)(1+1^3+1^3)^{-3}(3) = -2\cdot 3^{-1}$$
$$\left.\frac{\partial^2 f(x,y)}{\partial y\partial x}\right|_{\substack{x=1\\y=1}} = 3(1+1^3+1^3)^{-2} - 2(1-2+1^3)(1+1^3+1^3)^{-3}(3) = 3^{-1}$$
$$\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=1\\y=1}} = -6(1+1^3+1^3)^{-2} + 6(1+1^3+1^3)^{-3}(3) = 0$$

Finally by substituting these into ①, we get

$$f\,(x,y) = 3^{-1} + 0\cdot(x-1) - 3^{-1}\cdot(y-1) + \frac{1}{2!}\left[-2\cdot 3^{-1}(x-1)^2 + 2(x-1)(y-1)\cdot 3^{-1}\ 0\cdot(y-1)^2\right]$$
$$= 3^{-1} - 3^{-1}(y-1) + 3^{-1}(x-1)^2 + 3^{-1}(x-1)(y-1)$$
$$= \frac{1-(y-1)+(x-1)^2+(x-1)(y-1)}{3}$$

b) the linear approximation $f(x, y)$ of the Taylor series expansion around $(0, 0)$




In order to work out $f(x, y)$ , we need to find out $a$ and $b$ in Equation (83). Since

$$a = 0 \ ; \ \ b = 0$$

we get $a = 0$ and $b = 0$. This means we are going to find out the Taylor series expansion around $(x, y) = (0, 0)$ using Equation (83). We have already found $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$ previously. Since the quadratic approximation is required, we just need to find out $f(0, 0)$, $\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=0\\y=0}}$ , $\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=0\\y=0}}$ .

$$f(0,0) = 0 \ ; \ \left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=0\\y=0}} = 1 \ ; \ \left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=0\\y=0}} = 0$$

Now we find $f(x, y)$ using $(a, b) = (0, 0)$ as follows:

$$f(x,y) = f(0,0) + x\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=0\\y=0}} + y\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=0\\y=0}} = x \qquad ②$$

c) the value that the linear approximation gives for $f(0.2, -0.1)$ correct to 4 decimal places and the percentage error correct to 2 significant figures.

We have already worked out the approximation of $f(x, y)$ around $(0, 0)$. In order to find out $f(0.2, -0.1)$ using the approximation of $f(x, y)$, we need to know the exact value of $x - a$ and $y - b$.

$$x - a = 0.2 \ ; \ \ y - b = -0.1$$

Therefore we obtain $(x - a, y - b) = (0.2, -0.1)$ which satisfy the condition of $|x - a| \ll 1$ and $|y - b| \ll 1$. By substituting $(x, y) = (0.2, -0.1)$ into the degree one of ②, we get

$$f(0.2, -0.1) \simeq 0.2000$$

correct to 4 decimal places. Since $f(0.2, -0.1) = 0.19861 \simeq 0.1986$ correct to 4 decimal places, the percentage error is

$$\frac{0.2 - 0.1986}{0.1986} \times 100 = 0.704935\% \simeq 0.70\%$$

correct to 2 significant figures.

30) For the function

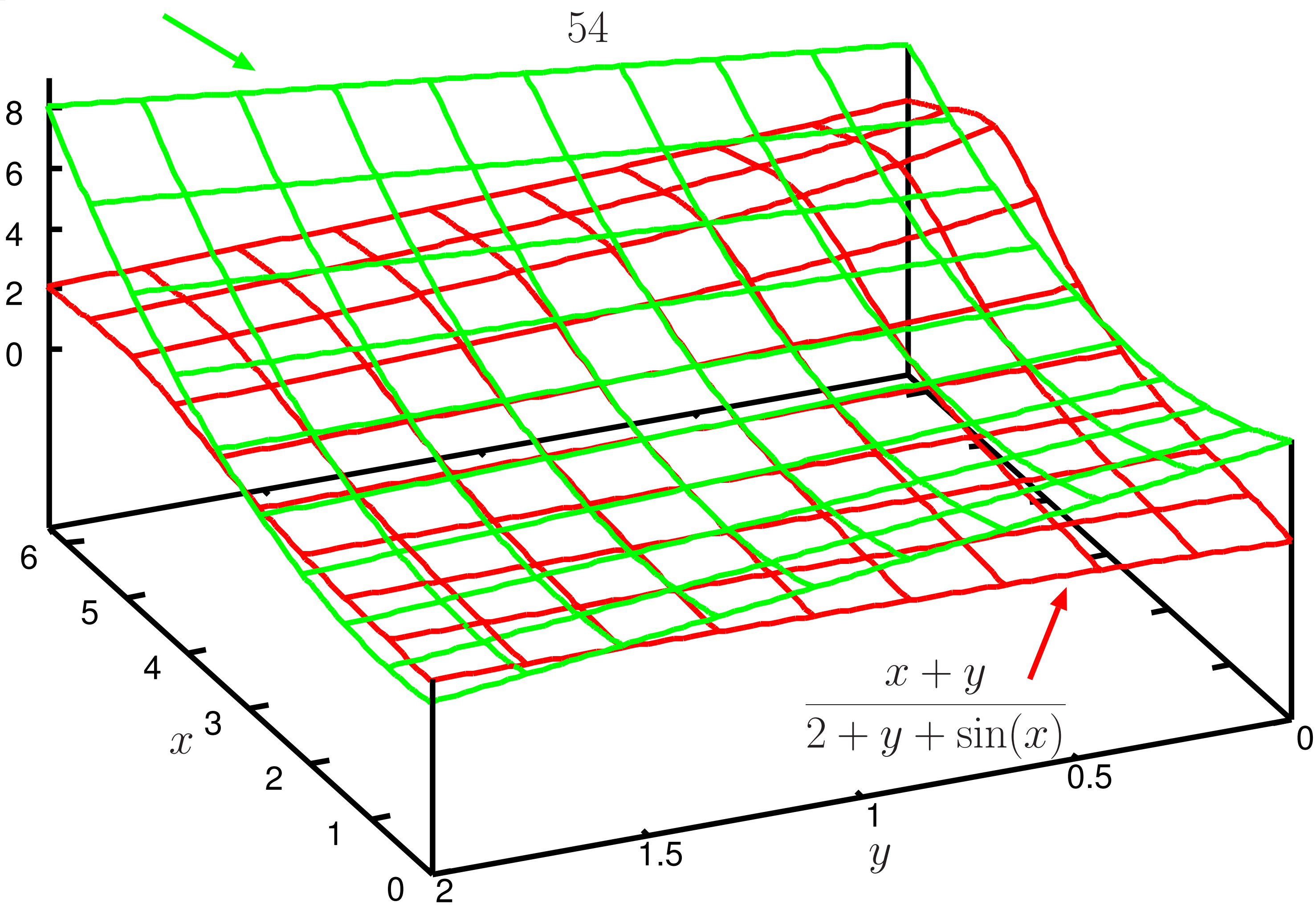
$$f(x, y) = \frac{x + y}{2 + y + \sin(x)}$$

find

a) the Taylor polynomial of degree two about the point $(x, y) = (\pi, 1)$

$$\frac{\pi+1}{3}+\frac{(\pi+4)(x-\pi)+(-\pi+2)(y-1)}{9}$$

$$+\frac{(2\pi+8)(x-\pi)^2+4(\pi+4)(x-\pi)(y-1)+(2\pi-4)(y-1)^2}{54}$$




We are going to use Equation (83) with $(a, b) = (\pi, 1)$ because we need to find out the approximation about the point $(x, y) = (\pi, 1)$

| | | |
|---|---|---|
| $a$ | $\rightarrow$ | $\pi$ |
| $b$ | $\rightarrow$ | $1$ |

Thus Equation (83) can be re-written as

$$f(x,y) = f(\pi,1) + (x-\pi)\frac{d\{f(x,y)\}}{dx}\bigg|_{\substack{x=\pi\\y=1}} + (y-1)\frac{d\{f(x,y)\}}{dy}\bigg|_{\substack{x=\pi\\y=1}}$$

$$+\frac{1}{2!}\left[(x-\pi)^2\frac{d^2f(x,y)}{dx^2}\bigg|_{\substack{x=\pi\\y=1}} + 2(x-\pi)(y-1)\frac{\partial^2 f(x,y)}{\partial y\partial x}\bigg|_{\substack{x=\pi\\y=1}} + (y-1)^2\frac{\partial^2 f(x,y)}{\partial y^2}\bigg|_{\substack{x=\pi\\y=1}}\right]$$

$$+\frac{1}{3!}\left[(x-\pi)^3\frac{\partial^3 f(x,y)}{\partial x^3}\bigg|_{\substack{x=\pi\\y=1}} + 3(x-\pi)^2(y-1)\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\bigg|_{\substack{x=\pi\\y=1}}\right.$$

$$+3(x-\pi)(y-1)^2\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\bigg|_{\substack{x=\pi\\y=1}}+(y-1)^3\frac{\partial^3 f(x,y)}{\partial y^3}\bigg|_{\substack{x=\pi\\y=1}}\Bigg] \quad ①$$

where $|x-\pi|\ll 1$ and $|y-1|\ll 1$.

First, we need to find out $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$.

$$f(x,y)=\frac{x+y}{2+y+\sin(x)}=(x+y)(2+y+\sin(x))^{-1}$$

$$\therefore \frac{d\{f(x,y)\}}{dx}=\frac{d\{(x+y)(2+y+\sin(x))^{-1}\}}{dx}$$

$$=\frac{d\{(x+y)\}}{dx}(2+y+\sin(x))^{-1}+(x+y)\frac{d\{(2+y+\sin(x))^{-1}\}}{dx}$$

$$=(2+y+\sin(x))^{-1}-(x+y)(2+y+\sin(x))^{-2}\cos(x)$$

$$\frac{d\{f(x,y)\}}{dy}=\frac{d\{(x+y)(2+y+\sin(x))^{-1}\}}{dy}$$

$$=\frac{d\{(x+y)\}}{dy}(2+y+\sin(x))^{-1}+(x+y)\frac{d\{(2+y+\sin(x))^{-1}\}}{dy}$$

$$=(2+y+\sin(x))^{-1}-(x+y)(2+y+\sin(x))^{-2}$$

$$\frac{d^2 f(x,y)}{dx^2}=\frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx}$$

$$=\frac{d\{(2+y+\sin(x))^{-1}-(x+y)(2+y+\sin(x))^{-2}\cos(x)\}}{dx}$$

$$=\frac{d\{(2+y+\sin(x))^{-1}\}}{dx}-\frac{d\{(x+y)\}}{dx}(2+y+\sin(x))^{-2}\cos(x)$$

$$-\frac{d\{(2+y+\sin(x))^{-2}\}}{dx}(x+y)\cos(x)-(x+y)(2+y+\sin(x))^{-2}\frac{d\{\cos(x)\}}{dx}$$

$$=-(2+y+\sin(x))^{-2}\cos(x)-(2+y+\sin(x))^{-2}\cos(x)$$

$$+2(2+y+\sin(x))^{-3}(x+y)\cos^2(x)+(x+y)(2+y+\sin(x))^{-2}\sin(x)$$

$$\frac{\partial^2 f(x,y)}{\partial y\partial x}=\frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dx}=\frac{d\{(2+y+\sin(x))^{-1}-(x+y)(2+y+\sin(x))^{-2}\}}{dx}$$

$$=\frac{d\{(2+y+\sin(x))^{-1}\}}{dx}-\frac{d\{(x+y)\}}{dx}(2+y+\sin(x))^{-2}-(x+y)\frac{d\{(2+y+\sin(x))^{-2}\}}{dx}$$

$$=-(2+y+\sin(x))^{-1}\cos(x)-(2+y+\sin(x))^{-2}+2(x+y)(2+y+\sin(x))^{-3}\cos(x)$$

$$\frac{\partial^2 f(x,y)}{\partial y^2}=\frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dy}=\frac{d\{(2+y+\sin(x))^{-1}-(x+y)(2+y+\sin(x))^{-2}\}}{dy}$$

$$=\frac{d\{(2+y+\sin(x))^{-1}\}}{dy}-\frac{d\{(x+y)\}}{dy}(2+y+\sin(x))^{-2}-(x+y)\frac{d\{(2+y+\sin(x))^{-2}\}}{dy}$$

$$=-(2+y+\sin(x))^{-2}-(2+y+\sin(x))^{-2}+2(x+y)(2+y+\sin(x))^{-3}$$

Second, we need to find out $f(\pi,1)$, $\frac{d\{f(x,y)\}}{dx}\Big|_{\substack{x=\pi\\y=1}}$, $\frac{d\{f(x,y)\}}{dy}\Big|_{\substack{x=\pi\\y=1}}$,

$$\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=\pi\\y=1}}, \left.\frac{\partial^2f(x,y)}{\partial y\partial x}\right|_{\substack{x=\pi\\y=1}}, \left.\frac{\partial^2f(x,y)}{\partial y^2}\right|_{\substack{x=\pi\\y=1}}.$$

$$f(\pi,1)=\frac{\pi+1}{2+1+\sin(\pi)}=\frac{\pi+1}{3}$$

$$\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=\pi\\y=1}}=(2+1+\sin(\pi))^{-1}-(\pi+1)(2+1+\sin(\pi))^{-2}\cos(\pi)=(3)^{-1}+(\pi+1)(3)^{-2}$$

$$=3\cdot(3)^{-2}+(\pi+1)(3)^{-2}=(\pi+4)(3)^{-2}$$

$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=\pi\\y=1}}=(2+1+\sin(\pi))^{-1}-(\pi+1)(2+1+\sin(\pi))^{-2}=3^{-1}-(\pi+1)(3)^{-2}$$

$$=3\cdot3^{-2}-(\pi+1)(3)^{-2}=(-\pi-1+3)(3)^{-2}=(-\pi+2)(3)^{-2}$$

$$\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=\pi\\y=1}}=-(2+1+\sin(\pi))^{-2}\cos(\pi)-(2+1+\sin(\pi))^{-2}\cos(\pi)$$

$$+2(2+1+\sin(\pi))^{-3}(\pi+1)\cos^2(\pi)+(\pi+1)(2+1+\sin(\pi))^{-2}\sin(\pi)$$

$$=(3)^{-2}+3^{-2}+2(3)^{-3}(\pi+1)=2\cdot(3)^{-2}+2(3)^{-3}(\pi+1)$$

$$=6\cdot(3)^{-3}+2(3)^{-3}(\pi+1)=(3)^{-3}(2\pi+8)$$

$$\left.\frac{\partial^2f(x,y)}{\partial y\partial x}\right|_{\substack{x=\pi\\y=1}}=-(2+1+\sin(\pi))^{-1}\cos(\pi)-(2+1+\sin(\pi))^{-2}+2(\pi+1)(2+1+\sin(\pi))^{-3}\cos(\pi)$$

$$=(3)^{-1}-3^{-2}+2(\pi+1)(3)^{-3}=9(3)^{-3}-3\cdot3^{-3}+2(\pi+1)(3)^{-3}$$

$$=(2\pi+2+9-3)(3)^{-3}=(2\pi+8)(3)^{-3}$$

$$\left.\frac{\partial^2f(x,y)}{\partial y^2}\right|_{\substack{x=\pi\\y=1}}=-(2+1+\sin(\pi))^{-2}$$

$$-(2+1+\sin(\pi))^{-2}+2(\pi+1)(2+1+\sin(\pi))^{-3}$$

$$=-3^{-2}-3^{-2}+2(\pi+1)(3)^{-3}=-2\cdot3^{-2}+2(\pi+1)(3)^{-3}$$

$$=-6\cdot3^{-3}+2(\pi+1)(3)^{-3}=(2\pi+2-6)(3)^{-3}=(2\pi-4)(3)^{-3}$$

Finally by substituting these into ①, we get

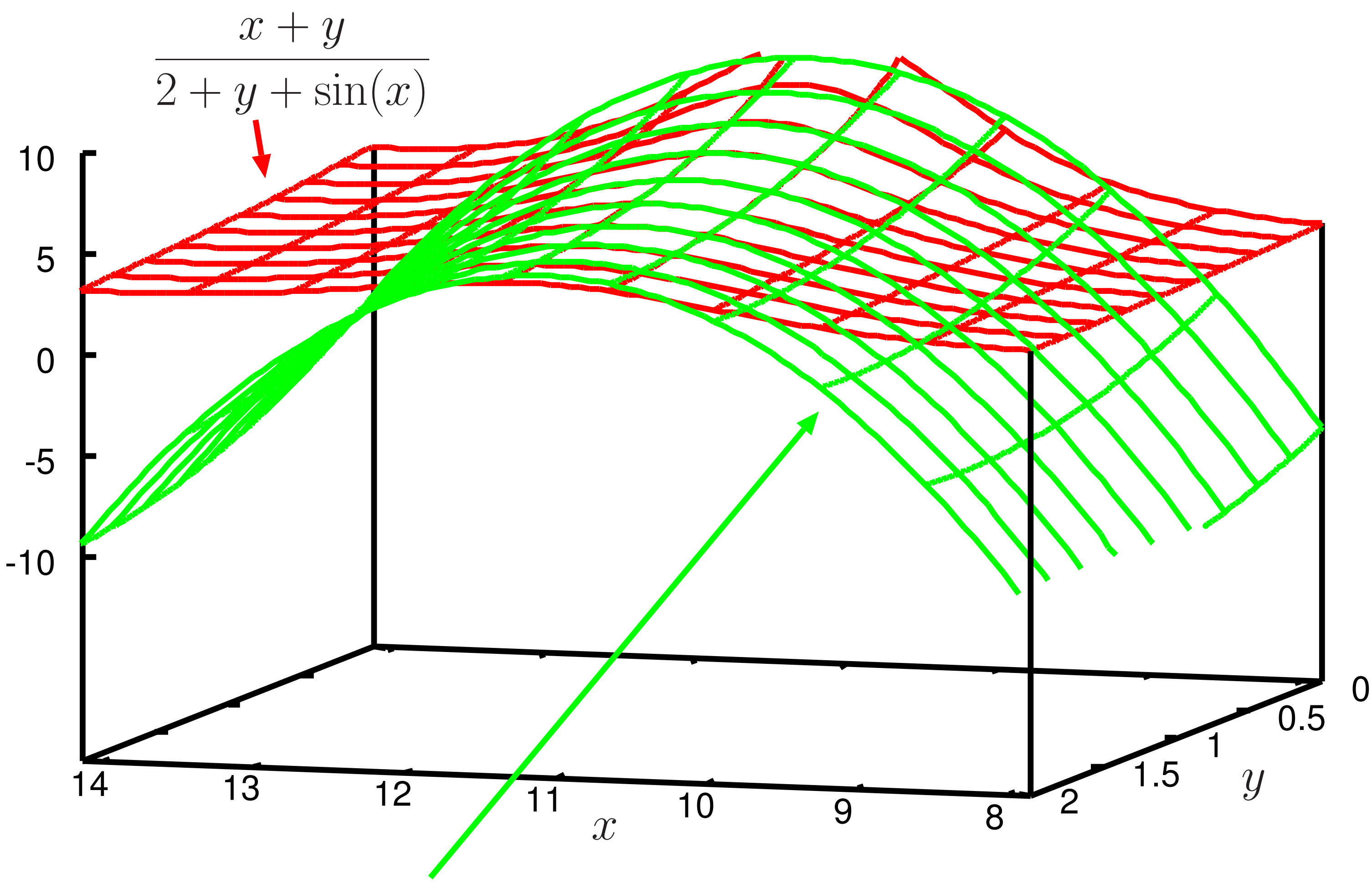
$$f(x,y)=\frac{\pi+1}{3}+(\pi+4)(3)^{-2}\cdot(x-\pi)+(-\pi+2)(3)^{-2}\cdot(y-1)+$$

$$\frac{1}{2!}\left[(3)^{-3}(2\pi+8)(x-\pi)^2+2(x-\pi)(y-1)\cdot(2\pi+8)(3)^{-3}+(2\pi-4)(3)^{-3}\cdot(y-1)^2\right]$$

$$=\frac{\pi+1}{3}+\frac{(\pi+4)(x-\pi)+(-\pi+2)(y-1)}{9}$$

$$+\frac{(2\pi+8)(x-\pi)^2+4(\pi+4)(x-\pi)(y-1)+(2\pi-4)(y-1)^2}{54}$$

b) the quadratic approximation $f(x,y)$ of the Taylor series expansion aroound $(3.5\pi,1)$




$$\frac{3.5\pi+1}{2}+\frac{x-3.5\pi}{2}+\frac{(1-3.5\pi)(y-1)}{2^2}$$

$$-\frac{(3.5\pi+1)(x-3.5\pi)^2}{2^3}-\frac{(x-3.5\pi)(y-1)}{2^2}+\frac{(7\pi-2)(y-1)^2}{2^4}$$

In order to work out $f(x,y)$ , we need to find out $a$ and $b$ in Equation (83). Since

$$a = 3.5\pi \;\; ; \;\; b = 1$$

we get $a = 3.5\pi$ and $b = 1$. This means we are going to find out the Taylor series expansion around $(x,y) = (3.5\pi, 1)$ using Equation (83). We have already found $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$ previously. Since the quadratic approximation is required, we just need to find out $f(3.5\pi,1)$, $\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=3.5\pi \\ y=1}}$ , $\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=3.5\pi \\ y=1}}$ , $\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=3.5\pi \\ y=1}}$ , $\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=3.5\pi \\ y=1}}$ , and $\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=3.5\pi \\ y=1}}$ .

$$f(3.5\pi,1) = \frac{3.5\pi+1}{2+1+\sin(3.5\pi)} = \frac{3.5\pi+1}{2+1-1} = \frac{3.5\pi+1}{2}$$

$$\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=3.5\pi\\y=1}} = (2+1+\sin(3.5\pi))^{-1} - (3.5\pi+1)(2+1+\sin(3.5\pi))^{-2}\cos(3.5\pi) = (2)^{-1}$$

$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=3.5\pi\\y=1}} = (2+1+\sin(3.5\pi))^{-1} - (3.5\pi+1)(2+1+\sin(3.5\pi))^{-2}$$

$$= (2)^{-1} - (3.5\pi+1)(2)^{-2} = 2\cdot(2)^{-2} - (3.5\pi+1)(2)^{-2} = (2-3.5\pi-1)(2)^{-2} = (1-3.5\pi)(2)^{-2}$$

$$\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=3.5\pi\\y=1}} = -(2+1+\sin(3.5\pi))^{-2}\cos(3.5\pi) - (2+1+\sin(3.5\pi))^{-2}\cos(3.5\pi)$$

$$+2(2+1+\sin(3.5\pi))^{-3}(3.5\pi+1)\cos^2(3.5\pi) + (3.5\pi+1)(2+1+\sin(3.5\pi))^{-2}\sin(3.5\pi)$$

$$= -(3.5\pi+1)(2)^{-2}$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y\partial x}\right|_{\substack{x=3.5\pi\\y=1}} = -(2)^{-2}$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=3.5\pi\\y=1}} = -(2)^{-2} - (2)^{-2} + 2(3.5\pi+1)(2)^{-3}$$

$$= -2(2)^{-2} + 2(3.5\pi+1)(2)^{-3} = -4(2)^{-3} + 2(3.5\pi+1)(2)^{-3} = (7\pi+2-4)(2)^{-3} = (7\pi-2)(2)^{-3}$$

Now we find $f(x,y)$ using $(a,b) = (3.5\pi, 1)$ as follows:

$$f(x,y) = f(3.5\pi,1) + (x-3.5\pi)\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=3.5\pi\\y=1}} + (y-1)\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=3.5\pi\\y=1}}$$

$$+\frac{1}{2!}\left[(x-3.5\pi)^2\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=3.5\pi\\y=1}} + 2(x-3.5\pi)(y-1)\left.\frac{\partial^2 f(x,y)}{\partial y\partial x}\right|_{\substack{x=3.5\pi\\y=1}} + (y-1)^2\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=3.5\pi\\y=1}}\right]$$

$$= \frac{3.5\pi+1}{2} + \frac{x-3.5\pi}{2} + \frac{(1-3.5\pi)(y-1)}{2^2}$$

$$+\frac{1}{2!}\left[-\frac{(3.5\pi+1)(x-3.5\pi)^2}{2^2} - \frac{(x-3.5\pi)(y-1)}{2} + \frac{(7\pi-2)(y-1)^2}{2^3}\right] \qquad ②$$

c) the value that the linear approximation gives for $f(3.5\pi, 1.3)$ correct to 4 decimal places and the percentage error correct to 2 significant figures.
We have already worked out the approximation of $f(x,y)$ around $(3.5\pi, 1)$. In order to find out $f(3.5\pi, 1.3)$ using the approximation of $f(x,y)$, we need to know the exact value of $x-a$ and $y-b$.

$$x-a = 3.5\pi - 3.5\pi \;\; ; \;\; y-b = 1.3-1 = 0.3$$

Therefore we obtain $(x-a, y-b) = (0, 0.3)$ which satisfy the condition of $|x-a| \ll 1$ and $|y-b| \ll 1$. By substituting $(x,y) = (3.5\pi, 1.3)$ into the degree one of ②, we get

$$f(3.5\pi, 1.3) \simeq 5.2481$$

correct to 4 decimal places. Since $f(3.5\pi, 1.3) = 5.3459$ correct to 4 decimal places, the percentage error is

$$\frac{5.2481-5.3459}{5.3459}\times 100 = -1.82944\% \simeq -1.8\%$$

correct to 2 significant figures.

d) the value that the linear approximation gives for $f(3.6\pi, 1)$ correct to 4 decimal places and the percentage error correct to 2 significant figures.
We have already worked out the approximation of $f(x,y)$ around $(3.5\pi, 1)$. In order to find out $f(3.6\pi, 1)$ using the approximation of $f(x,y)$, we need to know the exact value of $x-a$ and $y-b$.

$$x-a = 3.6\pi - 3.5\pi = 0.1\pi \;\; ; \;\; y-b = 1-1 = 0$$

Therefore we obtain $(x-a, y-b) = (0.1\pi, 0)$ which satisfy the condition of $|x-a| \ll 1$ and $|y-b| \ll 1$. By substituting $(x, y) = (3.6\pi, 1)$ into the degree one of ②, we get

$$f(3.6\pi, 1) \simeq 6.15486 \simeq 6.1549$$

correct to 4 decimal places. Since $f(3.6\pi, 1) = 6.00785 \simeq 6.0079$ correct to 4 decimal places, the percentage error is

$$\frac{6.1549 - 6.0079}{6.0079} \times 100 = 2.44678\% \simeq 2.4\%$$

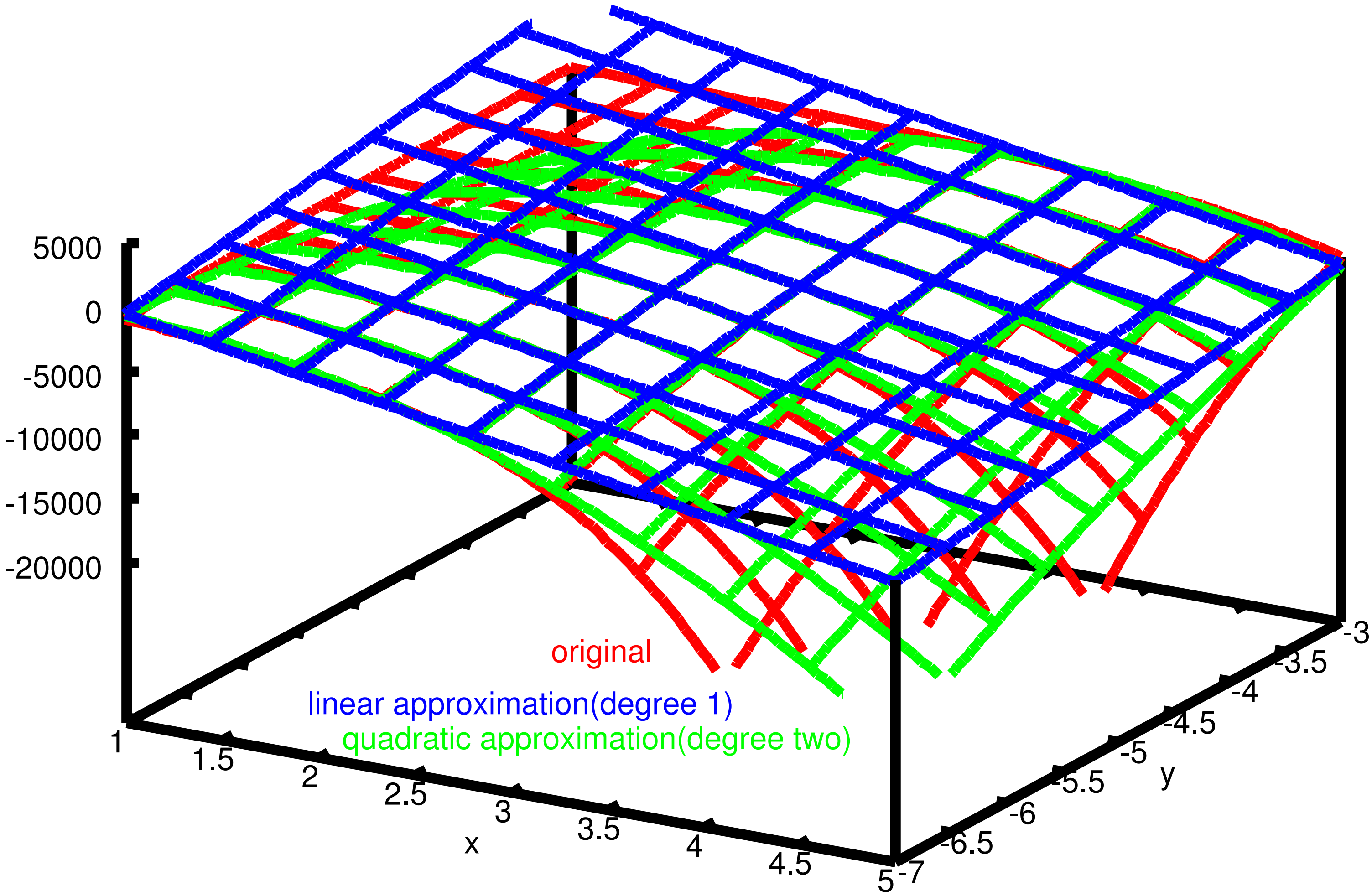
correct to 2 significant figures.

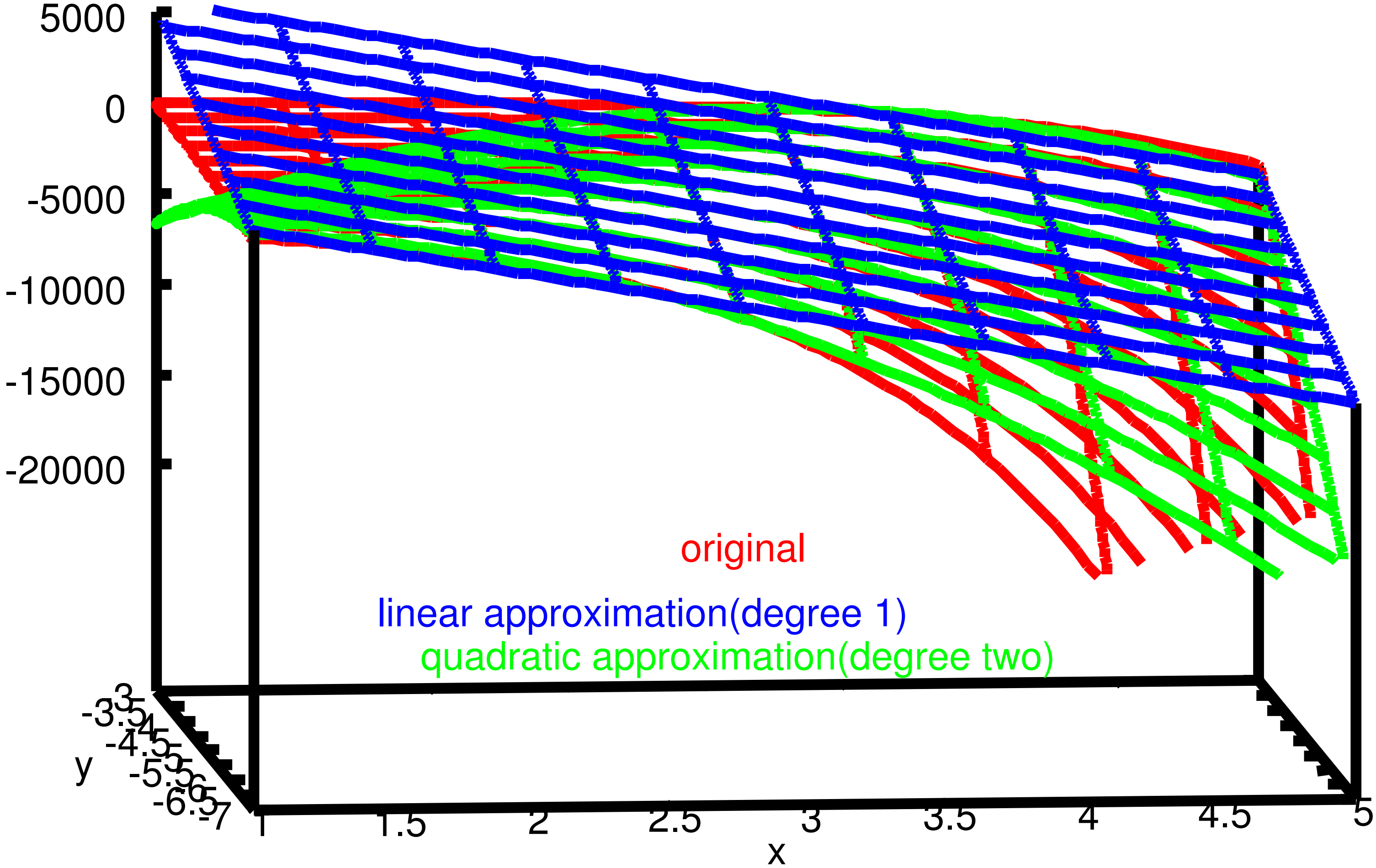
31) For the function

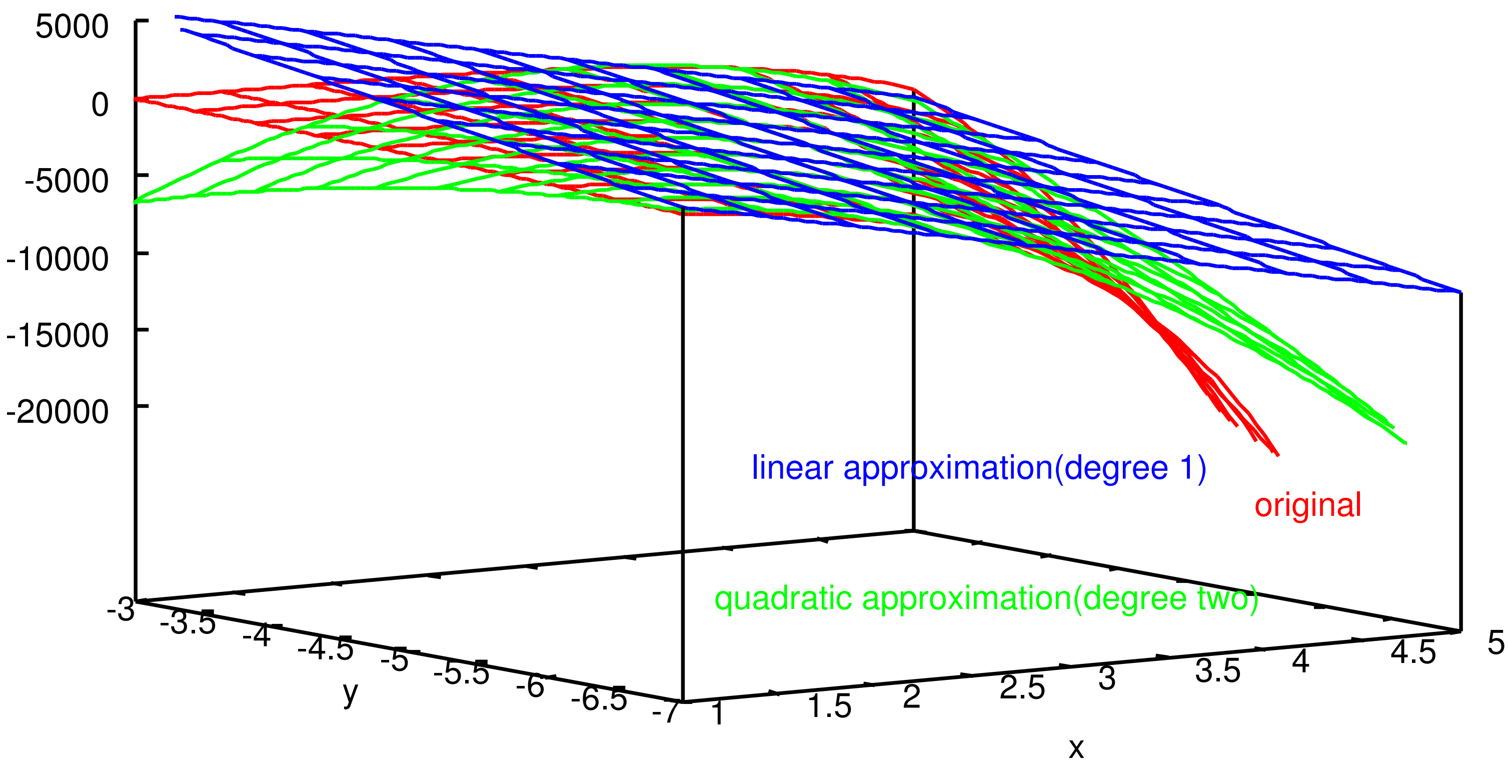
$$f(x, y) = \mathfrak{e}^x y^3$$

find

a) the Taylor polynomial of degree two about the point $(x, y) = (3, -5)$






We are going to use Equation (83) with $(a, b) = (3, -5)$ because we need to find out the approximation about the point $(x, y) = (3, -5)$

| | | |
|---|---|---|
| $a$ | $\rightarrow$ | $3$ |
| $b$ | $\rightarrow$ | $-5$ |

Thus Equation (83) can be re-written as

$$f(x,y) = f(3,-5) + (x-3)\frac{d\{f(x,y)\}}{dx}\bigg|_{\substack{x=3\\y=-5}} + (y+5)\frac{d\{f(x,y)\}}{dy}\bigg|_{\substack{x=3\\y=-5}}$$
$$+\frac{1}{2!}\left[(x-3)^2\frac{d^2f(x,y)}{dx^2}\bigg|_{\substack{x=3\\y=-5}} + 2(x-3)(y+5)\frac{\partial^2 f(x,y)}{\partial y\partial x}\bigg|_{\substack{x=3\\y=-5}} + (y+5)^2\frac{\partial^2 f(x,y)}{\partial y^2}\bigg|_{\substack{x=3\\y=-5}}\right]$$
$$+\frac{1}{3!}\left[(x-3)^3\frac{\partial^3 f(x,y)}{\partial x^3}\bigg|_{\substack{x=3\\y=-5}} + 3(x-3)^2(y+5)\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\bigg|_{\substack{x=3\\y=-5}}\right.$$
$$\left.+3(x-3)(y+5)^2\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\bigg|_{\substack{x=3\\y=-5}} + (y+5)^3\frac{\partial^3 f(x,y)}{\partial y^3}\bigg|_{\substack{x=3\\y=-5}}\right] \quad ①$$

where $|x-3| \ll 1$ and $|y+5| \ll 1$.

First, we need to find out $\dfrac{d\{f(x,y)\}}{dx}$, $\dfrac{d\{f(x,y)\}}{dy}$, $\dfrac{d^2f(x,y)}{dx^2}$, $\dfrac{\partial^2 f(x,y)}{\partial y\partial x}$, $\dfrac{\partial^2 f(x,y)}{\partial y^2}$.

$$f(x,y) = \mathfrak{e}^x y^3$$
$$\therefore \frac{d\{f(x,y)\}}{dx} = \frac{d\{\mathfrak{e}^x y^3\}}{dx} = \mathfrak{e}^x y^3$$
$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{\mathfrak{e}^x y^3\}}{dy} = 3\mathfrak{e}^x y^2$$
$$\frac{d^2f(x,y)}{dx^2} = \frac{d\left\{\dfrac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\{\mathfrak{e}^x y^3\}}{dx} = \mathfrak{e}^x y^3$$
$$\frac{\partial^2 f(x,y)}{\partial y\partial x} = \frac{d\left\{\dfrac{d\{f(x,y)\}}{dx}\right\}}{dy} = \frac{d\{\mathfrak{e}^x y^3\}}{dy} = 3\mathfrak{e}^x y^2$$
$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\dfrac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{3\mathfrak{e}^x y^2\}}{dy} = 6\mathfrak{e}^x y$$

Second, we need to find out $f(3,-5)$, $\dfrac{d\{f(x,y)\}}{dx}\bigg|_{\substack{x=3\\y=-5}}$, $\dfrac{d\{f(x,y)\}}{dy}\bigg|_{\substack{x=3\\y=-5}}$, $\dfrac{d^2f(x,y)}{dx^2}\bigg|_{\substack{x=3\\y=-5}}$, $\dfrac{\partial^2 f(x,y)}{\partial y\partial x}\bigg|_{\substack{x=3\\y=-5}}$, $\dfrac{\partial^2 f(x,y)}{\partial y^2}\bigg|_{\substack{x=3\\y=-5}}$.

$$f(3,-5) = \mathfrak{e}^3(-5)^3 \;\; ; \;\; \frac{d\{f(x,y)\}}{dx}\bigg|_{\substack{x=3\\y=-5}} = \mathfrak{e}^3(-5)^3$$
$$\frac{d\{f(x,y)\}}{dy}\bigg|_{\substack{x=3\\y=-5}} = 3\mathfrak{e}^3(-5)^2$$
$$\frac{d^2f(x,y)}{dx^2}\bigg|_{\substack{x=3\\y=-5}} = \mathfrak{e}^3(-5)^3$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=3\\y=-5}} = 3\mathfrak{e}^3(-5)^2$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=3\\y=-5}} = 6\mathfrak{e}^3(-5)$$

Finally by substituting these into ①, we get

$$f(x,y) = \mathfrak{e}^3(-5)^3 + \mathfrak{e}^3(-5)^3 \cdot (x-3) + 3\mathfrak{e}^3(-5)^2 \cdot (y+5)$$
$$+\frac{1}{2!}\left[\mathfrak{e}^3(-5)^3(x-3)^2 + 2(x-3)(y+5)\cdot 3\mathfrak{e}^3(-5)^2 - 30\mathfrak{e}^3\cdot(y+5)^2\right] \quad ②$$

b) the value that the linear approximation gives for $f(2.9,-4.9)$ correct to 2 decimal places and the percentage error correct to 2 significant figures.
We have already worked out the approximation of $f(x,y)$ around $(3,-5)$. In order to find out $f(2.9,-4.9)$ using the approximation of $f(x,y)$, we need to know the exact value of $x-a$ and $y-b$.

$$x-a = 2.9-3 \;\; ; \;\; y-b = -4.9+5$$
$$\therefore x-a = -0.1 \;\; ; \;\; y-b = 0.1$$

Therefore we obtain $(x-a, y-b) = (-0.1, 0.1)$ which satisfy the condition of $|x-a| \ll 1$ and $|y-b| \ll 1$. By substituting $(x,y) = (2.9,-4.9)$ into the degree one of ②, we get

$$f(3-0.1, -5+0.1) \simeq -2108.98$$

correct to 2 decimal places. Since $f(2.9,-4.9) = -2138.17$ correct to 2 decimal places, the percentage error is

$$\frac{-2108.98+2138.17}{-2138.17} \times 100 = -1.36519\% \simeq -1.37\%$$

correct to 2 significant figures.

**DAY3**

32) For the function

$$f(x,y)=\mathfrak{e}^{x^2}\sin y$$

for $-1<x<1$ and $|y|<\pi$ find

a) the Taylor polynomial of degree three about the point $(x,y)=(0,\frac{\pi}{2})$

We are going to use Equation (83) with $(a,b)=(0,\frac{\pi}{2})$ because we need to find out the approximation about the point $(x,y)=(0,\frac{\pi}{2})$.

| | | |
|---|---|---|
| $a$ | $\rightarrow$ | $0$ |
| $b$ | $\rightarrow$ | $\frac{\pi}{2}$ |

Thus Equation (83) can be re-written as

$$f(x,y)=f(0,\frac{\pi}{2})+x\frac{d\{f(x,y)\}}{dx}\bigg|_{\substack{x=0\\y=\frac{\pi}{2}}}+(y-\frac{\pi}{2})\frac{d\{f(x,y)\}}{dy}\bigg|_{\substack{x=0\\y=\frac{\pi}{2}}}$$
$$+\frac{1}{2!}\left[x^2\frac{d^2f(x,y)}{dx^2}\bigg|_{\substack{x=0\\y=\frac{\pi}{2}}}+2x(y-\frac{\pi}{2})\frac{\partial^2f(x,y)}{\partial y\partial x}\bigg|_{\substack{x=0\\y=\frac{\pi}{2}}}+(y-\frac{\pi}{2})^2\frac{\partial^2f(x,y)}{\partial y^2}\bigg|_{\substack{x=0\\y=\frac{\pi}{2}}}\right]$$
$$+\frac{1}{3!}\left[x^3\frac{\partial^3f(x,y)}{\partial x^3}\bigg|_{\substack{x=0\\y=\frac{\pi}{2}}}+3x^2(y-\frac{\pi}{2})\frac{\partial^3f(x,y)}{\partial y\partial x^2}\bigg|_{\substack{x=0\\y=\frac{\pi}{2}}}\right.$$
$$\left.+3x(y-\frac{\pi}{2})^2\frac{\partial^3f(x,y)}{\partial y^2\partial x}\bigg|_{\substack{x=0\\y=\frac{\pi}{2}}}+(y-\frac{\pi}{2})^3\frac{\partial^3f(x,y)}{\partial y^3}\bigg|_{\substack{x=0\\y=\frac{\pi}{2}}}\right] \quad ①$$

where $|x|\ll 1$ and $|y-\frac{\pi}{2}|\ll 1$. First, we need to find out $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2f(x,y)}{dx^2}$, $\frac{\partial^2f(x,y)}{\partial y\partial x}$, $\frac{\partial^2f(x,y)}{\partial y^2}$, $\frac{\partial^3f(x,y)}{\partial x^3}$, $\frac{\partial^3f(x,y)}{\partial y\partial x^2}$, $\frac{\partial^3f(x,y)}{\partial y^2\partial x}$, and $\frac{\partial^3f(x,y)}{\partial y^3}$

$$f(x,y)=\mathfrak{e}^{x^2}\sin y\ ;\quad\therefore\ \frac{d\{f(x,y)\}}{dx}=\frac{d\left\{\mathfrak{e}^{x^2}\sin y\right\}}{dx}=2x\mathfrak{e}^{x^2}\sin y$$

$$\frac{d\{f(x,y)\}}{dy}=\frac{d\left\{\mathfrak{e}^{x^2}\sin y\right\}}{dy}=\mathfrak{e}^{x^2}\cos y$$

$$\frac{d^2f(x,y)}{dx^2}=\frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx}=\frac{d\left\{2x\mathfrak{e}^{x^2}\sin y\right\}}{dx}$$

$$=\frac{d\{2x\}}{dx}\mathfrak{e}^{x^2}\sin y+2x\frac{d\left\{\mathfrak{e}^{x^2}\sin y\right\}}{dx}=(2+(2x)^2)\mathfrak{e}^{x^2}\sin y$$

$$\frac{\partial^2f(x,y)}{\partial y\partial x}=\frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dx}=\frac{d\left\{\mathfrak{e}^{x^2}\cos y\right\}}{dx}=2x\mathfrak{e}^{x^2}\cos y(=\frac{\partial^2f(x,y)}{\partial x\partial y})$$

$$\frac{\partial^2f(x,y)}{\partial y^2}=\frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dy}=\frac{d\left\{\mathfrak{e}^{x^2}\cos y\right\}}{dy}=-\mathfrak{e}^{x^2}\sin y$$

$$\frac{\partial^3f(x,y)}{\partial x^3}=\frac{d\left\{\frac{d^2f(x,y)}{dx^2}\right\}}{dx}=\frac{d\left\{(2+(2x)^2)\mathfrak{e}^{x^2}\sin y\right\}}{dx}$$

$$= \frac{d\{(2+(2x)^2)\}}{dx}\mathfrak{e}^{x^2}\sin y + (2+(2x)^2)\frac{d\left\{\mathfrak{e}^{x^2}\sin y\right\}}{dx}$$

$$= 8x\mathfrak{e}^{x^2}\sin y + (2+(2x)^2)(2x)\mathfrak{e}^{x^2}\sin y$$

$$\frac{\partial^3 f(x,y)}{\partial y\partial x^2} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y\partial x}\right\}}{dx} = \frac{d\left\{2x\mathfrak{e}^{x^2}\cos y\right\}}{dx}$$

$$= \frac{d\{2x\}}{dx}\mathfrak{e}^{x^2}\cos y + 2x\frac{d\left\{\mathfrak{e}^{x^2}\cos y\right\}}{dx} = (2+(2x)^2)\mathfrak{e}^{x^2}\cos y$$

$$\frac{\partial^3 f(x,y)}{\partial y^2\partial x} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dx} = \frac{d\left\{-\mathfrak{e}^{x^2}\sin y\right\}}{dx} = -2x\mathfrak{e}^{x^2}\sin y$$

$$\frac{\partial^3 f(x,y)}{\partial y^3} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dy} = \frac{d\left\{-\mathfrak{e}^{x^2}\sin y\right\}}{dy} = -\mathfrak{e}^{x^2}\cos y$$

Second, we need to find out $f(0,\frac{\pi}{2})$, $\left.\frac{d\{f(x,y)\}}{dx}\right|_{x=0, y=\frac{\pi}{2}}$, $\left.\frac{d\{f(x,y)\}}{dy}\right|_{x=0, y=\frac{\pi}{2}}$, $\left.\frac{d^2 f(x,y)}{dx^2}\right|_{x=0, y=\frac{\pi}{2}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y\partial x}\right|_{x=0, y=\frac{\pi}{2}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{x=0, y=\frac{\pi}{2}}$, $\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{x=0, y=\frac{\pi}{2}}$, $\left.\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\right|_{x=0, y=\frac{\pi}{2}}$, $\left.\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\right|_{x=0, y=\frac{\pi}{2}}$, and $\left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{x=0, y=\frac{\pi}{2}}$.

$$f(0,\frac{\pi}{2}) = \mathfrak{e}^0\sin\frac{\pi}{2} = 1 \;\;;\;\; \left.\frac{d\{f(x,y)\}}{dx}\right|_{x=0, y=\frac{\pi}{2}} = 0$$

$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{x=0, y=\frac{\pi}{2}} = 0 \;\;;\;\; \left.\frac{d^2 f(x,y)}{dx^2}\right|_{x=0, y=\frac{\pi}{2}} = 2$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y\partial x}\right|_{x=0, y=\frac{\pi}{2}} = 0 \;\;;\;\; \left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{x=0, y=\frac{\pi}{2}} = -1$$

$$\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{x=0, y=\frac{\pi}{2}} = 0 \;\;;\;\; \left.\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\right|_{x=0, y=\frac{\pi}{2}} = 0$$

$$\left.\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\right|_{x=0, y=\frac{\pi}{2}} = 0 \;\;;\;\; \left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{x=0, y=\frac{\pi}{2}} = 0$$

Finally by substituting these into ①, we get

$$f(x,y) = 1 + x\cdot 0 + (y-\frac{\pi}{2})\cdot 0 + \frac{1}{2!}\left[x^2\cdot 2 + 2x(y-\frac{\pi}{2})\cdot 0 + (y-\frac{\pi}{2})^2\cdot(-1)\right]$$
$$+\frac{1}{3!}\left[x^3\cdot 0 + 3x^2(y-\frac{\pi}{2})\cdot 0 + 3x(y-\frac{\pi}{2})^2 0 + (y-\frac{\pi}{2})^3\cdot 0\right]$$
$$= 1 + \frac{1}{2}(2x^2 - (y-\frac{\pi}{2})^2)$$

b) the approximation $f(x,y)$ of the Taylor series expansion of degree three around $(0,0)$
In order to work out $f(x,y)$, we need to find out $a$ and $b$ in Equation (83). Since

$$a = 0 \;\;;\;\; b = 0$$

we get $a = 0$ and $b = 0$. This means we are going to find out the Taylor series expansion around $(x,y) = (0,0)$ using Equation (83). We have already found $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$,

$\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y \partial x^2}$, $\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$ previously. Thus we just need to find out $f(\pi, 0)$, $\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=0\\y=0}}$, $\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=0\\y=0}}$, $\left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=0\\y=0}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=0\\y=0}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=0\\y=0}}$, $\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=0\\y=0}}$, $\left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=0\\y=0}}$, $\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=0\\y=0}}$, and $\left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=0\\y=0}}$.

$$f(0,0) = 0 \ ; \ \left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=0\\y=0}} = 0$$

$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=0\\y=0}} = 1 \ ; \ \left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=0\\y=0}} = 0$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=0\\y=0}} = 0 \ ; \ \left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=0\\y=0}} = 0$$

$$\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=0\\y=0}} = 0 \ ; \ \left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=0\\y=0}} = 2$$

$$\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=0\\y=0}} = 0 \ ; \ \left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=0\\y=0}} = -1$$

Now we find $f(x,y)$ using $(a,b) = (0,0)$ as follows:

$$\begin{aligned} f(x,y) = f(0,0) &+ x\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=0\\y=0}} + y\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=0\\y=0}} \\ &+ \frac{1}{2!}\left[x^2\left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=0\\y=0}} + 2xy\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=0\\y=0}} + y^2\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=0\\y=0}}\right] \\ &+ \frac{1}{3!}\left[x^3\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=0\\y=0}} + 3x^2y\left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=0\\y=0}} \right. \\ &\left. + 3xy^2\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=0\\y=0}} + y^3\left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=0\\y=0}}\right] \\ &= 0 + x \cdot 0 + y \cdot 1 \\ &+ \frac{1}{2!}\left[x^2 \cdot 0 + 2xy \cdot 0 + y^2 \cdot 0\right] \\ &+ \frac{1}{3!}\left[x^3 \cdot 0 + 3x^2y \cdot 2 \right. \\ &\left. + 3xy^2 \cdot 0 + y^3 \cdot (-1)\right] = y + \frac{1}{6}(6x^2y - y^3) \qquad ② \end{aligned}$$

c) Find the value that the cubic approximation gives for $f(-0.1, \frac{\pi}{10})$ correct to 5 decimal places and the percentage error correct to 3 significant figures.
We have already worked out the approximation of $f(x,y)$ around $(0,0)$. In order to find out $f(-0.1, \frac{\pi}{10})$ using the approximation of $f(x,y)$, we need to know the exact value of $x-a$ and $y-b$.

$$x - a = -0.1 - 0 = -0.1 y - b = \frac{\pi}{10} - 0 = \frac{\pi}{10}$$

Therefore we obtain $(x-a, y-b) = (-0.1, \frac{\pi}{10})$ which satisfy the condition of $|x-a| \ll 1$ and $|y-b| \ll 1$. By substituting $(x,y) = (-0.1, \frac{\pi}{10})$ into ②, we get

$$\frac{\pi}{10} + \frac{1}{6}(6(-0.1)^2(\frac{\pi}{10}) - (\frac{\pi}{10})^3) = 0.312133 \simeq 0.31213$$

correct to 5 decimal places. Since $f(-0.1, \frac{\pi}{10}) = 0.312122 \simeq 0.31212$, the percentage error is

$$\frac{0.31213 - 0.31212}{0.31212} \times 100 = 0.0032039\% \simeq 0.00320\%$$

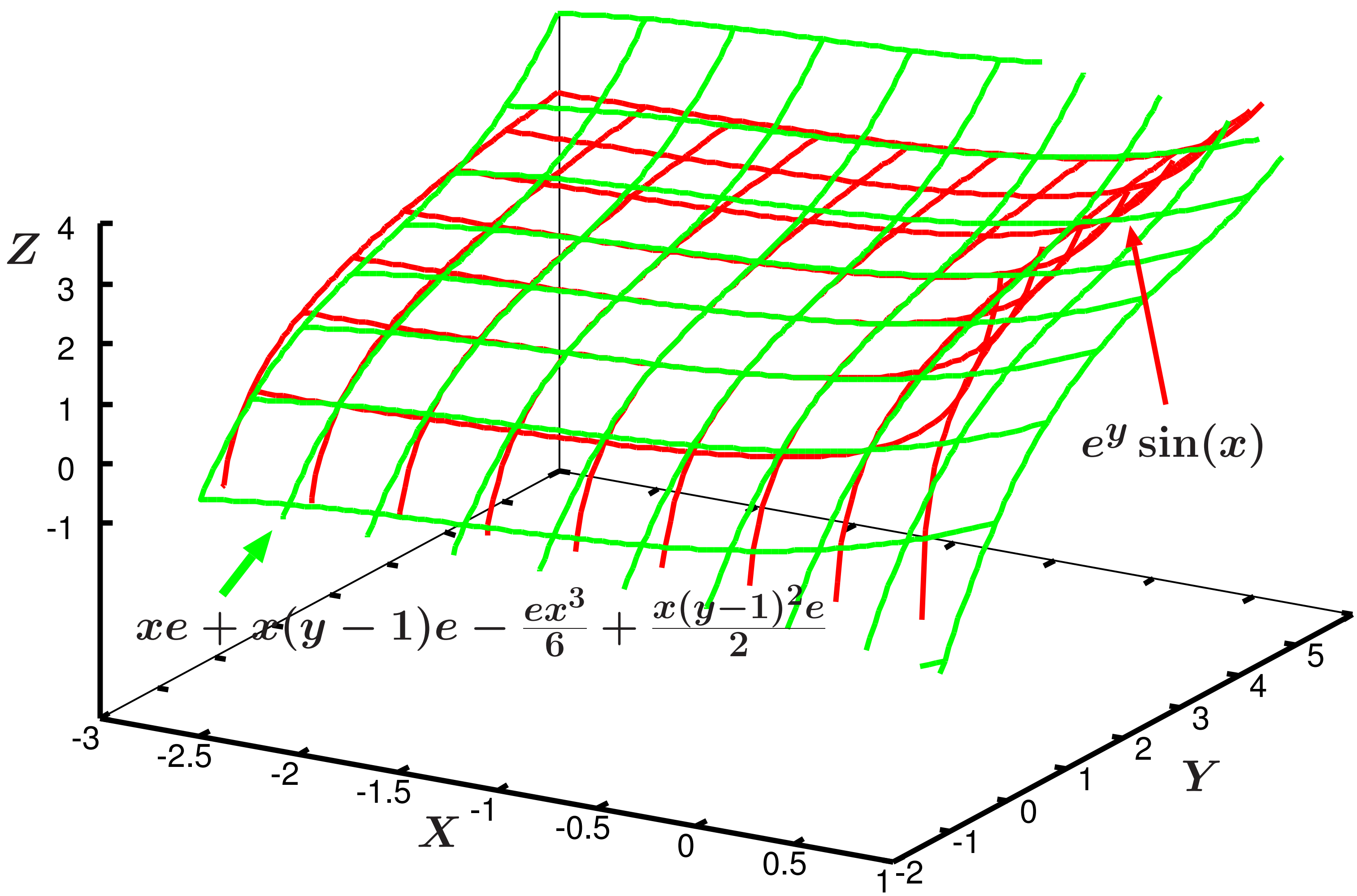
correct to 3 significant figures.

33) For the function

$$f(x, y) = \mathfrak{e}^y \sin x$$

find

a) the Taylor polynomial of degree three about the point $(x, y) = (0, 1)$




We are going to use Equation (83) with $(a, b) = (0, 1)$ because we need to find out the approximation about the point $(x, y) = (0, 1)$.

| | | |
|---|---|---|
| $a$ | $\rightarrow$ | 0 |
| $b$ | $\rightarrow$ | 1 |

Thus Equation (83) can be re-written as

$$f(x,y) = f(0,1) + x\frac{d\{f(x,y)\}}{dx}\bigg|_{\substack{x=0\\y=1}} + (y-1)\frac{d\{f(x,y)\}}{dy}\bigg|_{\substack{x=0\\y=1}}$$

$$+\frac{1}{2!}\left[x^2\frac{d^2 f(x,y)}{dx^2}\bigg|_{\substack{x=0\\y=1}} + 2x(y-1)\frac{\partial^2 f(x,y)}{\partial y \partial x}\bigg|_{\substack{x=0\\y=1}} + (y-1)^2\frac{\partial^2 f(x,y)}{\partial y^2}\bigg|_{\substack{x=0\\y=1}}\right]$$

$$+\frac{1}{3!}\left[x^3\frac{\partial^3 f(x,y)}{\partial x^3}\bigg|_{\substack{x=0\\y=1}} + 3x^2(y-1)\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\bigg|_{\substack{x=0\\y=1}}\right.$$

$$+3x(y-1)^2\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\bigg|_{\substack{x=0\\y=1}} + (y-1)^3\frac{\partial^3 f(x,y)}{\partial y^3}\bigg|_{\substack{x=0\\y=1}}\Bigg] \qquad ①$$

where $|x| \ll 1$ and $|y-1| \ll 1$. First, we need to find out $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y\partial x^2}$, $\frac{\partial^3 f(x,y)}{\partial y^2\partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$

$$f(x,y) = \mathfrak{e}^y \sin x \;\;;\;\; \therefore \frac{d\{f(x,y)\}}{dx} = \frac{d\{\mathfrak{e}^y \sin x\}}{dx} = \mathfrak{e}^y\frac{d\{\sin x\}}{dx} = \mathfrak{e}^y \cos x$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{\mathfrak{e}^y \sin x\}}{dy} = \sin x\frac{d\{\mathfrak{e}^y\}}{dy} = \mathfrak{e}^y \sin x$$

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{\dfrac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\{\mathfrak{e}^y \cos x\}}{dx} = \mathfrak{e}^y\frac{d\{\cos x\}}{dx} = -\mathfrak{e}^y \sin x$$

$$\frac{\partial^2 f(x,y)}{\partial y\partial x} = \frac{d\left\{\dfrac{d\{f(x,y)\}}{dy}\right\}}{dx} = \frac{d\{\mathfrak{e}^y \sin x\}}{dx} = \mathfrak{e}^y\frac{d\{\sin x\}}{dx} = \mathfrak{e}^y \cos x (= \frac{\partial^2 f(x,y)}{\partial x\partial y})$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\dfrac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{\mathfrak{e}^y \sin x\}}{dy} = \sin x\frac{d\{\mathfrak{e}^y\}}{dy} = \mathfrak{e}^y \sin x$$

$$\frac{\partial^3 f(x,y)}{\partial x^3} = \frac{d\left\{\dfrac{d^2 f(x,y)}{dx^2}\right\}}{dx} = \frac{d\{-\mathfrak{e}^y \sin x\}}{dx} = -\mathfrak{e}^y\frac{d\{\sin x\}}{dx} = -\mathfrak{e}^y \cos x$$

$$\frac{\partial^3 f(x,y)}{\partial y\partial x^2} = \frac{d\left\{\dfrac{\partial^2 f(x,y)}{\partial y\partial x}\right\}}{dx} = \frac{d\{\mathfrak{e}^y \cos x\}}{dx} = \mathfrak{e}^y\frac{d\{\cos x\}}{dx} = -\mathfrak{e}^y \sin x$$

$$\frac{\partial^3 f(x,y)}{\partial y^2\partial x} = \frac{d\left\{\dfrac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dx} = \frac{d\{\mathfrak{e}^y \sin x\}}{dx} = \mathfrak{e}^y\frac{d\{\sin x\}}{dx} = \mathfrak{e}^y \cos x$$

$$\frac{\partial^3 f(x,y)}{\partial y^3} = \frac{d\left\{\dfrac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dy} = \frac{d\{\mathfrak{e}^y \sin x\}}{dy} = \sin x\frac{d\{\mathfrak{e}^y\}}{dy} = \mathfrak{e}^y \sin x$$

Second, we need to find out $f(0,1)$, $\frac{d\{f(x,y)\}}{dx}\Big|_{\substack{x=0\\y=1}}$, $\frac{d\{f(x,y)\}}{dy}\Big|_{\substack{x=0\\y=1}}$, $\frac{d^2 f(x,y)}{dx^2}\Big|_{\substack{x=0\\y=1}}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}\Big|_{\substack{x=0\\y=1}}$, $\frac{\partial^2 f(x,y)}{\partial y^2}\Big|_{\substack{x=0\\y=1}}$, $\frac{\partial^3 f(x,y)}{\partial x^3}\Big|_{\substack{x=0\\y=1}}$, $\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\Big|_{\substack{x=0\\y=1}}$, $\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\Big|_{\substack{x=0\\y=1}}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}\Big|_{\substack{x=0\\y=1}}$.

$$f(0,1) = \mathfrak{e}^1 \sin 0 = 0 \;\;;\;\; \frac{d\{f(x,y)\}}{dx}\bigg|_{\substack{x=0\\y=1}} = \mathfrak{e}^1 \cos 0 = \mathfrak{e}$$

$$\frac{d\{f(x,y)\}}{dy}\bigg|_{\substack{x=0\\y=1}} = \mathfrak{e}^1 \sin 0 = 0 \;\;;\;\; \frac{d^2 f(x,y)}{dx^2}\bigg|_{\substack{x=0\\y=1}} = -\mathfrak{e}^1 \sin 0 = 0$$

$$\frac{\partial^2 f(x,y)}{\partial y\partial x}\bigg|_{\substack{x=0\\y=1}} = \mathfrak{e}^1 \cos 0 = \mathfrak{e} \;\;;\;\; \frac{\partial^2 f(x,y)}{\partial y^2}\bigg|_{\substack{x=0\\y=1}} = \mathfrak{e}^1 \sin 0 = 0$$

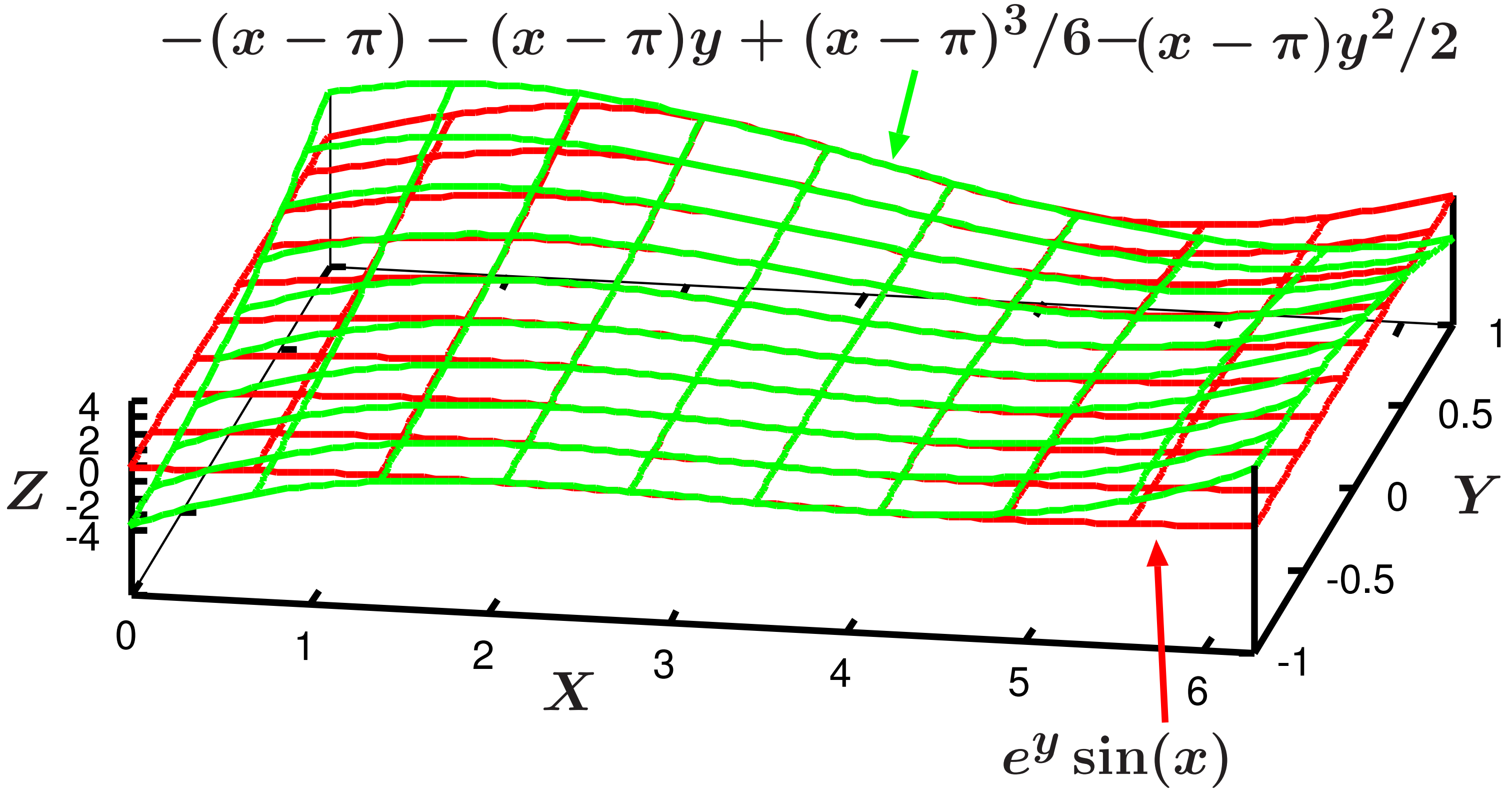
$$\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=0\\y=1}} = -\mathfrak{e}^1\cos 0 = -\mathfrak{e} \;\; ; \;\; \left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=0\\y=1}} = -\mathfrak{e}^1\sin 0 = 0$$

$$\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=0\\y=1}} = \mathfrak{e}^1\cos 0 = \mathfrak{e} \;\; ; \;\; \left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=0\\y=1}} = \mathfrak{e}^1\sin 0 = 0$$

Finally by substituting these into ①, we get

$$\begin{aligned} f(x,y) &= 0 + x\mathfrak{e} + (y-1)\cdot 0 + \frac{1}{2!}\left[x^2\cdot 0 + 2x(y-1)\mathfrak{e} + (y-1)^2\cdot 0\right] \\ &+\frac{1}{3!}\left[x^3\cdot(-\mathfrak{e}) + 3x^2(y-1)\cdot 0 + 3x(y-1)^2\mathfrak{e} + (y-1)^3\cdot 0\right] \\ &= x\mathfrak{e} + \frac{1}{2!}\left[2x(y-1)\mathfrak{e}\right] + \frac{1}{3!}\left[-\mathfrak{e}x^3 + 3x(y-1)^2\mathfrak{e}\right] \\ &= x\mathfrak{e} + x(y-1)\mathfrak{e} - \frac{\mathfrak{e}x^3}{6} + \frac{x(y-1)^2\mathfrak{e}}{2} \end{aligned}$$

b) the approximation $f(x,y)$ of the Taylor series expansion of degree three around $(\pi, 0)$




In order to work out $f(x,y)$, we need to find out $a$ and $b$ in Equation (83). Since

$$a = \pi \;\; ; \;\; b = 0$$

we get $a = \pi$ and $b = 0$. This means we are going to find out the Taylor series expansion around $(x,y) = (\pi, 0)$ using Equation (83). We have already found $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y \partial x^2}$, $\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$ previously. Thus we just need to find out $f(\pi,0)$, $\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=\pi\\y=0}}$, $\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=\pi\\y=0}}$, $\left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=\pi\\y=0}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=\pi\\y=0}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=\pi\\y=0}}$,

$$\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=\pi\\y=0}}, \left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=\pi\\y=0}}, \left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=\pi\\y=0}}, \text{ and } \left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=\pi\\y=0}}.$$

$$f(\pi,0) = \mathfrak{e}^0 \sin\pi = 0 \;;\; \left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=\pi\\y=0}} = \mathfrak{e}^0 \cos\pi = -1$$

$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=\pi\\y=0}} = \mathfrak{e}^0 \sin\pi = 0 \;;\; \left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=\pi\\y=0}} = -\mathfrak{e}^0 \sin\pi = 0$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=\pi\\y=0}} = \mathfrak{e}^0 \cos\pi = -1 \;;\; \left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=\pi\\y=0}} = \mathfrak{e}^0 \sin\pi = 0$$

$$\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=\pi\\y=0}} = -\mathfrak{e}^0 \cos\pi = 1 \;;\; \left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=\pi\\y=0}} = -\mathfrak{e}^0 \sin\pi = 0$$

$$\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=\pi\\y=0}} = \mathfrak{e}^0 \cos\pi = -1 \;;\; \left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=\pi\\y=0}} = \mathfrak{e}^0 \sin\pi = 0$$

Now we find $f(x,y)$ using $(a,b) = (\pi,0)$ as follows:

$$f(x,y) = f(\pi,0) + (x-\pi)\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=\pi\\y=0}} + y\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=\pi\\y=0}}$$

$$+\frac{1}{2!}\left[(x-\pi)^2\left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=\pi\\y=0}} + 2(x-\pi)y\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=\pi\\y=0}} + y^2\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=\pi\\y=0}}\right]$$

$$+\frac{1}{3!}\left[(x-\pi)^3\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=\pi\\y=0}} + 3(x-\pi)^2 y\left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=\pi\\y=0}}\right.$$

$$\left. +3(x-\pi)y^2\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=\pi\\y=0}} + y^3\left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=\pi\\y=0}}\right]$$

$$= 0 + (x-\pi)\cdot(-1) + y\cdot 0 + \frac{1}{2!}\left[(x-\pi)^2\cdot 0 + 2(x-\pi)y\cdot(-1) + y^2\cdot 0\right]$$

$$+\frac{1}{3!}\left[(x-\pi)^3\cdot(1) + 3(x-\pi)^2 y\cdot 0 + 3(x-\pi)y^2\cdot(-1) + y^3\cdot 0\right]$$

$$= -(x-\pi) + \frac{1}{2!}\left[-2(x-\pi)y\right] + \frac{1}{3!}\left[(x-\pi)^3 - 3(x-\pi)y^2\right]$$

$$= -(x-\pi) - (x-\pi)y + \frac{(x-\pi)^3}{6} - \frac{(x-\pi)y^2}{2} \qquad ②$$

c) Find the value that the cubic approximation gives for $f(1.1\pi, -0.1)$ correct to 4 decimal places and the percentage error correct to 3 significant figures.
We have already worked out the approximation of $f(x,y)$ around $(\pi,0)$. In order to find out $f(1.1\pi,-0.1)$ using the approximation of $f(x,y)$, we need to know the exact value of $x-a$ and $y-b$.

$$x-a = 1.1\pi - \pi = 0.1\pi \;;\; y-b = -0.1-0 = -0.1$$

Therefore we obtain $(x-a, y-b) = (0.1\pi, -0.1)$ which satisfy the condition of $|x-a| \ll 1$ and $|y-b| \ll 1$. By substituting $(x,y) = (1.1\pi,-0.1)$ into ②, we get

$$-(x-\pi) - (x-\pi)y + \frac{(x-\pi)^3}{6} - \frac{(x-\pi)y^2}{2}$$

$$= -0.1\pi - 0.1\pi\cdot(-0.1) + \frac{(0.1\pi)^3}{6} - \frac{0.1\pi(-0.1)^2}{2} = -0.1\pi + 0.01\pi + \frac{0.001\pi^3}{6} - 0.0005\pi$$

$$= (-0.1 + 0.01 - 0.0005)\pi + \frac{0.001\pi^3}{6} = -0.0905\pi + \frac{0.001\pi^3}{6} = -0.279146 \simeq -0.2791$$

correct to 4 decimal places. Since $f(1.1\pi, -0.1) = -0.279608 \simeq -0.2796$, the percentage error is

$$\frac{-0.2791 + 0.2796}{-0.2796} \times 100 = -0.178827\% \simeq -0.179\%$$

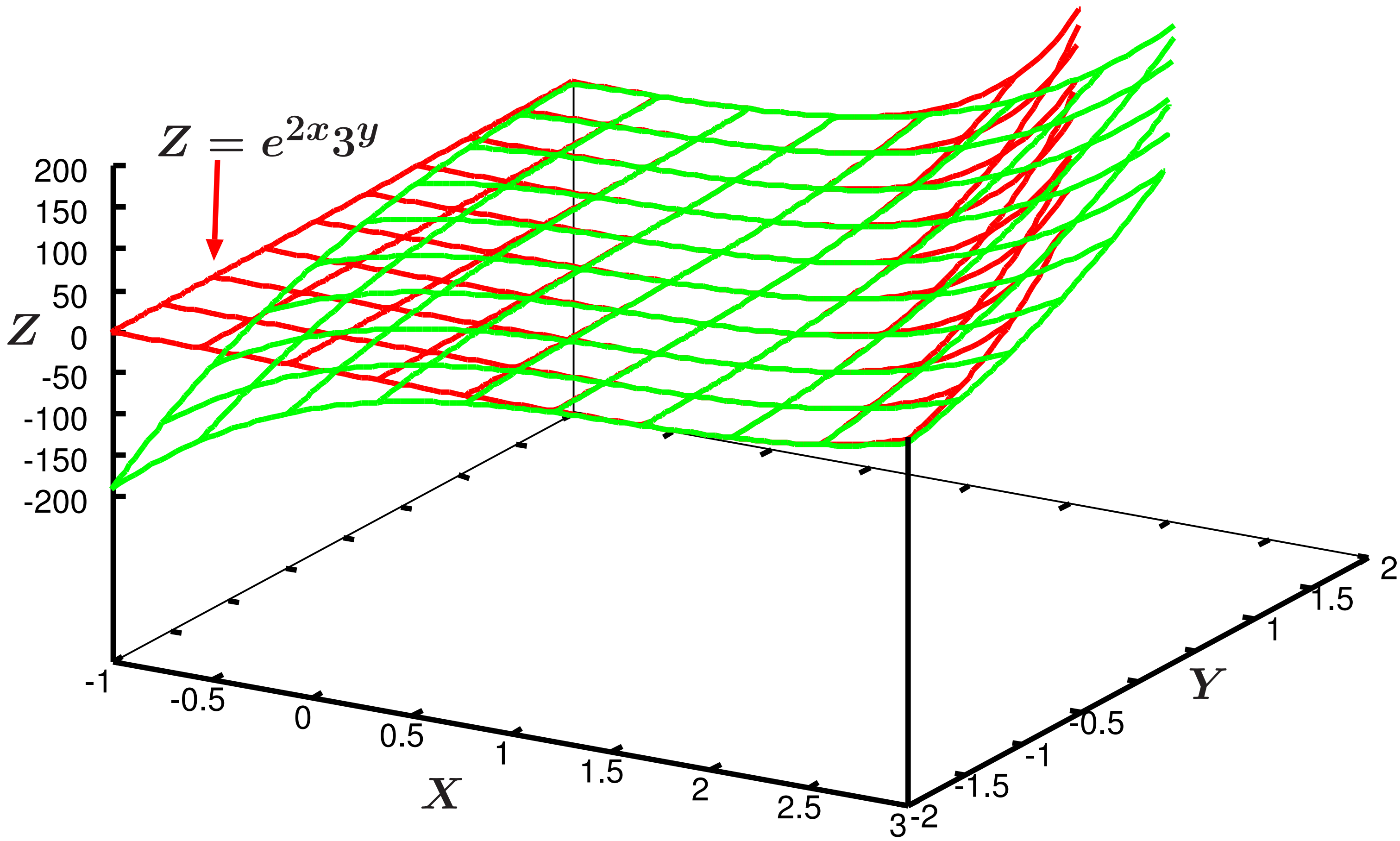
correct to 3 significant figures.

34) For the function

$$f(x, y) = \mathfrak{e}^{2x} 3^y$$

find

a) the Taylor polynomial of degree three about the point $(x, y) = (1, 0)$




We are going to use Equation (83) with $(a, b) = (1, 0)$ because we need to find out the approximation about the point $(x, y) = (1, 0)$.

| | | |
|---|---|---|
| $a$ | $\rightarrow$ | 1 |
| $b$ | $\rightarrow$ | 0 |

Thus Equation (83) can be re-written as

$$f(x,y) = f(1,0) + (x-1)\frac{d\{f(x,y)\}}{dx}\bigg|_{\substack{x=1\\y=0}} + y\frac{d\{f(x,y)\}}{dy}\bigg|_{\substack{x=1\\y=0}}$$

$$+\frac{1}{2!}\left[(x-1)^2\frac{d^2 f(x,y)}{dx^2}\bigg|_{\substack{x=1\\y=0}} + 2(x-1)y\frac{\partial^2 f(x,y)}{\partial y \partial x}\bigg|_{\substack{x=1\\y=0}} + y^2\frac{\partial^2 f(x,y)}{\partial y^2}\bigg|_{\substack{x=1\\y=0}}\right]$$

$$+\frac{1}{3!}\left[(x-1)^3\frac{\partial^3 f(x,y)}{\partial x^3}\bigg|_{\substack{x=1\\y=0}} + 3(x-1)^2 y\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\bigg|_{\substack{x=1\\y=0}}\right.$$

$$\left. +3(x-1)y^2\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\bigg|_{\substack{x=1\\y=0}} + y^3\frac{\partial^3 f(x,y)}{\partial y^3}\bigg|_{\substack{x=1\\y=0}}\right] \quad ①$$

where $|x-1| \ll 1$ and $|y| \ll 1$.

First, we need to find out $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y \partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y \partial x^2}$, $\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$

$$f(x,y) = \mathfrak{e}^{2x}3^y$$

$$\therefore \frac{d\{f(x,y)\}}{dx} = \frac{d\{\mathfrak{e}^{2x}3^y\}}{dx} = 3^y \frac{d\{\mathfrak{e}^{2x}\}}{dx} = 2 \cdot 3^y \mathfrak{e}^{2x}$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{\mathfrak{e}^{2x}3^y\}}{dy} = \mathfrak{e}^{2x}\frac{d\{3^y\}}{dy} = \mathfrak{e}^{2x}3^y \ln 3$$

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\{2 \cdot 3^y \mathfrak{e}^{2x}\}}{dx} = 2 \cdot 3^y \frac{d\{\mathfrak{e}^{2x}\}}{dx} = 4 \cdot 3^y \mathfrak{e}^{2x}$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dx} = \frac{d\{\mathfrak{e}^{2x}3^y \ln 3\}}{dx} = 3^y \ln 3 \frac{d\{\mathfrak{e}^{2x}\}}{dx} = 2 \cdot 3^y \mathfrak{e}^{2x} \ln 3$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{\mathfrak{e}^{2x}3^y \ln 3\}}{dy} = \mathfrak{e}^{2x} \ln 3 \frac{d\{3^y\}}{dy} = \mathfrak{e}^{2x}(\ln 3)^2 3^y$$

$$\frac{\partial^3 f(x,y)}{\partial x^3} = \frac{d\left\{\frac{d^2 f(x,y)}{dx^2}\right\}}{dx} = \frac{d\{4 \cdot 3^y \mathfrak{e}^{2x}\}}{dx} = 4 \cdot 3^y \frac{d\{\mathfrak{e}^{2x}\}}{dx} = 8 \cdot 3^y \mathfrak{e}^{2x}$$

$$\frac{\partial^3 f(x,y)}{\partial y \partial x^2} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y \partial x}\right\}}{dx} = \frac{d\{2 \cdot 3^y \mathfrak{e}^{2x} \ln 3\}}{dx} = 2 \cdot 3^y \ln 3 \frac{d\{\mathfrak{e}^{2x}\}}{dx} = 4 \cdot 3^y \ln 3 \mathfrak{e}^{2x}$$

$$\frac{\partial^3 f(x,y)}{\partial y^2 \partial x} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dx} = \frac{d\{\mathfrak{e}^{2x}(\ln 3)^2 3^y\}}{dx} = (\ln 3)^2 3^y \frac{d\{\mathfrak{e}^{2x}\}}{dx} = 2(\ln 3)^2 3^y \mathfrak{e}^{2x}$$

$$\frac{\partial^3 f(x,y)}{\partial y^3} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dy} = \frac{d\{\mathfrak{e}^{2x}(\ln 3)^2 3^y\}}{dy} = \mathfrak{e}^{2x}(\ln 3)^2 \frac{d\{3^y\}}{dy} = \mathfrak{e}^{2x}(\ln 3)^3 3^y$$

Second, we need to find out $f(1,0)$, $\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=1\\y=0}}$, $\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=1\\y=0}}$, $\left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=1\\y=0}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=1\\y=0}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=1\\y=0}}$, $\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=1\\y=0}}$, $\left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=1\\y=0}}$, $\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=1\\y=0}}$, and $\left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=1\\y=0}}$

$$f(1,0) = \mathfrak{e}^2 3^0 = \mathfrak{e}^2 \;\; ; \;\; \left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=1\\y=0}} = 2 \cdot 3^0 \mathfrak{e}^2 = 2\mathfrak{e}^2$$

$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=1\\y=0}} = \mathfrak{e}^2 3^0 \ln 3 = \mathfrak{e}^2 \ln 3 \;\; ; \;\; \left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=1\\y=0}} = 4 \cdot 3^0 \mathfrak{e}^2 = 4\mathfrak{e}^2$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=1\\y=0}} = 2 \cdot 3^0 \mathfrak{e}^2 \ln 3 = 2\mathfrak{e}^2 \ln 3 \;\; ; \;\; \left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=1\\y=0}} = \mathfrak{e}^2(\ln 3)^2 3^0 = \mathfrak{e}^2(\ln 3)^2$$

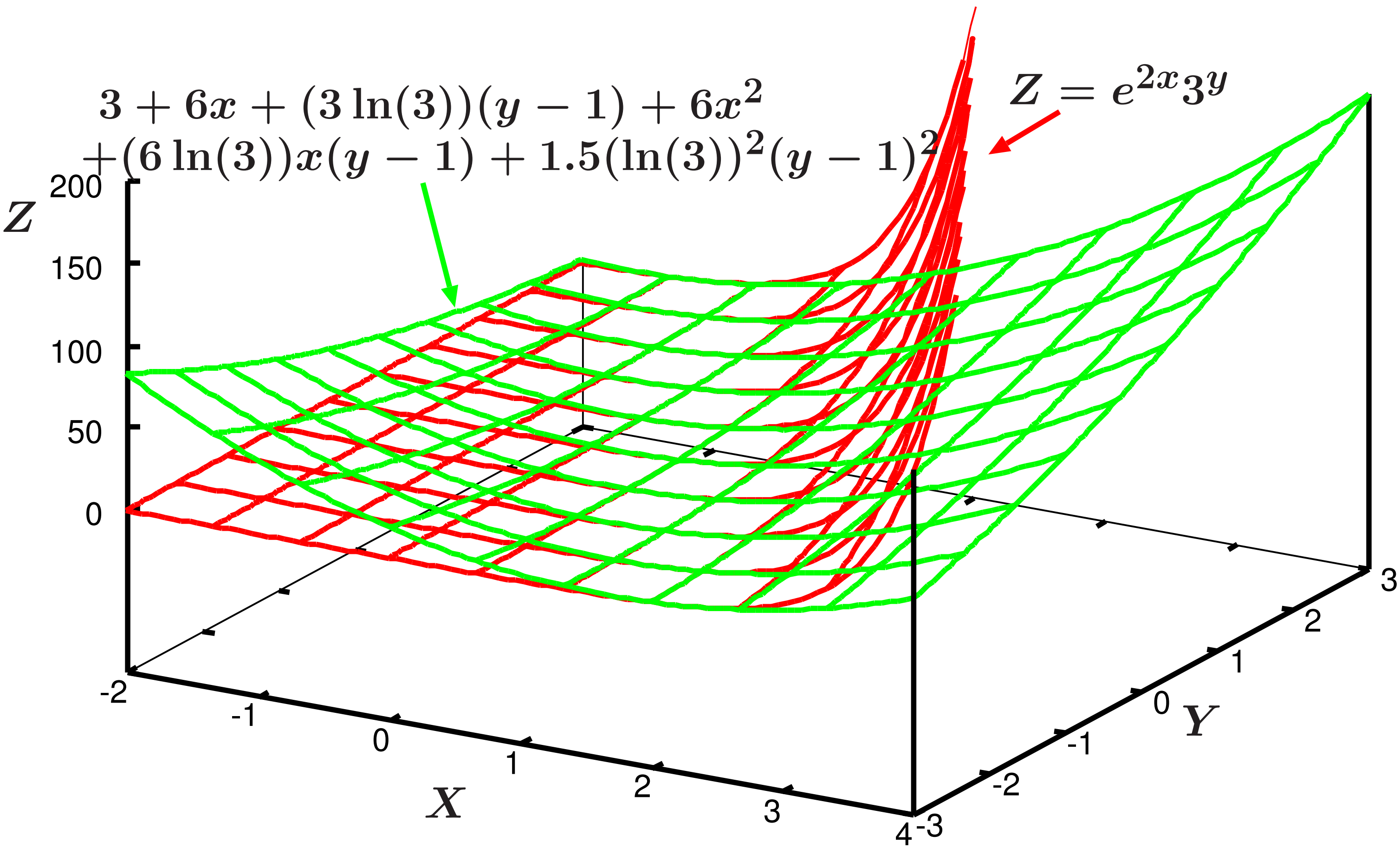
$$\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=1\\y=0}} = 8 \cdot 3^0 \mathfrak{e}^2 = 8\mathfrak{e}^2 \;\; ; \;\; \left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=1\\y=0}} = 4 \cdot 3^0 \mathfrak{e}^2 \ln 3 = 4\mathfrak{e}^2 \ln 3$$

$$\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=1\\y=0}} = 2(\ln 3)^2 3^0 \mathfrak{e}^2 = 2(\ln 3)^2 \mathfrak{e}^2 \;\; ; \;\; \left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=1\\y=0}} = \mathfrak{e}^2(\ln 3)^3 3^0 = \mathfrak{e}^2(\ln 3)^3$$

Finally by substituting these into ①, we get

$$\begin{aligned} f(x,y) = \mathfrak{e}^2 + (x-1)\cdot 2\mathfrak{e}^2 + y\cdot \mathfrak{e}^2 \ln 3 + \frac{1}{2!}\left[(x-1)^2\cdot 4\mathfrak{e}^2 + 2(x-1)y\cdot 2\mathfrak{e}^2 \ln 3 + y^2 \cdot \mathfrak{e}^2(\ln 3)^2\right] \\ + \frac{1}{3!}\left[(x-1)^3\cdot 8\mathfrak{e}^2 + 3(x-1)^2 y\cdot 4\ln 3\mathfrak{e}^2 + 3(x-1)y^2\cdot 2(\ln 3)^2\mathfrak{e}^2 + y^3\cdot \mathfrak{e}^2(\ln 3)^3\right] \\ = \mathfrak{e}^2 + \mathfrak{e}^2(2h(x-1) + y\ln 3) + 2\mathfrak{e}^2(x-1)^2 + 2(x-1)y\cdot \mathfrak{e}^2\ln 3 + \frac{\mathfrak{e}^2(\ln 3)^2 y^2}{2} \\ + \frac{4\mathfrak{e}^2(x-1)^3}{3} + (x-1)^2 y\cdot 2\mathfrak{e}^2\ln 3 + (\ln 3)^2\mathfrak{e}^2(x-1)y^2 + \frac{\mathfrak{e}^2(\ln 3)^3 y^3}{6} \end{aligned}$$

b) the quadratic approximation $f(x,y)$ of the Taylor series expansion around $(0,1)$




In order to work out $f(x,y)$ , we need to find out $a$ and $b$ in Equation (83). Since

$$a = 0 \;\; ; \;\; b = 1$$

we get $a = 0$ and $b = 1$. This means we are going to find out the Taylor series expansion around $(x,y) = (0,1)$ using Equation (83). We have already found $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2 f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y\partial x^2}$, $\frac{\partial^3 f(x,y)}{\partial y^2\partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$ previously. Since the quadratic approximation is required, we just need to find out $f(0,1)$, $\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=0\\y=1}}$, $\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=0\\y=1}}$, $\left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x\\y}}$

$$\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=0\\y=1}} \text{, and } \left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=0\\y=1}} .$$

$$f(0,1) = \mathfrak{e}^0 3^1 = 3 \;\; ; \;\; \left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=0\\y=1}} = 2\cdot 3^1 \mathfrak{e}^0 = 6$$

$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=0\\y=1}} = \mathfrak{e}^0 3^1 \ln 3 = 3\ln 3 \;\; ; \;\; \left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=0\\y=1}} = 4\cdot 3^1 \mathfrak{e}^0 = 12$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=0\\y=1}} = 2\cdot 3^1 \mathfrak{e}^0 \ln 3 = 6\ln 3 \;\; ; \;\; \left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=0\\y=1}} = \mathfrak{e}^0 (\ln 3)^2 3^1 = 3(\ln 3)^2$$

Now we find $f(x,y)$ using $(a,b) = (0,1)$ as follows:

$$f(x,y) = f(0,1) + x\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=0\\y=1}} + (y-1)\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=0\\y=1}}$$

$$+\frac{1}{2!}\left[x^2 \left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=0\\y=1}} + 2x(y-1)\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=0\\y=1}} + (y-1)^2 \left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=0\\y=1}}\right]$$

$$= 3 + 6x + (y-1)\cdot 3\ln 3 + \frac{1}{2!}\left[12x^2 + 2x(y-1)\cdot 6\ln 3 + 3(\ln 3)^2 (y-1)^2\right]$$

$$= 3 + 6x + (3\ln 3)(y-1) + 6x^2 + (6\ln 3)x(y-1) + 1.5(\ln 3)^2 (y-1)^2 \qquad ②$$

c) the value that the linear approximation gives for $f(0.1, 0.9)$ correct to 4 decimal places and the percentage error correct to 2 significant figures.
We have already worked out the approximation of $f(x,y)$ around $(0,1)$. In order to find out $f(0.1, 0.9)$ using the approximation of $f(x,y)$, we need to know the exact value of $x-a$ and $y-b$.

$$x - a = 0.1 - 0 = 0.1 \;\; ; \;\; y - b = 0.9 - 1 = -0.1$$

Therefore we obtain $(x-a, y-b) = (0.1, -0.1)$ which satisfy the condition of $|x-a| \ll 1$ and $|y-b| \ll 1$. By substituting $(x,y) = (0.1, 0.9)$ into the degree one of ②, we get

$$f(0.1,0.9) = 3 + 6x + (3\ln 3)(y-1) = 3 + 6\cdot 0.1 + (3\ln 3)\cdot(-0.1) = 3.6 - 0.3\ln 3 = 3.27042 \simeq 3.2704$$

correct to 4 decimal places. Since $f(0.1, 0.9) = 3.28298 \simeq 3.2830$ correct to 4 decimal places, the percentage error is

$$\frac{3.2704 - 3.2830}{3.2830} \times 100 = -0.383795\% \simeq -0.38\%$$

correct to 2 significant figures.

35) Find $f(2)$ when $f(x) = \ln\left|\frac{8}{x}\right| + x^3 + 12$ .

$$f(2) = \ln|\frac{8}{2}| + 2^3 + 12 = \ln|4| + 8 + 12 = \ln|4| + 20 = 2\ln|2| + 20$$

36) Simplify

$$\frac{(a^5 b^{20})^0 (4a^2 b^3)^2}{8a^3 b^3}$$

$$\frac{(a^5 b^{20})^0 (4a^2 b^3)^2}{8a^3 b^3} = \frac{1\cdot 4^2 a^{2\times 2} b^{3\times 2}}{8a^3 b^3} = \frac{16a^4 b^6}{8a^3 b^3} = 2a^{4-3} b^{6-3} = 2a^1 b^3 = 2ab^3$$

37) Find $f(\pi)$ when $f(x) = \cos 2x + \sin x + 4\tan 2x$

$$f(\pi) = \cos 2\pi + \sin \pi + 4\tan 2\pi = 1 + 0 + 4 \times 0 = 1$$

38) Simplify

$$(5 - x(5-2) + 3x)^2$$

$$(5 - x(5-2) + 3x)^2 = (5 - x(3) + 3x)^2 = (5 - 3x + 3x)^2 = 5^2 = 25$$

39) Solve the following equations

$$\frac{x+2}{6} - \frac{x-3}{3} = \frac{x}{6} + 2$$

$$\frac{x+2}{6} - \frac{x-3}{3} = \frac{x}{6} + 2 \;\; ; \quad \therefore x + 2 - 2(x-3) = x + 12 \;\; ; \quad \therefore x + 2 - 2x + 6 = x + 12$$
$$\therefore x - 2x - x = 12 - 2 - 6 \;\; ; \quad \therefore -2x = 4 \;\; ; \quad \therefore x = -2$$

40) Solve the following inequalities

$$-4 < 2x - 4 \leq 2$$

$$-4 < 2x - 4 \leq 2 \;\; ; \quad \therefore -4 + 4 < 2x \leq 2 + 4 \;\; ; \quad \therefore 0 < 2x \leq 6 \;\; ; \quad \therefore 0 < x \leq 3$$

41) Simplify

$$|-5-3|$$

$$|-5-3| = |-8| = 8$$

42) What is the summation from the first term to infinity of the sequence : $2, 1, 0.5...$
From the sequence we can tell that $r = 0.5$ $a = 2$. Therefore using Equation (81) we get.

$$S_\infty = \frac{2}{0.5} = 4$$

43) What is the summation from the first term to infinity of the sequence : $8, 2, \frac{1}{2}, \frac{1}{8}...$
From the sequence we can tell that $r = 0.25$ $a = 8$. Therefore using Equation (81) we get.

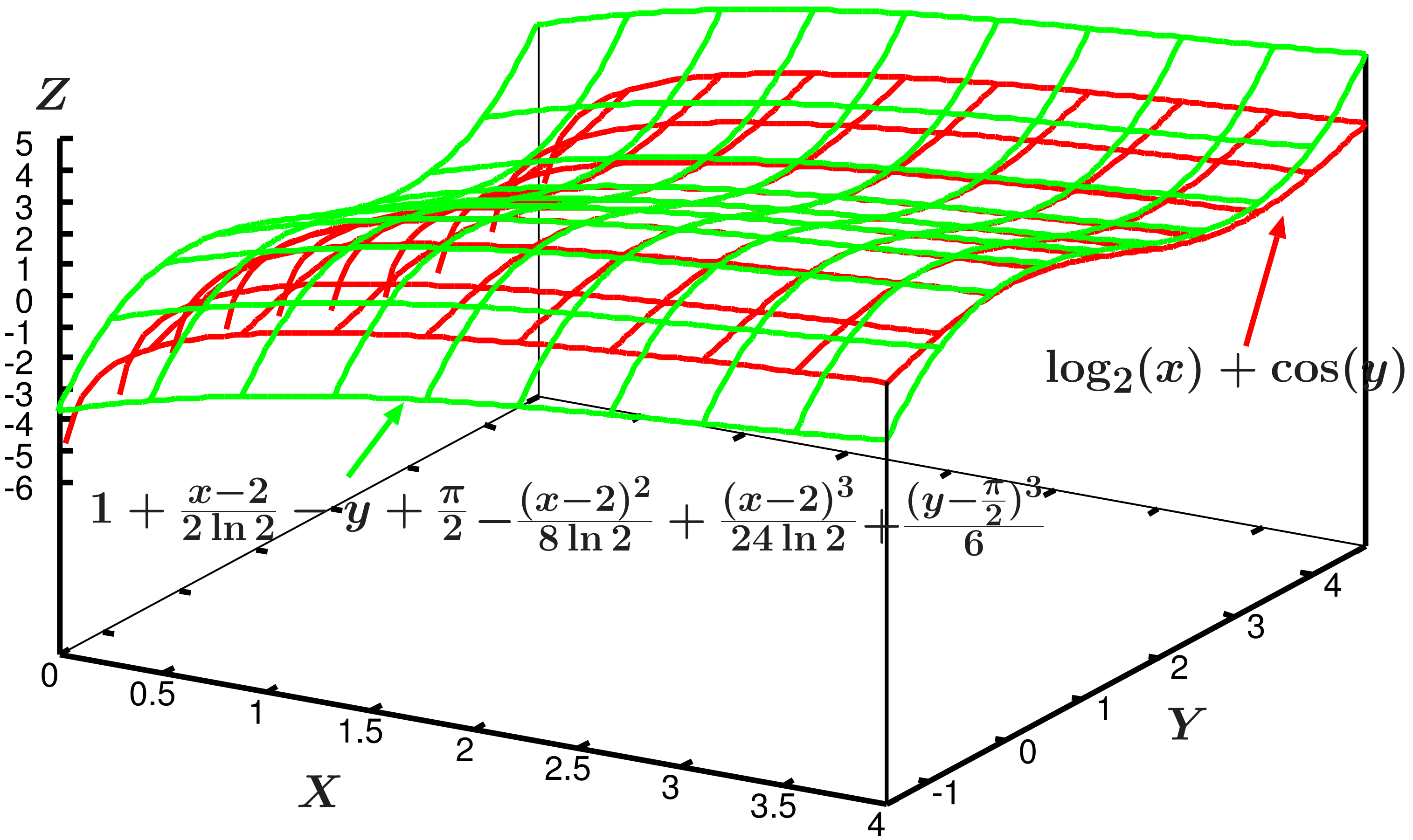
$$S_\infty = \frac{8}{1 - 0.25} = 32/3$$

44) For the function

$$f(x, y) = \log_2 x + \cos y$$

find

a) the Taylor polynomial of degree three about the point $(x, y) = (2, \frac{\pi}{2})$




We are going to use Equation (83) with $(a, b) = (2, \frac{\pi}{2})$ because we need to find out the approximation about the point $(x, y) = (2, \frac{\pi}{2})$.

| | | |
|---|---|---|
| $a$ | $\to$ | $2$ |
| $b$ | $\to$ | $\frac{\pi}{2}$ |

Thus Equation (83) can be re-written as

$$f(x,y) = f(2,\frac{\pi}{2}) + (x-2)\frac{d\{f(x,y)\}}{dx}\bigg|_{\substack{x=2\\y=\frac{\pi}{2}}} + (y-\frac{\pi}{2})\frac{d\{f(x,y)\}}{dy}\bigg|_{\substack{x=2\\y=\frac{\pi}{2}}}$$
$$+\frac{1}{2!}\left[(x-2)^2\frac{d^2f(x,y)}{dx^2}\bigg|_{\substack{x=2\\y=\frac{\pi}{2}}} + 2(x-2)(y-\frac{\pi}{2})\frac{\partial^2 f(x,y)}{\partial y\partial x}\bigg|_{\substack{x=2\\y=\frac{\pi}{2}}} + (y-\frac{\pi}{2})^2\frac{\partial^2 f(x,y)}{\partial y^2}\bigg|_{\substack{x=2\\y=\frac{\pi}{2}}}\right]$$
$$+\frac{1}{3!}\left[(x-2)^3\frac{\partial^3 f(x,y)}{\partial x^3}\bigg|_{\substack{x=2\\y=\frac{\pi}{2}}} + 3(x-2)^2(y-\frac{\pi}{2})\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\bigg|_{\substack{x=2\\y=\frac{\pi}{2}}}\right.$$
$$\left.+3(x-2)(y-\frac{\pi}{2})^2\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\bigg|_{\substack{x=2\\y=\frac{\pi}{2}}} + (y-\frac{\pi}{2})^3\frac{\partial^3 f(x,y)}{\partial y^3}\bigg|_{\substack{x=2\\y=\frac{\pi}{2}}}\right] \quad ①$$

where $|x-2| \ll 1$ and $|y-\frac{\pi}{2}| \ll 1$.

First, we need to find out $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y\partial x^2}$, $\frac{\partial^3 f(x,y)}{\partial y^2\partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$

$$f(x,y) = \log_2 x + \cos y$$

$$\therefore \frac{d\{f(x,y)\}}{dx} = \frac{d\{\log_2 x + \cos y\}}{dx} = \frac{d\{\log_2 x\}}{dx} + \frac{d\{\cos y\}}{dx} = \frac{d\left\{\frac{\ln x}{\ln 2}\right\}}{dx} + 0 = \frac{1}{\ln 2}\frac{d\{\ln x\}}{dx} = \frac{1}{x\ln 2}$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{\log_2 x + \cos y\}}{dy} = \frac{d\{\log_2 x\}}{dy} + \frac{d\{\cos y\}}{dy} = -\sin y$$

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{\dfrac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\left\{\frac{1}{x\ln 2}\right\}}{dx} = \frac{1}{\ln 2}\frac{d\{x^{-1}\}}{dx} = -\frac{1}{\ln 2}x^{-2}$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{d\left\{\dfrac{d\{f(x,y)\}}{dy}\right\}}{dx} = \frac{d\{-\sin y\}}{dx} = 0$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\dfrac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{-\sin y\}}{dy} = -\cos y$$

$$\frac{\partial^3 f(x,y)}{\partial x^3} = \frac{d\left\{\dfrac{d^2 f(x,y)}{dx^2}\right\}}{dx} = \frac{d\left\{-\frac{1}{\ln 2}x^{-2}\right\}}{dx} = -\frac{1}{\ln 2}\frac{d\{x^{-2}\}}{dx} = \frac{2}{\ln 2}x^{-3}$$

$$\frac{\partial^3 f(x,y)}{\partial y \partial x^2} = \frac{d\left\{\dfrac{\partial^2 f(x,y)}{\partial y \partial x}\right\}}{dx} = \frac{d\{0\}}{dx} = 0$$

$$\frac{\partial^3 f(x,y)}{\partial y^2 \partial x} = \frac{d\left\{\dfrac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dx} = \frac{d\{-\cos y\}}{dx} = 0$$

$$\frac{\partial^3 f(x,y)}{\partial y^3} = \frac{d\left\{\dfrac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dy} = \frac{d\{-\cos y\}}{dy} = \sin y$$

Second, we need to find out $f(2,\frac{\pi}{2})$, $\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=2\\y=\frac{\pi}{2}}}$, $\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=2\\y=\frac{\pi}{2}}}$, $\left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=2\\y=\frac{\pi}{2}}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=2\\y=\frac{\pi}{2}}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=2\\y=\frac{\pi}{2}}}$, $\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=2\\y=\frac{\pi}{2}}}$, $\left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=2\\y=\frac{\pi}{2}}}$, $\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=2\\y=\frac{\pi}{2}}}$, and $\left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=2\\y=\frac{\pi}{2}}}$.

$$f(2,\frac{\pi}{2}) = \log_2 2 + \cos\frac{\pi}{2} = 1 + 0 = 1 \;\;;\;\; \left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=2\\y=\frac{\pi}{2}}} = \frac{1}{2\ln 2}$$

$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=2\\y=\frac{\pi}{2}}} = -\sin\frac{\pi}{2} = -1 \;\;;\;\; \left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=2\\y=\frac{\pi}{2}}} = -\frac{1}{\ln 2}2^{-2}$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=2\\y=\frac{\pi}{2}}} = 0 \;\;;\;\; \left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=2\\y=\frac{\pi}{2}}} = -\cos\frac{\pi}{2} = 0$$

$$\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=2\\y=\frac{\pi}{2}}} = \frac{2}{\ln 2}2^{-3} \;\;;\;\; \left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=2\\y=\frac{\pi}{2}}} = 0$$

$$\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=2\\y=\frac{\pi}{2}}} = 0 \;\;;\;\; \left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=2\\y=\frac{\pi}{2}}} = \sin\frac{\pi}{2} = 1$$

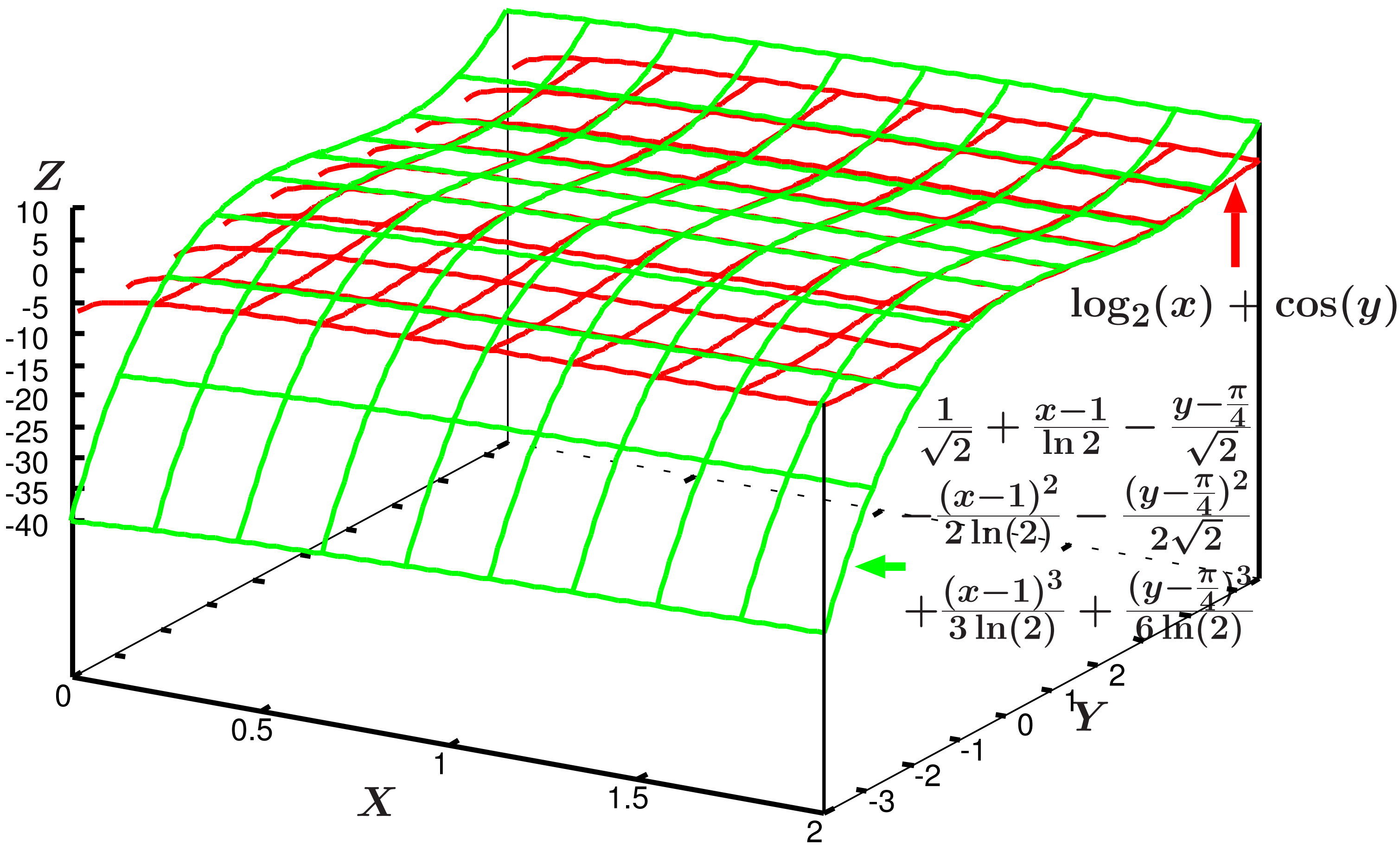
Finally by substituting these into ①, we get

$$f(x,y) = 1 + (x-2)\frac{1}{2\ln 2} + (y-\frac{\pi}{2})\cdot(-1)$$
$$+\frac{1}{2!}\left[(x-2)^2\cdot(-\frac{1}{\ln 2}2^{-2}) + 2(x-2)(y-\frac{\pi}{2})\cdot 0 + (y-\frac{\pi}{2})^2\cdot 0\right]$$

$$+\frac{1}{3!}\left[(x-2)^3\cdot\frac{2}{\ln 2}2^{-3}+3(x-2)^2(y-\frac{\pi}{2})\cdot 0+3(x-2)(y-\frac{\pi}{2})^2\cdot 0+(y-\frac{\pi}{2})^3\cdot 1\right]$$

$$=1+\frac{(x-2)}{2\ln 2}-(y-\frac{\pi}{2})-\frac{(x-2)^2}{8\ln 2}+\frac{(x-2)^3}{24\ln 2}+\frac{(y-\frac{\pi}{2})^3}{6}$$

b) the approximation $f(x,y)$ of the Taylor series expansion of degree three around $(1,\frac{\pi}{4})$




In order to work out $f(x,y)$ around $(1,\frac{\pi}{4})$ , we need to find out $a$ and $b$ in Equation (83). Since

$$a=1\ ;\ b=\frac{\pi}{4}$$

we get $a=1$ and $b=\frac{\pi}{4}$. This means we are going to find out the Taylor series expansion around $(x,y)=(1,\frac{\pi}{4})$ using Equation (83). We have already found $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y\partial x^2}$, $\frac{\partial^3 f(x,y)}{\partial y^2\partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$ previously. Thus we just need to find out $f(1,\frac{\pi}{4})$, $\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}$, $\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}$, $\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y\partial x}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}$, $\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}$, $\left.\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}$, $\left.\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}$, and $\left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}$.

$$f(1,\frac{\pi}{4})=\log_2 1+\cos\frac{\pi}{4}=0+\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\ ;\ \left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}=\frac{1}{\ln 2}$$

$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}=-\sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}}\ ;\ \left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}=-\frac{1}{\ln 2}$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=1\\y=\frac{\pi}{4}}} = 0 \ ; \ \left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=1\\y=\frac{\pi}{4}}} = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=1\\y=\frac{\pi}{4}}} = \frac{2}{\ln 2} \ ; \ \left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=1\\y=\frac{\pi}{4}}} = 0$$

$$\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=1\\y=\frac{\pi}{4}}} = 0 \ ; \ \left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=1\\y=\frac{\pi}{4}}} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Now we find $f(x,y)$ using $(a,b) = (1, \frac{\pi}{4})$ as follows:

$$f(x,y) = f(1,\frac{\pi}{4}) + (x-1)\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=1\\y=\frac{\pi}{4}}} + (y-\frac{\pi}{4})\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}$$

$$+\frac{1}{2!}\left[(x-1)^2\left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=1\\y=\frac{\pi}{4}}} + 2(x-1)(y-\frac{\pi}{4})\left.\frac{\partial^2 f(x,y)}{\partial y\partial x}\right|_{\substack{x=1\\y=\frac{\pi}{4}}} + (y-\frac{\pi}{4})^2\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}\right]$$

$$+\frac{1}{3!}\left[(x-1)^3\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=1\\y=\frac{\pi}{4}}} + 3(x-1)^2(y-\frac{\pi}{4})\left.\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}\right.$$

$$\left. +3(x-1)(y-\frac{\pi}{4})^2\left.\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\right|_{\substack{x=1\\y=\frac{\pi}{4}}} + (y-\frac{\pi}{4})^3\left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=1\\y=\frac{\pi}{4}}}\right]$$

$$= \frac{1}{\sqrt{2}} + (x-1)\cdot\frac{1}{\ln 2} + (y-\frac{\pi}{4})\cdot(-\frac{1}{\sqrt{2}})$$

$$+\frac{1}{2!}\left[(x-1)^2\cdot(-\frac{1}{\ln 2}) + 2(x-1)(y-\frac{\pi}{4})\cdot 0 + (y-\frac{\pi}{4})^2\cdot(-\frac{1}{\sqrt{2}})\right]$$

$$+\frac{1}{3!}\left[(x-1)^3\cdot\frac{2}{\ln 2} + 3(x-1)^2(y-\frac{\pi}{4})\cdot 0 + \cdot 0 + (y-\frac{\pi}{4})^3\cdot\frac{1}{\ln 2}\right]$$

$$= \frac{1}{\sqrt{2}} + \frac{x-1}{\ln 2} - \frac{y-\frac{\pi}{4}}{\sqrt{2}} - \frac{(x-1)^2}{2\ln 2} - \frac{(y-\frac{\pi}{4})^2}{2\sqrt{2}} + \frac{(x-1)^3}{3\ln 2} + \frac{(y-\frac{\pi}{4})^3}{6\ln 2} \qquad ②$$

c) the value that the cubic approximation gives for $f(0.9, \frac{\pi}{6})$ correct to 4 decimal places and the percentage error correct to 2 significant figures.
We have already worked out the approximation of $f(x,y)$ around $(1, \frac{\pi}{4})$. In order to find out $f(0.9, \frac{\pi}{6})$ using the approximation of $f(x,y)$, we need to know the exact value of $x-a$ and $y-b$.

$$x-a = 0.9-1 = -0.1 \ ; \ y-b = \frac{\pi}{6} - \frac{\pi}{4} = \frac{2\pi}{12} - \frac{3\pi}{12} = -\frac{\pi}{12}$$

Therefore we obtain $(x-a, y-b) = (-0.1, -\frac{\pi}{12})$ which satisfy the condition of $|x-a| \ll 1$ and $|y-b| \ll 1$. By substituting $(x,y) = (0.9, \frac{\pi}{6})$ into ②, we get

$$\frac{1}{\sqrt{2}} + \frac{x-1}{\ln 2} - \frac{y-\frac{\pi}{4}}{\sqrt{2}} - \frac{(x-1)^2}{2\ln 2} - \frac{(y-\frac{\pi}{4})^2}{2\sqrt{2}} + \frac{(x-1)^3}{3\ln 2} + \frac{(y-\frac{\pi}{4})^3}{6\ln 2}$$

$$= \frac{1}{\sqrt{2}} + \frac{(-0.1)}{\ln 2} - \frac{(-\frac{\pi}{12})}{\sqrt{2}} - \frac{(-0.1)^2}{2\ln 2} - \frac{(-\frac{\pi}{12})^2}{2\sqrt{2}} + \frac{(-0.1)^3}{3\ln 2} + \frac{(-\frac{\pi}{12})^3}{6\ln 2}$$

$$= \frac{1}{\sqrt{2}} - \frac{0.1}{\ln 2} + \frac{\pi}{12\sqrt{2}} - \frac{0.005}{\ln 2} - \frac{\pi^2}{288\sqrt{2}} - \frac{0.001}{3\ln 2} - \frac{\pi^3}{10368\ln 2}$$

$$= 0.711714 \simeq 0.7117$$

correct to 4 decimal places. Since $f(0.9, \frac{\pi}{6}) = 0.714023 \simeq 0.7140$, the percentage error is

$$\frac{0.7117 - 0.7140}{0.7140} \times 100 = -0.322129\% \simeq -0.32\%$$

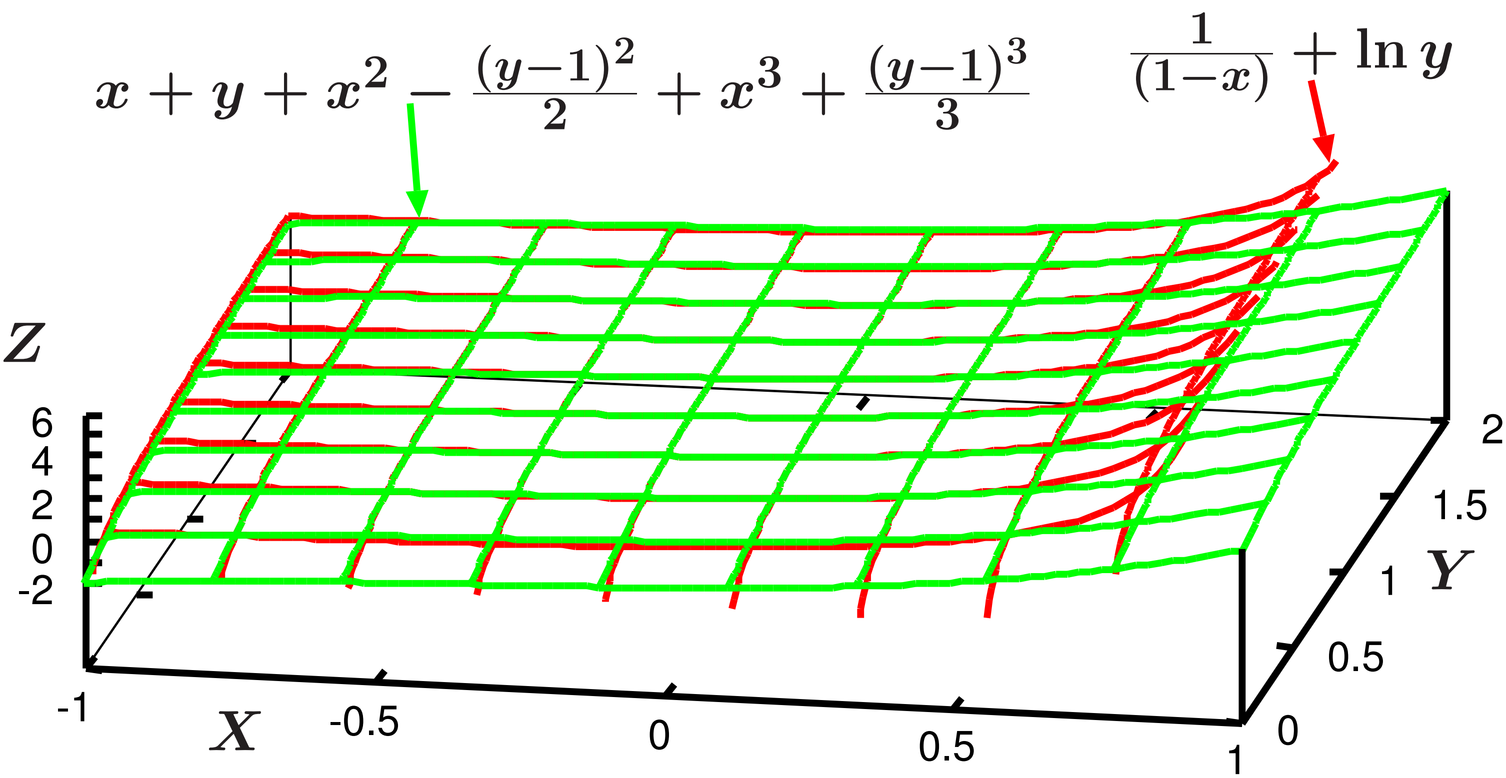
correct to 2 significant figures.

45) For the function

$$f(x,y) = \frac{1}{1-x} + \ln y$$

find

a) the Taylor polynomial of degree three about the point $(x,y) = (0,1)$




We are going to use Equation (83) with $(a,b) = (0,1)$ because we need to find out the approximation about the point $(x,y) = (0,1)$.

| | | |
|---|---|---|
| $a$ | $\rightarrow$ | 0 |
| $b$ | $\rightarrow$ | 1 |

Thus Equation (83) can be re-written as

$$f(x,y) = f(0,1) + x\frac{d\{f(x,y)\}}{dx}\bigg|_{\substack{x=0\\y=1}} + (y-1)\frac{d\{f(x,y)\}}{dy}\bigg|_{\substack{x=0\\y=1}}$$
$$+\frac{1}{2!}\left[x^2\frac{d^2f(x,y)}{dx^2}\bigg|_{\substack{x=0\\y=1}} + 2x(y-1)\frac{\partial^2 f(x,y)}{\partial y\partial x}\bigg|_{\substack{x=0\\y=1}} + (y-1)^2\frac{\partial^2 f(x,y)}{\partial y^2}\bigg|_{\substack{x=0\\y=1}}\right]$$
$$+\frac{1}{3!}\left[x^3\frac{\partial^3 f(x,y)}{\partial x^3}\bigg|_{\substack{x=0\\y=1}} + 3x^2(y-1)\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\bigg|_{\substack{x=0\\y=1}}\right.$$
$$\left.+3x(y-1)^2\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\bigg|_{\substack{x=0\\y=1}} + (y-1)^3\frac{\partial^3 f(x,y)}{\partial y^3}\bigg|_{\substack{x=0\\y=1}}\right] \quad ①$$

where $|x| \ll 1$ and $|y-1| \ll 1$.

First, we need to find out $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y\partial x^2}$,

$\dfrac{\partial^3 f(x,y)}{\partial y^2 \partial x}$, and $\dfrac{\partial^3 f(x,y)}{\partial y^3}$

$$f(x,y) = \frac{1}{1-x} + \ln y$$

$$\therefore \frac{d\{f(x,y)\}}{dx} = \frac{d\left\{\frac{1}{1-x} + \ln y\right\}}{dx} = \frac{d\{(1-x)^{-1}\}}{dx} + \frac{d\{\ln y\}}{dx} = \frac{d\{u\}}{dx}\frac{\partial\{u^{-1}\}}{\partial u} + 0(\because u \triangleq 1-x)$$
$$= \frac{d\{1-x\}}{dx}(-u^{-2}) = -(-u^{-2}) = (1-x)^{-2}$$

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\left\{\frac{1}{1-x} + \ln y\right\}}{dy} = \frac{d\left\{\frac{1}{1-x}\right\}}{dy} + \frac{d\{\ln y\}}{dy} = \frac{1}{y}$$

$$\frac{d^2 f(x,y)}{dx^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dx}\right\}}{dx} = \frac{d\{(1-x)^{-2}\}}{dx} = \frac{d\{u\}}{dx}\frac{\partial\{u^{-2}\}}{\partial u}$$
$$= \frac{d\{1-x\}}{dx}(-2u^{-3}) = -(-2u^{-3}) = 2(1-x)^{-3}$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dx} = \frac{d\left\{\frac{1}{y}\right\}}{dx} = 0$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} = \frac{d\left\{\frac{d\{f(x,y)\}}{dy}\right\}}{dy} = \frac{d\{y^{-1}\}}{dy} = -y^{-2}$$

$$\frac{\partial^3 f(x,y)}{\partial x^3} = \frac{d\left\{\frac{d^2 f(x,y)}{dx^2}\right\}}{dx} = \frac{d\{2(1-x)^{-3}\}}{dx} = \frac{d\{u\}}{dx}\frac{\partial\{2u^{-3}\}}{\partial u}$$
$$= \frac{d\{1-x\}}{dx}(-6u^{-4}) = -(-6u^{-4}) = 6(1-x)^{-4}$$

$$\frac{\partial^3 f(x,y)}{\partial y \partial x^2} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y \partial x}\right\}}{dx} = \frac{d\{0\}}{dx} = 0$$

$$\frac{\partial^3 f(x,y)}{\partial y^2 \partial x} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dx} = \frac{d\{-y^{-2}\}}{dx} = 0$$

$$\frac{\partial^3 f(x,y)}{\partial y^3} = \frac{d\left\{\frac{\partial^2 f(x,y)}{\partial y^2}\right\}}{dy} = \frac{d\{-y^{-2}\}}{dy} = 2y^{-3}$$

Second, we need to find out $f(0,1)$, $\left.\dfrac{d\{f(x,y)\}}{dx}\right|_{\substack{x=0\\y=1}}$, $\left.\dfrac{d\{f(x,y)\}}{dy}\right|_{\substack{x=0\\y=1}}$, $\left.\dfrac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=0\\y=1}}$, $\left.\dfrac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=0\\y=1}}$, $\left.\dfrac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=0\\y=1}}$, $\left.\dfrac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=0\\y=1}}$, $\left.\dfrac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=0\\y=1}}$, $\left.\dfrac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=0\\y=1}}$, and $\left.\dfrac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=0\\y=1}}$.

$$f(0,1) = \frac{1}{1-0} + \ln 1 = 1 + 0 = 1 \;\;;\;\; \left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=0\\y=1}} = (1-0)^{-2} = 1$$

$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=0\\y=1}} = \frac{1}{1} = 1 \;\;;\;\; \left.\frac{d^2 f(x,y)}{dx^2}\right|_{\substack{x=0\\y=1}} = 2(1-0)^{-3} = 2$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y \partial x}\right|_{\substack{x=0\\y=1}} = 0 \;\; ; \;\; \left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=0\\y=1}} = -1^{-2} = -1$$

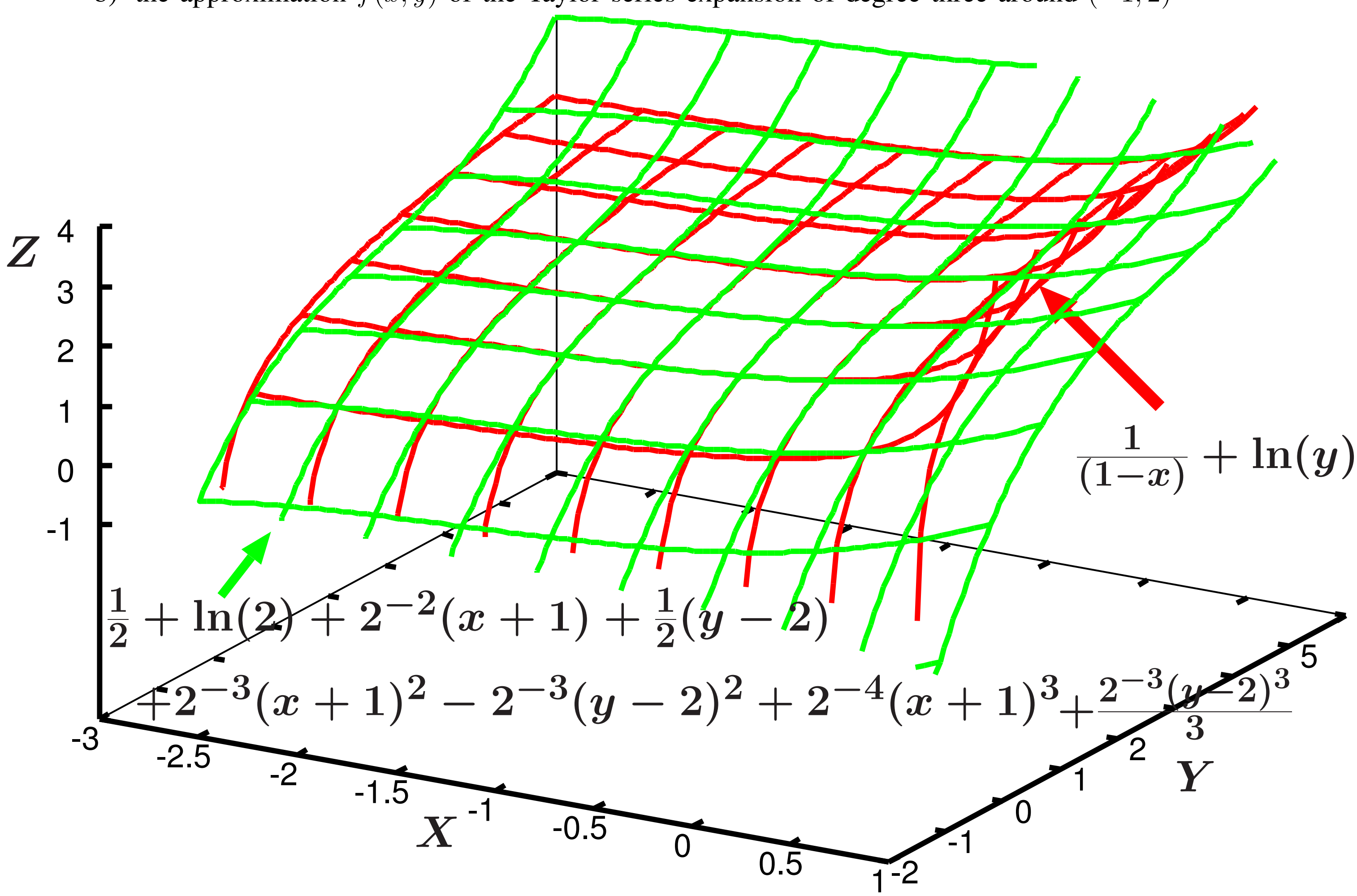
$$\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=0\\y=1}} = 6(1-0)^{-4} = 6 \;\; ; \;\; \left.\frac{\partial^3 f(x,y)}{\partial y \partial x^2}\right|_{\substack{x=0\\y=1}} = 0$$

$$\left.\frac{\partial^3 f(x,y)}{\partial y^2 \partial x}\right|_{\substack{x=0\\y=1}} = 0 \;\; ; \;\; \left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=0\\y=1}} = 2 \cdot 1^{-3} = 2$$

Finally by substituting these into ①, we get

$$\begin{aligned} f(x,y) &= 1 + x + (y-1) + \frac{1}{2!}\left[2x^2 + 2x(y-1)\cdot 0 - (y-1)^2\right] \\ &+ \frac{1}{3!}\left[6x^3 + 3x^2(y-1)\cdot 0 + 3x(y-1)^2 \cdot 0 + 2(y-1)^3\right] \\ &= 1 + x + (y-1) + x^2 - \frac{(y-1)^2}{2} + x^3 + \frac{(y-1)^3}{3} \\ &= x + y + x^2 - \frac{(y-1)^2}{2} + x^3 + \frac{(y-1)^3}{3} \end{aligned}$$

b) the approximation $f(x,y)$ of the Taylor series expansion of degree three around $(-1,2)$




In order to work out $f(x,y)$ around $(-1,2)$ , we need to find out $a$ and $b$ in Equation (83). Since

$$a = -1 \;\; ; \;\; b = 2$$

we get $a=-1$ and $b=2$. This means we are going to find out the Taylor series expansion around $(x,y)=(-1,2)$ using Equation (83). We have already found $\frac{d\{f(x,y)\}}{dx}$, $\frac{d\{f(x,y)\}}{dy}$, $\frac{d^2f(x,y)}{dx^2}$, $\frac{\partial^2 f(x,y)}{\partial y\partial x}$, $\frac{\partial^2 f(x,y)}{\partial y^2}$, $\frac{\partial^3 f(x,y)}{\partial x^3}$, $\frac{\partial^3 f(x,y)}{\partial y\partial x^2}$, $\frac{\partial^3 f(x,y)}{\partial y^2\partial x}$, and $\frac{\partial^3 f(x,y)}{\partial y^3}$ previously. Thus we just need to find out $f(-1,2)$, $\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=-1\\y=2}}$, $\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=-1\\y=2}}$, $\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=-1\\y=2}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y\partial x}\right|_{\substack{x=-1\\y=2}}$, $\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=-1\\y=2}}$, $\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=-1\\y=2}}$, $\left.\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\right|_{\substack{x=-1\\y=2}}$, $\left.\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\right|_{\substack{x=-1\\y=2}}$, and $\left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=-1\\y=2}}$.

$$f(-1,2)=\frac{1}{1-(-1)}+\ln 2=\frac{1}{2}+\ln 2\ ;\ \left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=-1\\y=2}}=(1-(-1))^{-2}=2^{-2}$$

$$\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=-1\\y=2}}=\frac{1}{2}\ ;\ \left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=-1\\y=2}}=2(1-(-1))^{-3}=2(2)^{-3}=2^{-2}$$

$$\left.\frac{\partial^2 f(x,y)}{\partial y\partial x}\right|_{\substack{x=-1\\y=2}}=0\ ;\ \left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=-1\\y=2}}=-2^{-2}$$

$$\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=-1\\y=2}}=6(1-(-1))^{-4}=3\cdot 2^{-3}\ ;\ \left.\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\right|_{\substack{x=-1\\y=2}}=0$$

$$\left.\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\right|_{\substack{x=-1\\y=2}}=0\ ;\ \left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=-1\\y=2}}=2\cdot 2^{-3}=2^{-2}$$

Now we find $f(x,y)$ using $(a,b)=(-1,2)$ as follows:

$$\begin{aligned}
f(x,y)&=f(-1,2)+(x+1)\left.\frac{d\{f(x,y)\}}{dx}\right|_{\substack{x=-1\\y=2}}+(y-2)\left.\frac{d\{f(x,y)\}}{dy}\right|_{\substack{x=-1\\y=2}}\\
&+\frac{1}{2!}\left[(x+1)^2\left.\frac{d^2f(x,y)}{dx^2}\right|_{\substack{x=-1\\y=2}}+2(x+1)(y-2)\left.\frac{\partial^2 f(x,y)}{\partial y\partial x}\right|_{\substack{x=-1\\y=2}}+(y-2)^2\left.\frac{\partial^2 f(x,y)}{\partial y^2}\right|_{\substack{x=-1\\y=2}}\right]\\
&+\frac{1}{3!}\left[(x+1)^3\left.\frac{\partial^3 f(x,y)}{\partial x^3}\right|_{\substack{x=-1\\y=2}}+3(x+1)^2(y-2)\left.\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\right|_{\substack{x=-1\\y=2}}\right.\\
&\left.+3(x+1)(y-2)^2\left.\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\right|_{\substack{x=-1\\y=2}}+(y-2)^3\left.\frac{\partial^3 f(x,y)}{\partial y^3}\right|_{\substack{x=-1\\y=2}}\right]\\
&=\frac{1}{2}+\ln 2+2^{-2}(x+1)+\frac{1}{2}(y-2)+\frac{1}{2!}\left[2^{-2}(x+1)^2+2(x+1)(y-2)\cdot 0-2^{-2}(y-2)^2\right]\\
&+\frac{1}{3!}\left[3\cdot 2^{-3}(x+1)^3+3(x+1)^2(y-2)\cdot 0+3(x+1)(y-2)^2\cdot 0+2^{-2}(y-2)^3\right]\\
&=\frac{1}{2}+\ln 2+2^{-2}(x+1)+\frac{1}{2}(y-2)+2^{-3}(x+1)^2-2^{-3}(y-2)^2+2^{-4}(x+1)^3+\frac{2^{-3}(y-2)^3}{3} \qquad ②
\end{aligned}$$

c) the value that the quadratic approximation gives for $f(-0.9,1.9)$ correct to 5 decimal places and the percentage error correct to 2 significant figures.
We have already worked out the approximation of $f(x,y)$ around $(-1,2)$. In order to find out $f(-0.9,1.9)$ using the approximation of $f(x,y)$, we need to know the exact value of $x-a$ and $y-b$.

$$x-a=-0.9+1=0.1\ ;\ y-b=1.9-2=-0.1$$

Therefore we obtain $(x-a, y-b) = (0.1, -0.1)$ which satisfy the condition of $|x-a| \ll 1$ and $|y-b| \ll 1$. By substituting $(x, y) = (-0.9, 1.9)$ into degree two of ②, we get

$$\begin{aligned}
&\frac{1}{2} + \ln 2 + 2^{-2}(x+1) + \frac{1}{2}(y-2) + 2^{-3}(x+1)^2 - 2^{-3}(y-2)^2 \\
&= \frac{1}{2} + \ln 2 + 2^{-2} \cdot (0.1) + \frac{1}{2} \cdot (-0.1) + 2^{-3} \cdot 0.1^2 - 2^{-3} \cdot (-0.1)^2 \\
&= \frac{1}{2} + \ln 2 + \frac{1}{40} - \frac{1}{20} + \frac{0.01}{8} - \frac{0.01}{8} \\
&= \frac{20}{40} + \ln 2 + \frac{1}{40} - \frac{2}{40} \\
&= \frac{19}{40} + \ln 2 \\
&= 1.16815
\end{aligned}$$

correct to 5 decimal places. Since $f(-0.9, 1.9) = 1.16817$, the percentage error is

$$\frac{1.16815 - 1.16817}{1.16817} \times 100 = -0.00171208\% \simeq -0.0017\%$$

correct to 2 significant figures.

**DAY4**

46) The current, $I$, is given by

$$I(V) = I_s \sinh(V)$$

where $V$ is the applied voltage and $I_s$ is a constant. If the operating voltage is given by $V_a = \pi$ (measured in Volts), find a second order Taylor approximation for $I(V)$ about this operating voltage.

A second order Taylor expansion for $I(V)$ about $V = \pi$ is

$$I(V) = I(\pi) + \underline{(V-\pi)} \left.\frac{dI}{dV}\right|_{V=\pi} + \frac{(V-\pi)^2}{\underline{2!}} \left.\frac{d^2I}{dV^2}\right|_{V=\pi}$$

We now need $\frac{dI}{dV}$ and $\frac{d^2I}{dV^2}$.

$$\begin{aligned} I(V) = I_s \sinh(V) &= I_s \frac{\mathfrak{e}^V - \mathfrak{e}^{-V}}{2} \\ \frac{dI}{dV} &= I_s \frac{\mathfrak{e}^V + \mathfrak{e}^{-V}}{2} \\ \frac{d^2I}{dV^2} &= I_s \frac{\mathfrak{e}^V - \mathfrak{e}^{-V}}{2} \end{aligned}$$

Therefore

$$\begin{aligned} I(V) &= I_s \sinh(\pi) + \underline{(V-\pi)}\, I_s \left.\frac{\mathfrak{e}^V + \mathfrak{e}^{-V}}{2}\right|_{V=\pi} + \frac{(V-\pi)^2}{\underline{2!}}\, I_s \left.\frac{\mathfrak{e}^V - \mathfrak{e}^{-V}}{2}\right|_{V=\pi} \\ &= I_s \sinh(\pi) + I_s \frac{\mathfrak{e}^\pi + \mathfrak{e}^{-\pi}}{2}(V-\pi) + I_s \frac{\mathfrak{e}^\pi - \mathfrak{e}^{-\pi}}{2} \frac{(V-\pi)^2}{2!} \\ &= I_s \sinh(\pi) + I_s \cosh(\pi)(V-\pi) + I_s \sinh(\pi) \frac{(V-\pi)^2}{2} \end{aligned}$$

47) The current, $I$, is given by

$$I(V,t) = \mathfrak{e}^{-V} \cos(\omega t)$$

where $V$ is the applied voltage and $t$ is time. Find the term in $t^2V^3$ in the Taylor series expansion around $t = 0, V = 0$.

The term that has $t^2V^3$ in it must be the term $\frac{1}{5!}\ {}_5C_2 \left.\frac{\partial^5 I}{\partial t^2 \partial V^3}\right|_{t=0,V=0} t^2V^3$ as the overall order is equal to 5. Therefore

$$\begin{aligned} I(V,t) &= \mathfrak{e}^{-V} \cos(\omega t) \\ \frac{\partial I}{\partial t} &= -\omega \mathfrak{e}^{-V} \sin(\omega t) \\ \frac{\partial^2 I}{\partial t^2} &= -\omega^2 \mathfrak{e}^{-V} \cos(\omega t) \\ \frac{\partial^3 I}{\partial V \partial t^2} = -(-1)\omega^2 \mathfrak{e}^{-V} \cos(\omega t) &= \omega^2 \mathfrak{e}^{-V} \cos(\omega t) \\ \frac{\partial^4 I}{\partial V^2 \partial t^2} &= -\omega^2 \mathfrak{e}^{-V} \cos(\omega t) \\ \frac{\partial^5 I}{\partial V^3 \partial t^2} &= \omega^2 \mathfrak{e}^{-V} \cos(\omega t) \end{aligned}$$

When we put this into $\frac{1}{5!}\ {}_5C_2 \left.\frac{\partial^5 I}{\partial t^2 \partial V^3}\right|_{t=0,V=0} t^2V^3$

$$\frac{1}{5!}\ {}_5C_2 \omega^2 \mathfrak{e}^{-V} \cos(\omega t)\Big|_{t=0,V=0} t^2V^3 = \frac{1}{5!}\ {}_5C_2 \omega^2 \mathfrak{e}^{-0} \cos(\omega 0) t^2V^3 = \frac{1}{5\cdot4\cdot3\cdot2\cdot1} \frac{5\cdot4}{2\cdot1} \omega^2 t^2 V^3 = \frac{1}{12}\omega^2 t^2 V^3$$

48) The current, $I$, is given by

$$I(V,R) = \frac{\mathfrak{e}^V}{R}$$

where $V$ is the applied voltage and $R$ is a variable.

a) Find the second order Taylor series for $I$ around $V = 0, R = 1$

The first derivatives of $I$ are

$$\frac{\partial I}{\partial V} = \frac{\mathfrak{e}^V}{R}$$
$$\frac{\partial I}{\partial R} = -\mathfrak{e}^V R^{-2}$$

The second derivatives of $I$ are

$$\frac{\partial^2 I}{\partial V^2} = \frac{\mathfrak{e}^V}{R}$$
$$\frac{\partial^2 I}{\partial R^2} = 2\mathfrak{e}^V R^{-3}$$
$$\frac{\partial^2 I}{\partial R \partial V} = -\mathfrak{e}^V R^{-2}$$

We evaluate these derivatives at $(V,R) = (0,1)$ as follows.

$$\left.\frac{\partial I}{\partial V}\right|_{(V,R)=(0,1)} = 1$$
$$\left.\frac{\partial I}{\partial R}\right|_{(V,R)=(0,1)} = -1$$
$$\left.\frac{\partial^2 I}{\partial V^2}\right|_{(V,R)=(0,1)} = 1$$
$$\left.\frac{\partial^2 I}{\partial R^2}\right|_{(V,R)=(0,1)} = 2$$
$$\left.\frac{\partial^2 I}{\partial R \partial V}\right|_{(V,R)=(0,1)} = -1$$

Therefore

$$I(V,R) = I(0,1) + (V-0)\left.\frac{\partial I}{\partial V}\right|_{(V,R)=(0,1)} + (R-1)\left.\frac{\partial I}{\partial R}\right|_{(V,R)=(0,1)}$$
$$+\frac{1}{2!}\left[(V-0)^2\left.\frac{\partial^2 I}{\partial V^2}\right|_{(V,R)=(0,1)} + 2(V-0)(R-1)\left.\frac{\partial^2 I}{\partial R \partial V}\right|_{(V,R)=(0,1)} + (R-1)^2\left.\frac{\partial^2 I}{\partial R^2}\right|_{(V,R)=(0,1)}\right]$$
$$= 1 + V - R + 1 + \frac{1}{2}\left[V^2 - 2V(R-1) - (R-1)^2\right] = 2 + V - R + \frac{1}{2}\left[V^2 - 2V(R-1) - (R-1)^2\right]$$

b) Using the series estimate $I(0.1, 0.9)$ and compare it with the exact value of $I(0.1, 0.9)$

We substitute $V = 0.1, R = 0.9$ into the second-order Taylor series we found in question 48a.

$$I(0.1, 0.9) == 2 + 0.1 - 0.9 + \frac{1}{2}\left[0.1^2 - 2 \cdot 0.1(0.9-1) - (0.9-1)^2\right] = 1.21$$

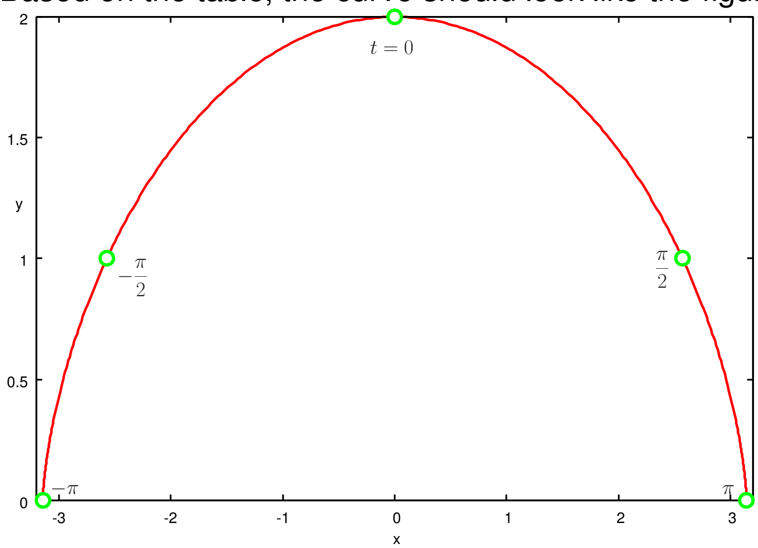
If we work it out manually using $I(V,R) = \frac{\mathfrak{e}^V}{R}$ it becomes $I(0.1, 0.9) = \frac{\mathfrak{e}^{0.1}}{0.9} = 1.22797$ Therefore the two results differ by 0.0179677.

1) **DAY1**
2) A curve is defined by $x(t) = t + \sin(t)$ and $y(t) = 1 + \cos(t)$.
Sketch this curve for $-\pi \le t \le \pi$. Find the length of this curve for $-\pi \le t \le \pi$.
**Hint:** you may need to use the trigonometric formulae:
$\cos^2(t) + \sin^2(t) = 1$ and $1 + \cos(t) = 2\cos^2(t/2)$ and $\cos(A - B) = \cos A \cos B + \sin A \sin B$

When $t = \frac{n\pi}{2}$ where $n = -2 \sim 2$, we obtain

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $t$ | $x$ | $y$ | $-\pi$ | $-\pi$ | 0 | $-\frac{\pi}{2}$ | $-\pi/2 - 1$ | 1 |
| 0 | 0 | 2 | $\frac{\pi}{2}$ | $\pi/2 + 1$ | 1 | $\pi$ | $\pi$ | 0 |

Based on the table, the curve should look like the figure.




The length of the curve, $L$ can be obtained by doing

$$L = \int_0^{2\pi} \sqrt{\left(\frac{d\{x\}}{dt}\right)^2 + \left(\frac{d\{y\}}{dt}\right)^2} dt$$

We now need $\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$.

$$x(t) = t + \sin(t); \therefore \frac{d\{x\}}{dt} = 1 + \cos(t)$$
$$y(t) = 1 + \cos(t); \therefore \frac{d\{y\}}{dt} = -\sin(t)$$

$$L = \int_{-\pi}^{\pi} \sqrt{(1 + \cos(t))^2 + (-\sin(t))^2} dt = \int_{-\pi}^{\pi} \sqrt{1 + \cos^2(t) + 2\cos(t) + \sin^2(t)} dt = \int_{-\pi}^{\pi} \sqrt{2 + 2\cos(t)} dt$$
$$= \int_{-\pi}^{\pi} \sqrt{2(1 + \cos(t))} dt = \int_{-\pi}^{\pi} \sqrt{2 \cdot 2\cos^2(t/2)} dt = \int_{-\pi}^{\pi} 2\cos(t/2) dt = 2\left[2\sin(t/2)\right]_{-\pi}^{\pi} = 4\left[1 - (-1)\right] = 8$$

3) A cone is generated by rotating the curve $y = \cosh(x) + 1$ about $x$-axis through $2\pi$ radians from $x = 0$ to $x = 1$. Calculate the suface area of the cone (excluding the two ends). Surface area is

$$\int_0^1 (2\pi y)\sqrt{dx^2 + dy^2}$$
$$= \int_0^1 (2\pi y)\sqrt{1 + \left(\frac{d\{y\}}{dx}\right)^2} dx$$

Since $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\dfrac{d\{\cosh x\}}{dx} = \dfrac{e^x - e^{-x}}{2}$. Therefore the surface area is

$$\int_0^1 (2\pi y)\sqrt{1 + \left(\frac{d\{y\}}{dx}\right)^2} dx = \int_0^1 (2\pi y)\sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2} dx = \int_0^1 (2\pi y)\sqrt{1 + \frac{e^{2x} + e^{-2x} - 2}{4}} dx$$

$$= \int_0^1 (2\pi y)\sqrt{\frac{4 + e^{2x} + e^{-2x} - 2}{4}} dx = \int_0^1 (2\pi y)\sqrt{\frac{e^{2x} + e^{-2x} + 2}{4}} dx = \int_0^1 (2\pi y)\sqrt{\frac{e^{2x} + e^{-2x} + 2}{4}} dx$$

$$= \int_0^1 (2\pi y)\sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2} dx = \int_0^1 (2\pi y) \times \frac{e^x + e^{-x}}{2} dx = \int_0^1 (2\pi(\frac{e^x + e^{-x}}{2} + 1)) \cdot \frac{e^x + e^{-x}}{2} dx$$

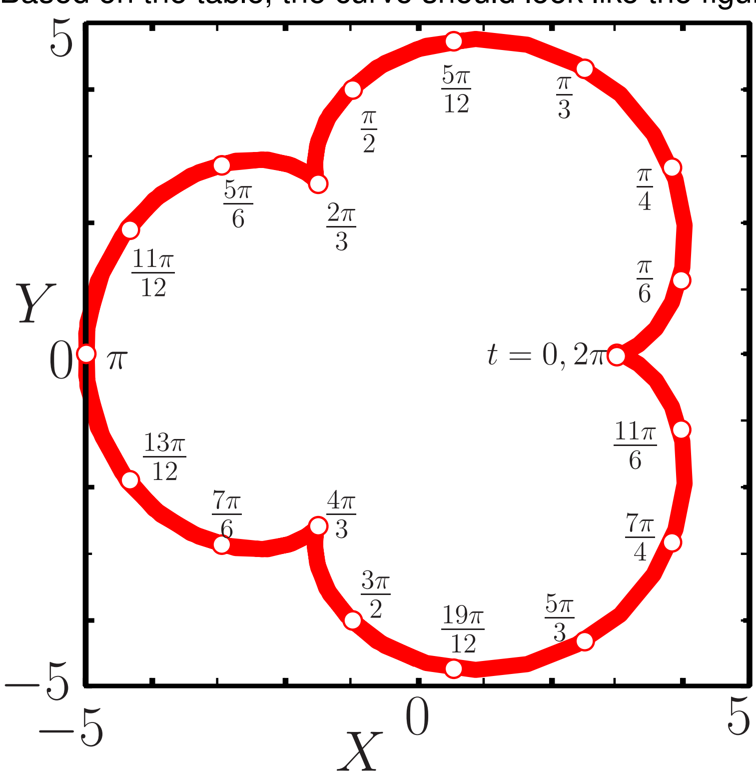
$$= \frac{\pi}{2}\int_0^1 (e^x + e^{-x} + 2) \cdot (e^x + e^{-x}) dx = \frac{\pi}{2}\int_0^1 e^{2x} + e^{-2x} + 2 + 2e^x + 2e^{-x} dx$$

$$= \frac{\pi}{2}\left[\frac{e^{2x}}{2} - \frac{e^{-2x}}{2} + 2x + 2e^x - 2e^{-x}\right]_0^1 = \frac{\pi}{2}\left[\frac{e^2}{2} - \frac{e^{-2}}{2} + 2 + 2e - 2e^{-1}\right]$$

4) A curve is defined by $x(t) = 4\cos(t) - \cos(4t)$ and $y(t) = 4\sin(t) - \sin(4t)$.
Sketch this curve for $0 \le t \le 2\pi$. Find the length of this curve for $0 \le t \le 2\pi$.
**Hint:** you may need to use the trigonometric formulae:
$\cos^2(t) + \sin^2(t) = 1$ and $1 - \cos(t) = 2\sin^2(t/2)$ and $\cos(A - B) = \cos A \cos B + \sin A \sin B$
When $t = \frac{n\pi}{12}$ where $n = 0 \sim 24$, we obtain

| $t$ | $x$ | $y$ | $\frac{0\pi}{12}$ | 3 | 0 | $\frac{2\pi}{12}$ | 3.9641 | 1.13397 |
|---|---|---|---|---|---|---|---|---|
| $\frac{3\pi}{12}$ | 3.82843 | 2.82842 | $\frac{4\pi}{12}$ | 2.5 | 4.33012 | $\frac{5\pi}{12}$ | 0.535284 | 4.72973 |
| $\frac{6\pi}{12}$ | -1 | 4 | $\frac{7\pi}{12}$ | -1.53528 | 2.99768 | $\frac{9\pi}{12}$ | -1.82842 | 2.82842 |
| $\frac{10\pi}{12}$ | -2.96409 | 2.86603 | $\frac{11\pi}{12}$ | -4.36369 | 1.90132 | $\frac{12\pi}{12}$ | -5 | 0 |
| $\frac{13\pi}{12}$ | -4.36372 | -1.90128 | $\frac{14\pi}{12}$ | -2.96412 | -2.86602 | $\frac{16\pi}{12}$ | -1.5 | -2.59808 |
| $\frac{18\pi}{12}$ | -1 | -4 | $\frac{19\pi}{12}$ | 0.535245 | -4.72972 | $\frac{20\pi}{12}$ | 2.49997 | -4.33014 |
| $\frac{21\pi}{12}$ | 3.82841 | -2.82846 | $\frac{22\pi}{12}$ | 3.96411 | -1.134 | | | |

Based on the table, the curve should look like the figure.



The length of the curve,$L$ can be obtained by doing

$$L = \int_0^{2\pi} \sqrt{\left(\frac{d\{x\}}{dt}\right)^2 + \left(\frac{d\{y\}}{dt}\right)^2} dt$$

We now need $\frac{d\{x\}}{dt}$ and $\frac{d\{y\}}{dt}$.

$$x(t) = 4\cos(t) - \cos(4t); \therefore \frac{d\{x\}}{dt} = -4\sin(t) + 4\sin(4t)$$
$$y(t) = 4\sin(t) - \sin(4t); \therefore \frac{d\{y\}}{dt} = 4\cos(t) - 4\cos(4t)$$

$$L = \int_0^{2\pi} \sqrt{(-4\sin(t) + 4\sin(4t))^2 + (4\cos(t) - 4\cos(4t))^2} dt$$
$$= \int_0^{2\pi} \sqrt{16(-\sin(t) + \sin(4t))^2 + 16(\cos(t) - \cos(4t))^2} dt$$
$$= 4\int_0^{2\pi} \sqrt{(-\sin(t) + \sin(4t))^2 + (\cos(t) - \cos(4t))^2} dt$$
$$= 4\int_0^{2\pi} \sqrt{\sin^2(t) + \sin^2(4t) - 2\sin(t)\sin(4t) + \cos^2(t) + \cos^2(4t) - 2\cos(t)\cos(4t)} dt$$
$$= 4\int_0^{2\pi} \sqrt{2 - 2\sin(t)\sin(4t) - 2\cos(t)\cos(4t)} dt$$
$$= 4\int_0^{2\pi} \sqrt{2 - 2(\sin(t)\sin(4t) + \cos(t)\cos(4t))} dt = 4\int_0^{2\pi} \sqrt{2 - 2\cos(3t)} dt$$
$$= 4\int_0^{2\pi} \sqrt{4\sin^2(3t/2)} dt = 4\int_0^{2\pi} 2\sin(3t/2) dt = 8\left[-\frac{2}{3}\cos(3t/2)\right]_0^{2\pi} = 8\left[-\frac{2}{3}(-1-1)\right] = \frac{32}{3}$$

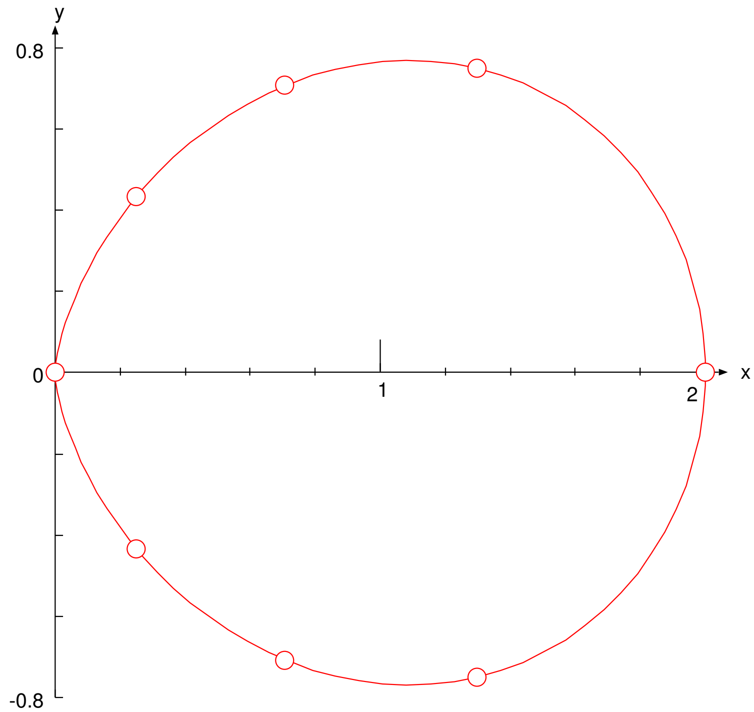
5) Sketch the curve described in Polar coordinates$(r, \theta)$ by

$$r = \cos(2\theta) + 1$$

for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and find the area that it encloses
First of all we scan $\theta$ and produce a table of

| $\theta$ | $r = \cos(2\theta) + 1$ | $x = r\cos\theta$ | $y = r\sin\theta$ |
|---|---|---|---|
| $-\frac{\pi}{2}$ | 0 | 0 | 0 |
| $-\frac{\pi}{3}$ | 0.5 | 0.25 | -0.43 |
| $-\frac{\pi}{4}$ | 1 | 0.7 | -0.7 |
| $-\frac{\pi}{6}$ | 1.5 | 1.3 | -0.75 |
| 0 | 2 | 2 | 0 |
| $\frac{\pi}{6}$ | 1.5 | 1.3 | 0.75 |
| $\frac{\pi}{4}$ | 1 | 0.7 | 0.7 |
| $\frac{\pi}{3}$ | 0.5 | 0.25 | 0.43 |
| $\frac{\pi}{2}$ | 0 | 0 | 0 |

and then we can produce the figure




Finally we calculate $A$ as

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^2}{2} d\theta = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos(2\theta))^2 d\theta = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(2\theta) + 1 + 2\cos(2\theta) d\theta$$

$$= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos(4\theta)}{2} + 1 + 2\cos(2\theta) d\theta = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(4\theta)}{2} + 1.5 + 2\cos(2\theta) d\theta$$

$$= \frac{1}{2} \left[ \frac{\sin(4\theta)}{8} + 1.5\theta + \sin(2\theta) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{3\pi}{4}$$

6) Sketch the curve described in Polar coordinates$(r, \theta)$ by

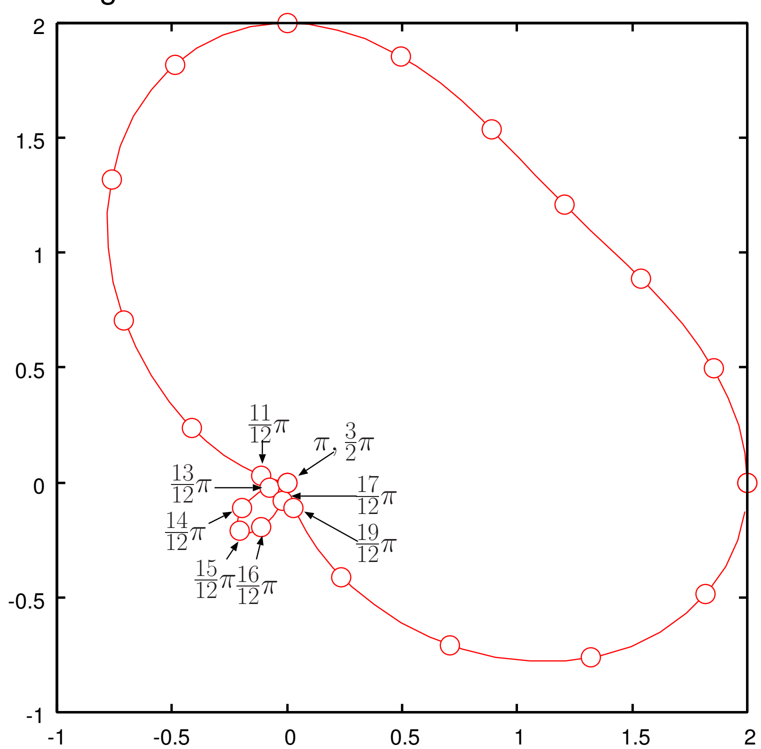
$$r = \cos^3(\theta) + \sin^3(\theta) + 1$$

for $0 \le \theta \le 2\pi$ and find the area that it encloses for $\pi \le \theta \le 1.5\pi$
Note: $4\cos^3(\theta) = 3\cos(\theta) + \cos(3\theta)$
$4\sin^3(\theta) = 3\sin(\theta) - \sin(3\theta)$
and $16\sin^6(\theta) = 5 - 7.5\cos(2\theta) + 3\cos(4\theta) - 0.5\cos(6\theta)$
$16\cos^6(\theta) = 5 + 7.5\cos(2\theta) + 3\cos(4\theta) + 0.5\cos(6\theta)$
and $16\sin^6(\theta) = 5 - 7.5\cos(2\theta) + 3\cos(4\theta) - 0.5\cos(6\theta)$
and $32\sin^3(\theta)\cos^3(\theta) = 3\sin(2\theta) - \sin(6\theta)$
and $16\left(\cos^3(\theta) + \sin^3(\theta)\right)^2 = 10 + 3\sin(2\theta) + 6\cos(4\theta) - \sin(6\theta)$
and $16\left(\cos^3(\theta) + \sin^3(\theta) + 1\right)^2 = 26 + 3\sin(2\theta) + 6\cos(4\theta) - \sin(6\theta) + 24\sin(\theta) - 8\sin(3\theta) + 24\cos(\theta) + 8\cos(3\theta)$

First of all we scan $\theta$ and produce a table of

| $\theta$ | $r = \cos^3(\theta) + \sin^3(\theta) + 1$ | $x = r\cos\theta$ | $y = r\sin\theta$ |
|---|---|---|---|
| $0/12\pi$ | 2 | 2 | 0 |
| $1/12\pi$ | 1.91856 | 1.85319 | 0.496559 |
| $2/12\pi$ | 1.77452 | 1.53678 | 0.887259 |
| $3/12\pi$ | 1.70711 | 1.20711 | 1.20711 |
| $4/12\pi$ | 1.77452 | 0.887261 | 1.53678 |
| $5/12\pi$ | 1.91856 | 0.496561 | 1.85318 |
| $6/12\pi$ | 2 | 0 | 2 |
| $7/12\pi$ | 1.88388 | -0.487582 | 1.81969 |
| $8/12\pi$ | 1.52452 | -0.762259 | 1.32028 |
| $9/12\pi$ | 1 | -0.707108 | 0.707111 |
| $10/12\pi$ | 0.475485 | -0.411781 | 0.237743 |
| $11/12\pi$ | 0.116119 | -0.112162 | 0.030054 |
| $12/12\pi$ | 0 | 0 | 0 |
| $13/12\pi$ | 0.0814398 | -0.0786649 | -0.021078 |
| $14/12\pi$ | 0.225479 | -0.195271 | -0.112739 |
| $15/12\pi$ | 0.292893 | -0.207107 | -0.207106 |
| $16/12\pi$ | 0.225483 | -0.112742 | -0.195273 |
| $17/12\pi$ | 0.0814433 | -0.0210794 | -0.0786681 |
| $18/12\pi$ | 0 | 0 | 0 |
| $19/12\pi$ | 0.116113 | 0.0300517 | -0.112156 |
| $20/12\pi$ | 0.475473 | 0.237735 | -0.411773 |
| $21/12\pi$ | 0.99999 | 0.707097 | -0.707103 |
| $22/12\pi$ | 1.52451 | 1.32026 | -0.762262 |
| $23/12\pi$ | 1.88388 | 1.81968 | -0.487593 |
| $24/12\pi$ | 2 | 2 | 0 |

and then we can produce the figure



Finally we calculate $A$ as

$$A = \int_{\pi}^{1.5\pi} \frac{r^2}{2} d\theta = \frac{1}{2} \int_{\pi}^{1.5\pi} \left(\cos^3(\theta) + \sin^3(\theta) + 1\right)^2 d\theta$$

$$= \frac{1}{2 \cdot 16} \int_{\pi}^{1.5\pi} 26 + 3\sin(2\theta) + 6\cos(4\theta) - \sin(6\theta) + 24\sin(\theta) - 8\sin(3\theta) + 24\cos(\theta) + 8\cos(3\theta) d\theta$$

$$= \frac{1}{32} \left[26\theta - \frac{3}{2}\cos(2\theta) + \frac{6}{4}\sin(4\theta) + \frac{1}{6}\cos(6\theta) - 24\cos(\theta) + \frac{8}{3}\cos(3\theta) + 24\sin(\theta) + \frac{8}{3}\sin(3\theta)\right]_{\pi}^{1.5\pi} = 0.0262813$$

7) Find the length of

$$y = 2x^{\frac{3}{2}}$$

for $0 \leq x \leq 1$
After we obtain

$$\frac{d\{y\}}{dx} = 3x^{0.5}$$

we calculate $L$ as

$$L = \int_0^1 \sqrt{1 + \left(\frac{d\{y\}}{dx}\right)^2} dx = \int_0^1 \sqrt{1 + (3x^{0.5})^2} dx = \int_0^1 \sqrt{1 + 9x} dx$$

When we introduce $t = 1 + 9x$

| | | | |
|---|---|---|---|
| $x$ | 0 | $\rightarrow$ | 1 |
| $t$ | 1 | $\rightarrow$ | 10 |

and $dt = 9dx$. Thus

$$L = \int_0^1 \sqrt{1 + 9x} dx = \int_1^{10} \sqrt{t} \frac{dt}{9} = \frac{1}{9} \left[\frac{t^2}{2}\right]_1^{10} = \frac{1}{9} \left[50 - \frac{1}{2}\right]$$

8) Find the length of

$$y = 2\sqrt{x}$$

for $0.5 \leq x \leq 1$
After we obtain

$$\frac{d\{y\}}{dx} = x^{-0.5}$$

we calculate $L$ as

$$L = \int_{0.5}^1 \sqrt{1 + \left(\frac{d\{y\}}{dx}\right)^2} dx = \int_{0.5}^1 \sqrt{1 + (x^{-0.5})^2} dx = \int_{0.5}^1 \sqrt{1 + x^{-1}} dx$$

When we introduce $t = \sqrt{1 + x^{-1}}$

| | | | |
|---|---|---|---|
| $x$ | 0.5 | $\rightarrow$ | 1 |
| $t$ | $\sqrt{3}$ | $\rightarrow$ | $\sqrt{2}$ |

and

$$t = \sqrt{1 + x^{-1}}; \therefore t^2 = 1 + x^{-1}; \therefore 2tdt = -x^{-2}dx$$
$$x^{-1} = t^2 - 1; \therefore x^{-2} = (t^2 - 1)^2; \therefore 2tdt = -(t^2 - 1)^2 dx = -(t + 1)^2(t - 1)^2 dx$$

Thus

$$L = \int_0^1 \sqrt{1 + x^{-1}} dx = \int_{\sqrt{3}}^{\sqrt{2}} t \frac{-2tdt}{(t + 1)^2(t - 1)^2} = \int_{\sqrt{3}}^{\sqrt{2}} \frac{-2t^2}{(t + 1)^2(t - 1)^2} dt$$

Now we assume

$$\frac{-2t^2}{(t+1)^2(t-1)^2} = \frac{A}{t+1} + \frac{B}{t-1} + \frac{C}{(t+1)^2} + \frac{D}{(t-1)^2} \qquad ①$$

where $A$, $B$, $C$, and $D$ are real constant numbers.
When $t = 1$ ① gives us $-2 = 4D \longrightarrow D = -0.5$.
When $t = -1$ ① gives us $-2 = 4C \longrightarrow C = -0.5$.
When $t = 0$ and $C = D = -0.5$ ① gives us $A - B = 1$.
When $t = 2$ and $C = D = -0.5$ ① gives us $A - 3B = -1$.
From these two equations we obtain $(A, B) = (0.5, -0.5)$ Finally

$$L = \int_{\sqrt{3}}^{\sqrt{2}} \frac{-2t^2}{(t+1)^2(t-1)^2} dt = \int_{\sqrt{3}}^{\sqrt{2}} \frac{A}{t+1} + \frac{B}{t-1} + \frac{C}{(t+1)^2} + \frac{D}{(t-1)^2} dt$$

$$= 0.5 \int_{\sqrt{3}}^{\sqrt{2}} \frac{1}{t+1} - \frac{1}{t-1} - \frac{1}{(t+1)^2} - \frac{1}{(t-1)^2} dt = 0.5 \left[ \ln|t+1| - \ln|t-1| + \frac{1}{t+1} + \frac{1}{t-1} \right]_{\sqrt{3}}^{\sqrt{2}}$$

$$= 0.5 \left[ \ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| - \ln\left|\frac{\sqrt{3}+1}{\sqrt{3}-1}\right| + \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{3}+1} - \frac{1}{\sqrt{3}-1} \right]$$

9) A solid object is generated by rotating the curve

$$y = x + 1$$

about $x-$ axis through $2\pi$ radians from $x = 0$ to $x = 1$. Calculate the surface area of the object exclusing the two ends
After we obtain

$$\frac{d\{y\}}{dx} = 1$$

we calculate $S$ as

$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{d\{y\}}{dx}\right)^2} dx = \int_0^1 2\pi(x+1)\sqrt{1+1^2} dx = 2\sqrt{2}\pi \int_0^1 (x+1) dx = 2\sqrt{2}\pi \left[\frac{x^2}{2} + x\right]_0^1 = 3\sqrt{2}\pi$$

10) A solid object is generated by rotating the curve

$$x = \sqrt{7}t^2 - 1$$
$$y = 2t^2 + 1$$

about $x-$ axis through $2\pi$ radians from $t = 0$ to $t = 1$. Calculate the surface area of the object exclusing the two ends
After we obtain

$$\frac{d\{x\}}{dt} = 2\sqrt{7}t$$
$$\frac{d\{y\}}{dt} = 4t$$

we calculate $S$ as

$$S = \int_0^1 2\pi y(t) \sqrt{\left(\frac{d\{x\}}{dt}\right)^2 + \left(\frac{d\{y\}}{dt}\right)^2} dt$$
$$= \int_0^1 2\pi(2t^2+1)\sqrt{\left(2\sqrt{7}t\right)^2 + (4t)^2} dt$$
$$= \int_0^1 2\pi(2t^2+1)\sqrt{28t^2+16t^2} dt$$
$$= 2\sqrt{44}\pi \int_0^1 (2t^2+1)t dt$$
$$= 2\sqrt{44}\pi \int_0^1 (2t^3+t) dt$$
$$= 2\sqrt{44}\pi \left[\frac{2t^4}{4} + \frac{t^2}{2}\right]_0^1$$
$$= 4\sqrt{11}\pi$$

11) A solid object is generated by rotating the curve

$$y = x + 1$$

about $y-$ axis through $2\pi$ radians from $y = 1$ to $y = 2$. Calculate the surface area of the object exclusing the two ends

After we obtain

$$\frac{d\{x\}}{dy} = 1$$

we calculate $S$ as

$$S = \int_1^2 2\pi x \sqrt{1 + \left(\frac{d\{y\}}{dx}\right)^2} dy = \int_1^2 2\pi(y-1)\sqrt{1+(1)^2} dy = 2\sqrt{2}\pi \int_1^2 (y-1) dy$$
$$= 2\sqrt{2}\pi \left[\frac{y^2}{2} - y\right]_1^2 = 2\sqrt{2}\pi \left[2 - 2 - \frac{1}{2} + 1\right] = \sqrt{2}\pi$$

12) A solid object is generated by rotating the curve

$$y = -x + 1$$

about $y-$ axis through $2\pi$ radians from $y = 0$ to $y = 2$. Calculate the surface area of the object exclusing the two ends

Please note that $x = 1 - y > 0$ for $0 < y < 1$ and $x = 1 - y < 0$ for $1 < y < 2$ . After we obtain

$$\frac{d\{x\}}{dy} = -1$$

we calculate $S$ as

$$S = \int_0^2 2\pi|x|\sqrt{1+\left(\frac{d\left\{y\right\}}{dx}\right)^2}dy$$
$$= \int_0^1 2\pi(x)\sqrt{1+(1)^2}dy + \int_1^2 2\pi(-x)\sqrt{1+(1)^2}dy$$
$$= \int_0^1 2\pi(1-y)\sqrt{1+(1)^2}dy + \int_1^2 2\pi(-1+y)\sqrt{1+(1)^2}dy$$
$$= 2\sqrt{2}\pi\left\{\int_0^1 (1-y)dy + \int_1^2 (-1+y)dy\right\} = 2\sqrt{2}\pi\left\{\left[y-\frac{y^2}{2}\right]_0^1 + \left[-y+\frac{y^2}{2}\right]_1^2\right\}$$
$$= 2\sqrt{2}\pi\left\{\frac{1}{2}-2+2+1-\frac{1}{2}\right\} = 2\sqrt{2}\pi$$

13) A solid object is generated by rotating the curve

$$x = \sqrt{7}t^2 - 1$$
$$y = 2t^2 + 1$$

about $y-$ axis through $2\pi$ radians from $t = 0$ to $t = 1$. Calculate the surface area of the object exclusing the two ends

After we obtain

$$\frac{d\left\{x\right\}}{dt} = 2\sqrt{7}t$$
$$\frac{d\left\{y\right\}}{dt} = 4t$$

we calculate $S$ as

$$S = \int_0^1 2\pi x(t)\sqrt{\left(\frac{d\left\{x\right\}}{dt}\right)^2 + \left(\frac{d\left\{y\right\}}{dt}\right)^2}dt$$
$$= \int_0^1 2\pi(\sqrt{7}t^2-1)\sqrt{\left(2\sqrt{7}t\right)^2 + (4t)^2}dt = \int_0^1 2\pi(\sqrt{7}t^2-1)\sqrt{28t^2+16t^2}dt$$
$$= 2\sqrt{44}\pi\int_0^1 2\pi(\sqrt{7}t^2-1)tdt = 2\sqrt{44}\pi\int_0^1 2\pi(\sqrt{7}t^3-t)dt = 2\sqrt{44}\pi\left[\frac{\sqrt{7}t^4}{4}-\frac{t^2}{2}\right]_0^1 = 2\sqrt{44}\pi\left[\frac{\sqrt{7}}{4}-\frac{1}{2}\right]$$

14) A solid object is generated by rotating the curve

$$x = t\sin t + \cos t$$
$$y = \sin t - t\cos t$$

about $x-$ axis through $2\pi$ radians from $t = 0$ to $t = \frac{\pi}{4}$. Calculate the surface area of the object exclusing the two ends

After we obtain

$$\frac{d\left\{x\right\}}{dt} = \sin t + t\cos t - \sin t = t\cos t$$
$$\frac{d\left\{y\right\}}{dt} = \cos t - \cos t - t(-\sin t) = t\sin t$$

we calculate $S$ as

$$S = \int_0^{\frac{\pi}{4}} 2\pi y(t) \sqrt{\left(\frac{d\{x\}}{dt}\right)^2 + \left(\frac{d\{y\}}{dt}\right)^2} dt$$

$$= \int_0^{\frac{\pi}{4}} 2\pi(\sin t - t\cos t)\sqrt{(t\cos t)^2 + (t\sin t)^2} = \int_0^{\frac{\pi}{4}} 2\pi(\sin t - t\cos t)\sqrt{t^2\left((\cos t)^2 + (\sin t)^2\right)} dt$$

$$= \int_0^{\frac{\pi}{4}} 2\pi(\sin t - t\cos t) t dt = 2\pi \int_0^{\frac{\pi}{4}} (t\sin t - t^2\cos t) dt \qquad ①$$

Here we find out $\int t\sin t dt$ and $\int t^2 \cos t dt$ as follows:

$$\int t\sin t dt = -t\cos t - \int(-\cos t)\cdot 1\cdot dt = -t\cos t + \int \cos t dt = -t\cos t + \sin t \qquad ②$$

$$\int t^2\cos t dt = t^2\sin t - \int 2t\sin t dt = t^2\sin t - 2\int t\sin t dt = t^2\sin t - 2(-t\cos t + \sin t)(\because ②)$$

$$= t^2\sin t + 2t\cos t - 2\sin t \qquad ③$$

Putting ② and ③ into ① we obtain

$$S = 2\pi \int_0^{\frac{\pi}{4}} (t\sin t - t^2\cos t) dt$$

$$= 2\pi\left[-t\cos t + \sin t - \left(t^2\sin t + 2t\cos t - 2\sin t\right)\right]_0^{\frac{\pi}{4}}$$

$$= 2\pi\left[-t\cos t + \sin t - t^2\sin t - 2t\cos t + 2\sin t\right]_0^{\frac{\pi}{4}}$$

$$= 2\pi\left[3\sin t - t^2\sin t - 3t\cos t\right]_0^{\frac{\pi}{4}} = \frac{2\pi}{\sqrt{2}}\left[3 - \left(\frac{\pi}{4}\right)^2 - \frac{3\pi}{4}\right]$$

15) A solid object is generated by rotating the curve

$$x = t\sin t + \cos t$$
$$y = \sin t - t\cos t$$

about $y-$ axis through $2\pi$ radians from $t = 0$ to $t = \frac{\pi}{4}$. Calculate the surface area of the object exclusing the two ends

After we obtain

$$\frac{d\{x\}}{dt} = \sin t + t\cos t - \sin t = t\cos t$$

$$\frac{d\{y\}}{dt} = \cos t - \cos t - t(-\sin t) = t\sin t$$

we calculate $S$ as

$$S = \int_0^{\frac{\pi}{4}} 2\pi x(t)\sqrt{\left(\frac{d\{x\}}{dt}\right)^2 + \left(\frac{d\{y\}}{dt}\right)^2} dt = \int_0^{\frac{\pi}{4}} 2\pi(t\sin t + \cos t)\sqrt{(t\cos t)^2 + (t\sin t)^2} dt$$

$$= \int_0^{\frac{\pi}{4}} 2\pi(t\sin t + \cos t) t dt = 2\pi \int_0^{\frac{\pi}{4}} \left(t^2\sin t + t\cos t\right) dt \qquad ①$$

Here we find out $\int t\cos t dt$ and $\int t^2\sin t dt$ as follows:

$$\int t\cos t dt = t\sin t - \int \sin t\cdot 1\cdot dt = t\sin t - (-\cos t) = t\sin t + \cos t \qquad ②$$

$$\int t^2 \sin t dt = -t^2 \cos t - \int 2t(-\cos t) dt = -t^2 \cos t + \int 2t \cos t dt$$
$$= -t^2 \cos t + 2 \int t \cos t dt = -t^2 \cos t + 2(t \sin t + \cos t)(\because ②) = -t^2 \cos t + 2t \sin t + 2 \cos t$$

Putting ② and ③ into ① we obtain

$$S = 2\pi \int_0^{\frac{\pi}{4}} \left(t^2 \sin t + t \cos t\right) dt$$
$$= 2\pi \left[-t^2 \cos t + 2t \sin t + 2 \cos t + t \sin t + \cos t\right]_0^{\frac{\pi}{4}}$$
$$= 2\pi \left[-t^2 \cos t + 3t \sin t + 3 \cos t\right]_0^{\frac{\pi}{4}} = \frac{2\pi}{\sqrt{2}} \left[-\left(\frac{\pi}{4}\right)^2 + \frac{3\pi}{4} + 3\right]$$

16) A solid object is generated by rotating the curve

$$x = \sqrt{7}t^2 - 1$$
$$y = 2t^2 + 1$$

about $x-$ axis through $2\pi$ radians from $t = 0$ to $t = 1$. Calculate the volume of the object
After we obtain

$$\frac{d\{x\}}{dt} = 2\sqrt{7}t$$

we calculate $V$ as

$$V = \int dV = \int_0^1 \pi (y(t))^2 \frac{d\{x\}}{dt} dt = \pi \int_0^1 \left(2t^2 + 1\right)^2 \cdot 2\sqrt{7}t \cdot dt = 2\sqrt{7}\pi \int_0^1 \left(2t^2 + 1\right)^2 t \cdot dt$$
$$= 2\sqrt{7}\pi \int_0^1 \left(4t^5 + t + 4t^3\right) \cdot dt = 2\sqrt{7} \left[\frac{4}{6}t^6 + \frac{1}{2}t^2 + t^4\right]_0^1$$
$$= 2\sqrt{7} \left[\frac{4}{6} + \frac{1}{2} + 1\right] = 2\sqrt{7} \left[\frac{4}{6} + \frac{3}{6} + \frac{6}{6}\right] = 2\sqrt{7} \left[\frac{13}{6}\right] = \frac{26\sqrt{7}}{6}$$

17) A solid object is generated by rotating the curve

$$x = \sqrt{7}t^2 - 1$$
$$y = 2t^2 + 1$$

about $y-$ axis through $2\pi$ radians from $t = 0$ to $t = 2$. Calculate the volume of the object
After we obtain

$$\frac{d\{y\}}{dt} = 4t$$

we calculate $V$ as

$$V = \int dV = \int_0^2 \pi (x(t))^2 \frac{d\{y\}}{dt} dt = \pi \int_0^2 \left(\sqrt{7}t^2 - 1\right)^2 \cdot 4t \cdot dt$$
$$= 4\pi \int_0^2 \left(7t^4 + 1 - 2\sqrt{7}t^2\right) \cdot t \cdot dt = 4\pi \int_0^2 \left(7t^5 + t - 2\sqrt{7}t^3\right) dt = 4\pi \left[\frac{7t^6}{6} + \frac{t^2}{2} - \frac{2\sqrt{7}t^4}{4}\right]_0^2$$
$$= 4\pi \left[\frac{7 \cdot 2^6}{6} + \frac{2^2}{2} - \frac{2\sqrt{7} \cdot 2^4}{4}\right] = 4\pi \left[\frac{7 \cdot 2^5}{3} + \frac{2}{1} - \frac{2\sqrt{7} \cdot 2^2}{1}\right] = 4\pi \left[\frac{224}{3} + \frac{6}{3} - \frac{24\sqrt{7}}{3}\right] = 4\pi \frac{230 - 24\sqrt{7}}{3}$$

18) A solid object is generated by rotating the curve

$$x = -4\cos t$$
$$y = \sin t$$

about $x-$ axis through $2\pi$ radians from $t = 0$ to $t = \frac{\pi}{4}$. Calculate the volume of the object
After we obtain

$$\frac{d\{x\}}{dt} = 4\sin t$$

we calculate $V$ as

$$V = \int dV = \int_0^{\frac{\pi}{4}} \pi (y(t))^2 \frac{d\{x\}}{dt} dt = \int_0^{\frac{\pi}{4}} \pi (\sin t)^2 \cdot (4\sin t)\, dt$$
$$= \int_0^{\frac{\pi}{4}} 4\pi \sin^3 t dt = 4\pi \int_0^{\frac{\pi}{4}} \frac{3\sin t - \sin 3t}{4} dt = \pi \int_0^{\frac{\pi}{4}} (3\sin t - \sin 3t) dt$$
$$= \pi \left[-3\cos t + \frac{1}{3}\cos 3t\right]_0^{\frac{\pi}{4}} = \pi \left[-3\frac{1}{\sqrt{2}} + \frac{1}{3}(-\frac{1}{\sqrt{2}}) - (-3 + \frac{1}{3})\right] = \pi \left[\frac{-3}{\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{9}{3} - \frac{1}{3}\right]$$
$$= \pi \left[\frac{-9}{3\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{9}{3} - \frac{1}{3}\right] = \pi \left[-\frac{10}{3\sqrt{2}} + \frac{8}{3}\right] = \pi \frac{16 - 10\sqrt{2}}{6}$$

19) A solid object is generated by rotating the curve

$$x = -4\cos t$$
$$y = \sin t$$

about $y-$ axis through $2\pi$ radians from $t = 0$ to $t = \frac{\pi}{4}$. Calculate the volume of the object
After we obtain

$$\frac{d\{y\}}{dt} = \cos t$$

we calculate $V$ as

$$V = \int dV = \int_0^{\frac{\pi}{4}} \pi (x(t))^2 \frac{d\{y\}}{dt} dt = \int_0^{\frac{\pi}{4}} \pi (-4\cos t)^2 \cdot \cos t dt$$
$$= 16\pi \int_0^{\frac{\pi}{4}} \cos^3 t dt = 16\pi \int_0^{\frac{\pi}{4}} \frac{3\cos t + \cos 3t}{4} dt = 4\pi \left[3\sin t + \frac{1}{3}\sin 3t\right]_0^{\frac{\pi}{4}}$$
$$= \frac{4\pi}{\sqrt{2}} \left[3 + \frac{1}{3}\right] = \frac{40\pi}{3\sqrt{2}}$$

20) **DAY2**

21) Evaluate the following integral

$$\iint_D dA$$

where $D$ is the region bounded by $0 \leq y \leq \sin x$ and $0 \leq x \leq \pi$.
The limit of $x$ and $y$ are given. As the limit of $x$ is fixed, the integral we evaluate is written as

$$\int_0^{\pi} \int_0^{\sin x} dy dx$$

The integral inside is

$$\int_0^{\sin x} dy = [y]_0^{\sin x} = \sin x$$

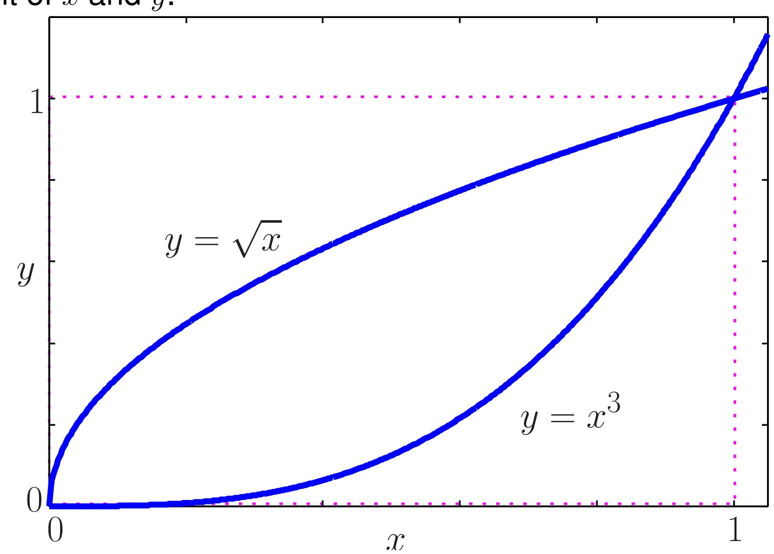
Putting this into the original integral,

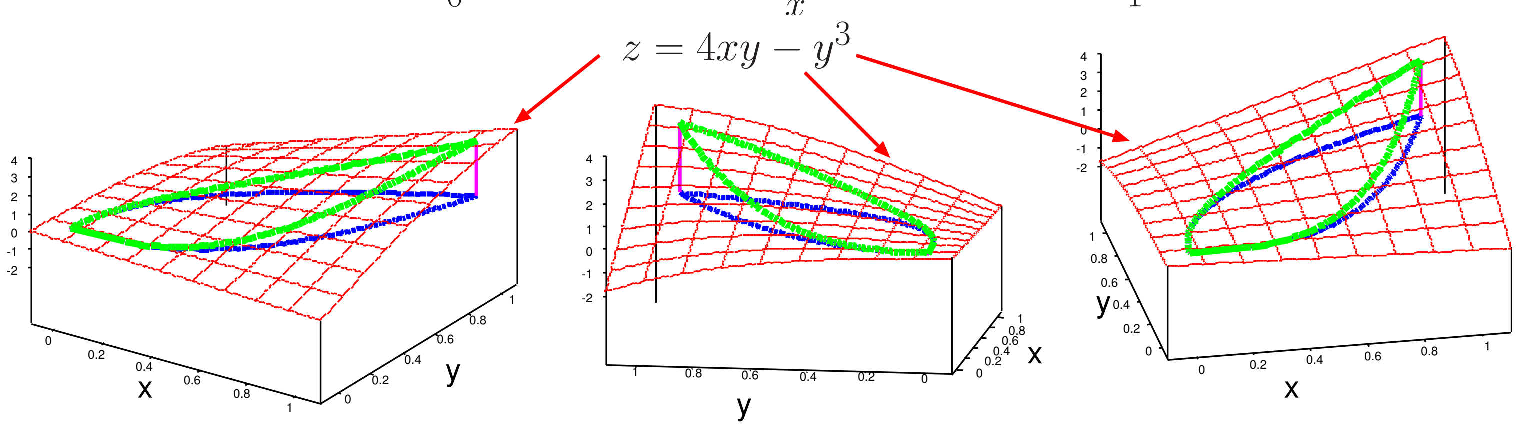
$$\int_0^{\pi} \sin x dx = [-\cos x]_0^{\pi} = -\cos \pi - (-\cos 0) = -(-1) + 1 = 2$$

22) Evaluate the following integral

$$\iint_D (4xy - y^3) dA$$

where $D$ is the region bounded by $y = \sqrt{x}$ and $y = x^3$.
We need to find the limit of $x$ and $y$.



From the sketch ,we can see that

$$x^3 \le y \le \sqrt{x} \text{ for } 0 \le x \le 1 \qquad ①$$

$$\text{or } y^2 \le x \le y^{\frac{1}{3}} \text{ for } 0 \le y \le 1 \qquad ②$$

When we take ②, the integral we evaluate is

$$\int_0^1 \int_{y^2}^{y^{\frac{1}{3}}} (4xy - y^3)dxdy$$

When we take ①, the integral we evaluate is

$$\int_0^1 \int_{x^3}^{\sqrt{x}} (4xy - y^3)dydx$$

The integral inside is

$$\int_{x^3}^{\sqrt{x}} (4xy - y^3)dy = \left[4x \cdot \frac{y^2}{2} - \frac{y^4}{4}\right]_{x^3}^{\sqrt{x}} = 2x(\sqrt{x})^2 - (\sqrt{x})^4/4 - \left(2x(x^3)^2 - (x^3)^4/4\right)$$
$$= 2x^2 - x^2/4 - \left(2x^{1+2\cdot 3} - x^{3\cdot 4}/4\right) = \frac{7}{4}x^2 - 2x^7 + \frac{x^{12}}{4}$$

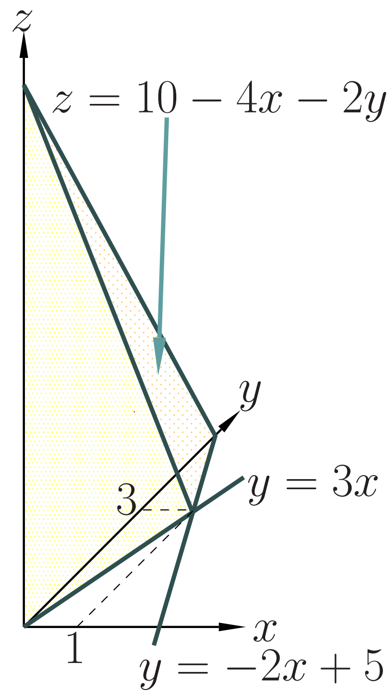
Putting this into the original integral,

$$\int_0^1 (\frac{7}{4}x^2 - 2x^7 + \frac{x^{12}}{4})dx = \left[\frac{7}{4} \cdot \frac{1}{3}x^3 - 2\frac{1}{8}x^8 + \frac{x^{13}}{4} \cdot \frac{1}{13}\right]_0^1 = \frac{7}{12} - \frac{1}{4} + \frac{1}{4 \cdot 13}$$
$$= \frac{7 \cdot 13}{12 \cdot 13} - \frac{13 \cdot 3}{4 \cdot 13 \cdot 3} + \frac{3}{4 \cdot 13 \cdot 3} = \frac{91 - 39 + 3}{12 \cdot 13} = \frac{55}{156}$$

23) Find the volume of the solid enclosed by the planes

$$\begin{aligned} 4x + 2y + z &= 10 \\ y &= 3x \\ z &= 0 \\ x &= 0 \end{aligned}$$

The plane $4x+2y+z = 10$ intersects $z = 0$ plane on the line $4x+2y+0 = 10$, in other words, $y = -2x+5$. Thus these four planes can make a solid like the following figure:




The integral limits of the $x$ and $y$ directions are

$$0 \le x \le 1 \;\; ; \;\; 3x \le y \le -2x+5$$

Therefore the volume of the solid is

$$\int_0^1 \int_{3x}^{-2x+5} z\,dy\,dx = \int_0^1 \int_{3x}^{-2x+5} (10-4x-2y)\,dy\,dx = \int_0^1 \left[(10-4x)y - y^2\right]_{3x}^{-2x+5} dx$$

$$= \int_0^1 \left\{(10-4x)(-2x+5) - (-2x+5)^2) - (10-4x)(3x) + (3x)^2\right\} dx$$

$$= \int_0^1 \left\{(10-4x+2x-5)(-2x+5) - (10-4x-3x)(3x)\right\} dx$$

$$= \int_0^1 \left\{(5-2x)(-2x+5) - (10-7x)(3x)\right\} dx = \int_0^1 \left\{4x^2 + 25 - 20x - 30x + 21x^2\right\} dx$$

$$= \int_0^1 \left\{25x^2 + 25 - 50x\right\} dx = \left[\frac{25x^3}{3} + 25x - 25x^2\right]_0^1 = \frac{25}{3} + 25 - 25 = \frac{25}{3}$$

24) Find the $\int t^{\frac{1}{2}} dt$.

$$\int t^{\frac{1}{2}} dt = \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2t^{\frac{3}{2}}}{3} + c$$

25) Calculate $\frac{12}{43} \div \frac{3}{5} \times 11$.

$$\frac{12}{43} \div \frac{3}{5} \times 11 = \frac{12 \times 5}{43 \times 3} \times 11 = \frac{60}{129} \times 11 = \frac{660}{129}$$

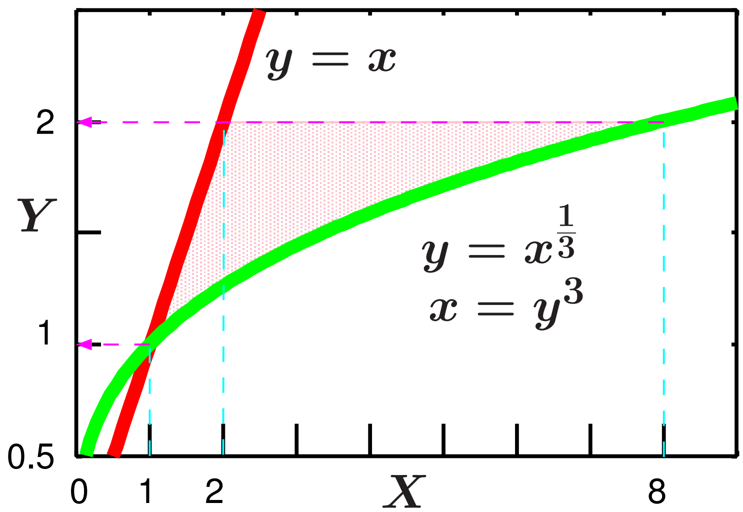
26) Simplify $\dfrac{\dfrac{12x}{5}}{\dfrac{4+x}{x^2}}$

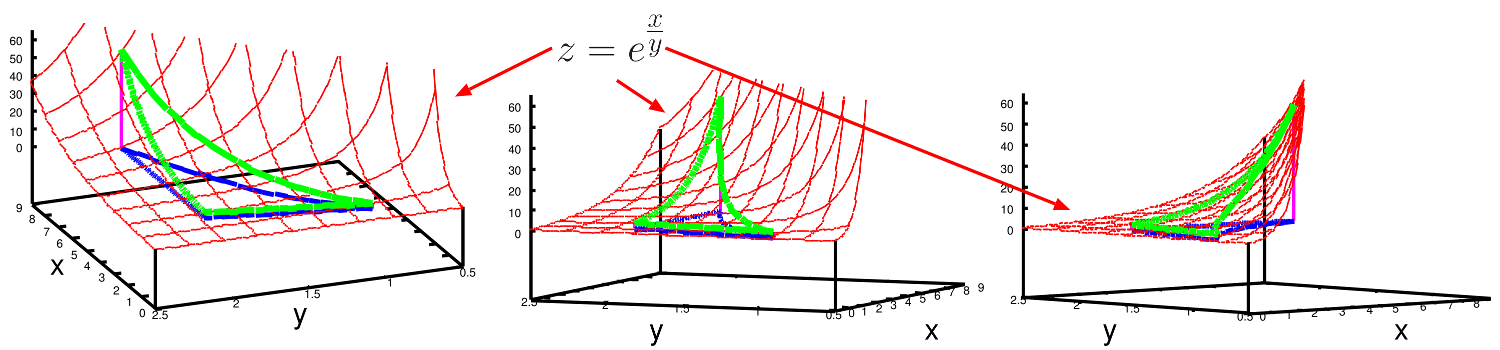
$$\frac{\dfrac{12x}{5}}{\dfrac{4+x}{x^2}} = \frac{12x \times x^2}{5 \times (4+x)} = \frac{12x^3}{(20+5x)}$$

27) Evaluate the following integral over the given region $D$

$$\iint_D \mathfrak{e}^{\frac{x}{y}} dA, D = \left\{(x,y)|1 \le y \le 2, y \le x \le y^3\right\}$$

The region of integral is




From the figure, we can write down the integral region in the following ways:

$x^{\frac{1}{3}} \le y \le x$ for $1 \le x \le 2$ and $x^{\frac{1}{3}} \le y \le 2$ for $2 \le x \le 8$ ①

or $y \le x \le y^3$ for $1 \le y \le 2$ ②

① is the order of integration reversed. ② is the original integral range. Since $y$ 's limit is fixed(=without any variables), integration for $y$ becomes the second integral and thus integration for $x$ becomes the first integral. The original problem can be re-written as

$$\int_1^2 \int_y^{y^3} \mathfrak{e}^{\frac{x}{y}} dxdy$$

First , we perform the integral inside as follows:

$$\int_y^{y^3} \mathfrak{e}^{\frac{x}{y}} dx = \left[\frac{1}{\frac{1}{y}} \mathfrak{e}^{\frac{x}{y}}\right]_{x=y}^{x=y^3} = y\left[\mathfrak{e}^{\frac{x}{y}}\right]_{x=y}^{x=y^3} = y\left[\mathfrak{e}^{\frac{y^3}{y}} - \mathfrak{e}^{\frac{y}{y}}\right] = y\left[\mathfrak{e}^{y^2} - \mathfrak{e}\right] = y\mathfrak{e}^{y^2} - y\mathfrak{e}$$

We now return this into the original integral:

$$\int_1^2 \int_y^{y^3} \mathfrak{e}^{\frac{x}{y}} dx dy = \int_1^2 (y\mathfrak{e}^{y^2} - y\mathfrak{e}) dy$$

In order to evaluate $\int y\mathfrak{e}^{y^2} dy$, we set $t \triangleq y^2$. Then $dt = 2ydy$.

$$\int y\mathfrak{e}^{y^2} dy = \int \mathfrak{e}^t \frac{dt}{2} = \frac{\mathfrak{e}^t}{2} = \frac{\mathfrak{e}^{y^2}}{2}$$

Thus

$$\int_1^2 (y\mathfrak{e}^{y^2} - y\mathfrak{e}) dy = \left[\frac{\mathfrak{e}^{y^2}}{2} - \frac{y^2}{2}\mathfrak{e}\right]_1^2 = (\frac{\mathfrak{e}^{2^2}}{2} - \frac{2^2}{2}\mathfrak{e}) - (\frac{\mathfrak{e}^{1^2}}{2} - \frac{1^2}{2}\mathfrak{e}) = (\frac{\mathfrak{e}^4}{2} - 2\mathfrak{e}) - (\frac{\mathfrak{e}}{2} - \frac{1}{2}\mathfrak{e})$$

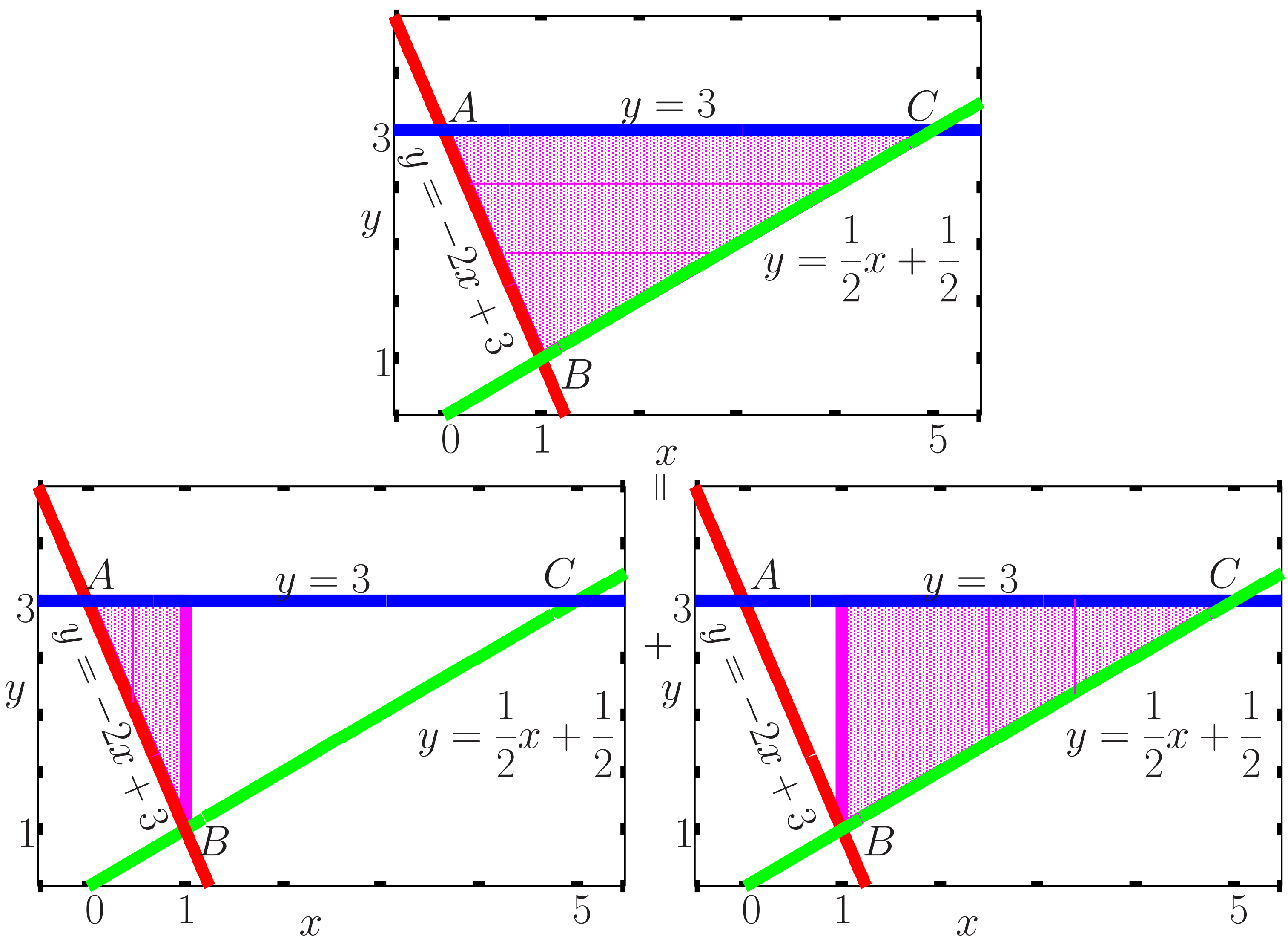
$$= \frac{\mathfrak{e}^4}{2} - 2\mathfrak{e} - \frac{\mathfrak{e}}{2} + \frac{1}{2}\mathfrak{e} = \frac{\mathfrak{e}^4}{2} - 2\mathfrak{e}$$

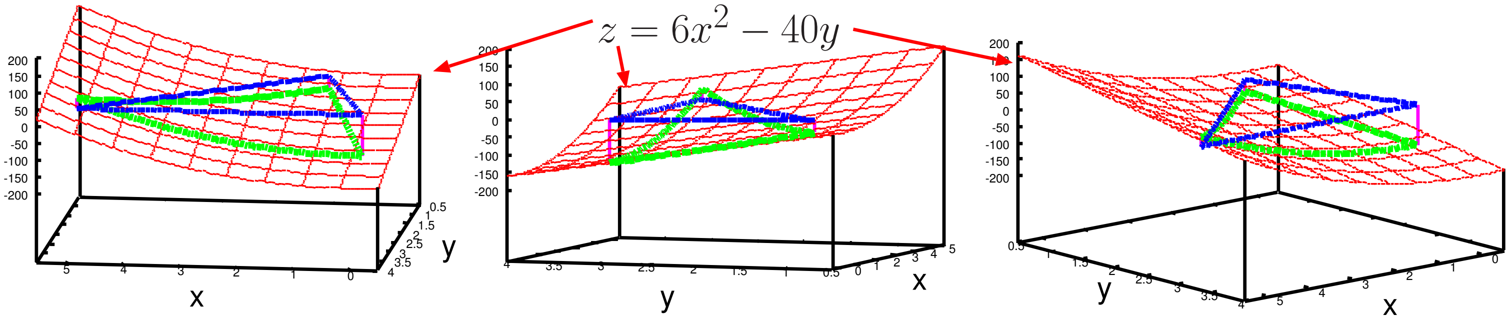
28) Evaluate the following integral

$$\iint_D (6x^2 - 40y) dA$$

where $D$ is the triangle with vertices $A(0,3)$, $B(1,1)$, $C(5,3)$

The sketch of the triangle with vertices $A(0,3)$, $B(1,1)$, and $C(5,3)$ is






The line $AB$ is expressed as $y = ax + b$, by putting $(x, y) = (0, 3)$ into $y = ax + b$, we obtain $b = 3$. Furthermore $(x, y) = (1, 1)$ is put into $y = ax + 3$ and $a = -2$ is obtained. Thus the line $AB$ is $y = -2x + 3$, in other words, $x = \frac{3-y}{2}$. In the same way, the line $AC$ is obtained as $y = 3$. As for the line $BC$, using $(x, y) = (1, 1)$, we obtain $1 = a + b$. Using $(x, y) = (5, 3)$, we obtain $3 = 5a + b$. By subtracting these two equations to remove $b$, we obtain $2 = 4a$, *i.e.,* $a = \frac{1}{2}$. Then, from $1 = a + b$, we obtain $b = \frac{1}{2}$. Thus, the line $BC$ is obtained as $y = \frac{x}{2} + \frac{1}{2}$, in other words, $x = 2y - 1$. From the figure, the range of the $x$ and $y$ values are expressed in the following two ways:

$$-2x + 3 \le y \le 3 \text{ for } 0 \le x \le 1 \text{ and } \frac{x}{2} + \frac{1}{2} \le y \le 3 \text{ for } 1 \le x \le 5$$

or

$$\frac{3-y}{2} \le x \le 2y - 1 \text{ for } 1 \le y \le 3$$

Since the second approach have only one integral, we go for the second approach.

$$\int_1^3 \int_{\frac{3-y}{2}}^{2y-1} (6x^2 - 40y)dxdy = \int_1^3 \left[\frac{6}{3}x^3 - 40yx\right]_{\frac{3-y}{2}}^{2y-1} dy$$

$$= \int_1^3 \left[2(2y-1)^3 - 40y(2y-1) - (2(\frac{3-y}{2})^3 - 40y\frac{3-y}{2})\right] dy$$

$$= \int_1^3 \left[2(2y-1)^3 - 40y(2y-1) - 2(\frac{3-y}{2})^3 + 40y\frac{3-y}{2}\right] dy$$

$$= \int_1^3 \left[2 \cdot 2^3(y - \frac{1}{2})^3 - 80(y^2 - \frac{y}{2}) - 2 \cdot \frac{1}{(-2)^3}(y-3)^3 + 20(3y - y^2)\right] dy$$

$$= \left[2 \cdot 2^3 \frac{1}{4}(y - \frac{1}{2})^4 - 80(\frac{1}{3}y^3 - \frac{1}{2} \cdot \frac{1}{2}y^2) - 2 \cdot \frac{1}{(-2)^3}\frac{1}{4}(y-3)^4 + 20(\frac{3}{2}y^2 - \frac{1}{3}y^3)\right]_1^3$$

$$= \left[4(y - \frac{1}{2})^4 - \frac{100}{3}y^3 + 50y^2 + \frac{1}{16}(y-3)^4\right]_1^3$$

$$= \left[4(3 - \frac{1}{2})^4 - \frac{100}{3}3^3 + 50 \cdot 3^2\right] - \left[4(1 - \frac{1}{2})^4 - \frac{100}{3} + 50 + \frac{1}{16}(1-3)^4\right]$$

$$= \left[4(\frac{5}{2})^4 - \frac{100}{3}3^3 + 50 \cdot 3^2\right] - \left[4(\frac{1}{2})^4 - \frac{100}{3} + 50 + \frac{1}{16}(2)^4\right] = \frac{5^4}{4} - 900 + 450 - \frac{1}{4} + \frac{100}{3} - 50 - 1$$

$$= \frac{624}{4} - 501 + \frac{100}{3} = 156 - 501 + \frac{100}{3} = -345 + \frac{100}{3} = -\frac{1035}{3} + \frac{100}{3} = -\frac{935}{3}$$

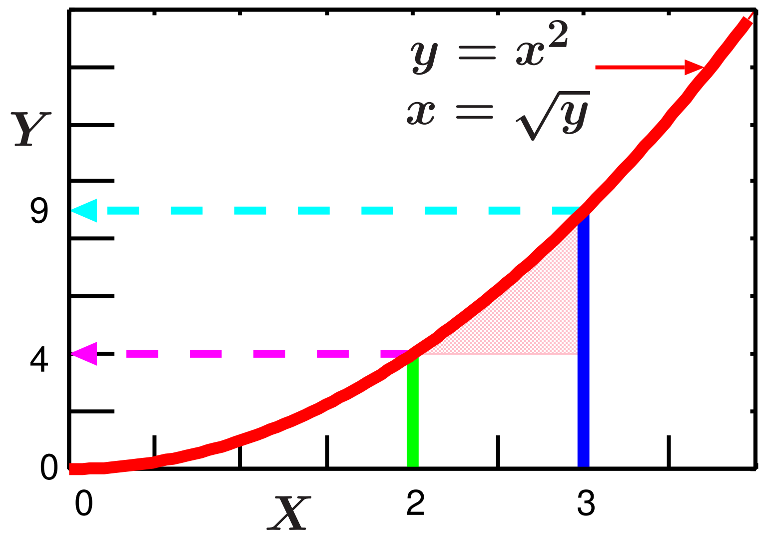
**DAY3**

29) Sketch the region of integration of the double integral of

$$\int_4^9 \int_{\sqrt{y}}^3 x dx dy$$

and rewrite the integral with the order of integration reversed and evaluate the integral

The region of integral is




From the figure, we can write down the integral region in the following ways:

$$4 \le y \le x^2 \text{ for } 2 \le x \le 3 \qquad ①$$
$$\text{or } \sqrt{y} \le x \le 3 \text{ for } 4 \le y \le 9 \qquad ②$$

② is the original integral range. ① is the order of integration reversed. Since $x$ 's limit is fixed(=without any variables), integration for $x$ becomes the second integral and thus integration for $y$ becomes the first integral. The integral can be re-written as

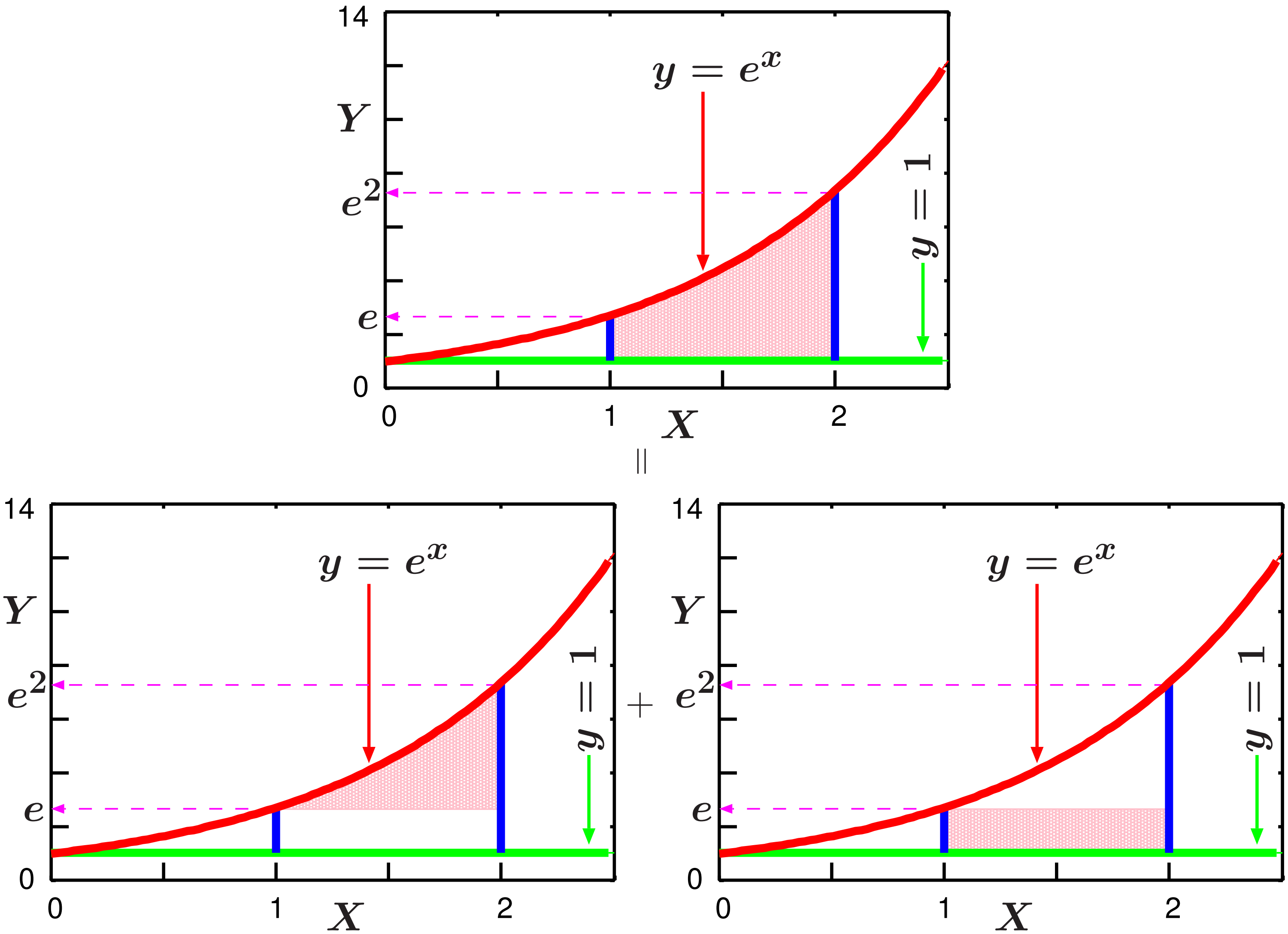
$$\int_2^3 \int_4^{x^2} x dy dx = \int_2^3 x[y]_4^{x^2} dx = \int_2^3 x\left(x^2 - 4\right) dx = \int_2^3 \left(x^3 - 4x\right) dx = \left[\frac{x^4}{4} - 2x^2\right]_2^3$$
$$= \frac{3^4}{4} - 2 \cdot 3^2 - \left(\frac{2^4}{4} - 2 \cdot 2^2\right) = \frac{81}{4} - 2 \cdot 9 - \frac{16}{4} + 2 \cdot 4 = \frac{65}{4} - 2 \cdot 5 = \frac{65 - 40}{4} = \frac{25}{4}$$

30) Sketch the region of integration of the double integral of

$$\int_1^2 \int_1^{e^x} y dy dx$$

and rewrite the integral with the order of integration reversed and evaluate the integral

The region of integral is




From the figure, we can write down the integral region in the following ways:

$$\ln(y) \le x \le 2 \text{ for } \mathfrak{e} \le y \le \mathfrak{e}^2 \text{ and } 1 \le x \le 2 \text{ for } 1 \le y \le \mathfrak{e} \qquad ①$$

$$\text{or } 1 \le y \le \mathfrak{e}^x \text{ for } 1 \le x \le 2 \qquad ②$$

② is the original integral range. ① is the order of integration reversed. Since $y$ 's limit is fixed(=without any variables), integration for $y$ becomes the second integral and thus integration for $x$ becomes the first integral. The integral can be re-written as

$$\int_1^{\mathfrak{e}}\int_1^2 ydxdy + \int_{\mathfrak{e}}^{\mathfrak{e}^2}\int_{\ln(y)}^2 ydxdy \qquad ③$$

The first term of ③ is

$$\int_1^{\mathfrak{e}}\int_1^2 ydxdy = \int_1^{\mathfrak{e}} y[x]_1^2 dy = \int_1^{\mathfrak{e}} (2-1)ydy = \int_1^{\mathfrak{e}} ydy = \left[\frac{y^2}{2}\right]_1^{\mathfrak{e}} = \frac{\mathfrak{e}^2}{2} - \frac{1^2}{2} = \frac{\mathfrak{e}^2 - 1}{2} \qquad ④$$

The second term of ③ is

$$\int_{\mathfrak{e}}^{\mathfrak{e}^2}\int_{\ln(y)}^2 ydxdy = \int_{\mathfrak{e}}^{\mathfrak{e}^2} y\,[x]_{\ln(y)}^2\,dy = \int_{\mathfrak{e}}^{\mathfrak{e}^2} y(2-\ln(y))dy = \int_{\mathfrak{e}}^{\mathfrak{e}^2} 2ydy - \int_{\mathfrak{e}}^{\mathfrak{e}^2} y\ln(y)dy \qquad ⑤$$

Now we deal with $\int y\ln(y)dy$ in ⑤ as follows:

$$\int y\ln(y)dy = \ln(y)\int ydy - \int\left(\frac{d\{\ln(y)\}}{dy}\int ydy\right)dy = \ln(y)\cdot\frac{y^2}{2} - \int\left(\frac{1}{y}\cdot\frac{y^2}{2}\right)dy$$
$$= \frac{y^2\ln(y)}{2} - \int\left(\frac{y}{2}\right)dy = \frac{y^2\ln(y)}{2} - \frac{y^2}{4} = \frac{2y^2\ln(y)-y^2}{4} \quad ⑥$$

Therefore using ⑥, we can re-write ⑤ as

$$\int_{\mathfrak{e}}^{\mathfrak{e}^2}\int_{\ln(y)}^{2} ydxdy = \int_{\mathfrak{e}}^{\mathfrak{e}^2} 2ydy - \int_{\mathfrak{e}}^{\mathfrak{e}^2} y\ln(y)dy = \left[y^2 - \frac{2y^2\ln(y)-y^2}{4}\right]_{\mathfrak{e}}^{\mathfrak{e}^2} = \left[\frac{4y^2-2y^2\ln(y)+y^2}{4}\right]_{\mathfrak{e}}^{\mathfrak{e}^2}$$
$$= \left[\frac{5y^2-2y^2\ln(y)}{4}\right]_{\mathfrak{e}}^{\mathfrak{e}^2} = \frac{5\mathfrak{e}^4-2\mathfrak{e}^4\ln(\mathfrak{e}^2)}{4} - \frac{5\mathfrak{e}^2-2\mathfrak{e}^2\ln(\mathfrak{e})}{4} = \frac{5\mathfrak{e}^4-4\mathfrak{e}^4}{4} - \frac{5\mathfrak{e}^2-2\mathfrak{e}^2}{4} = \frac{\mathfrak{e}^4-3\mathfrak{e}^2}{4} \quad ⑦$$

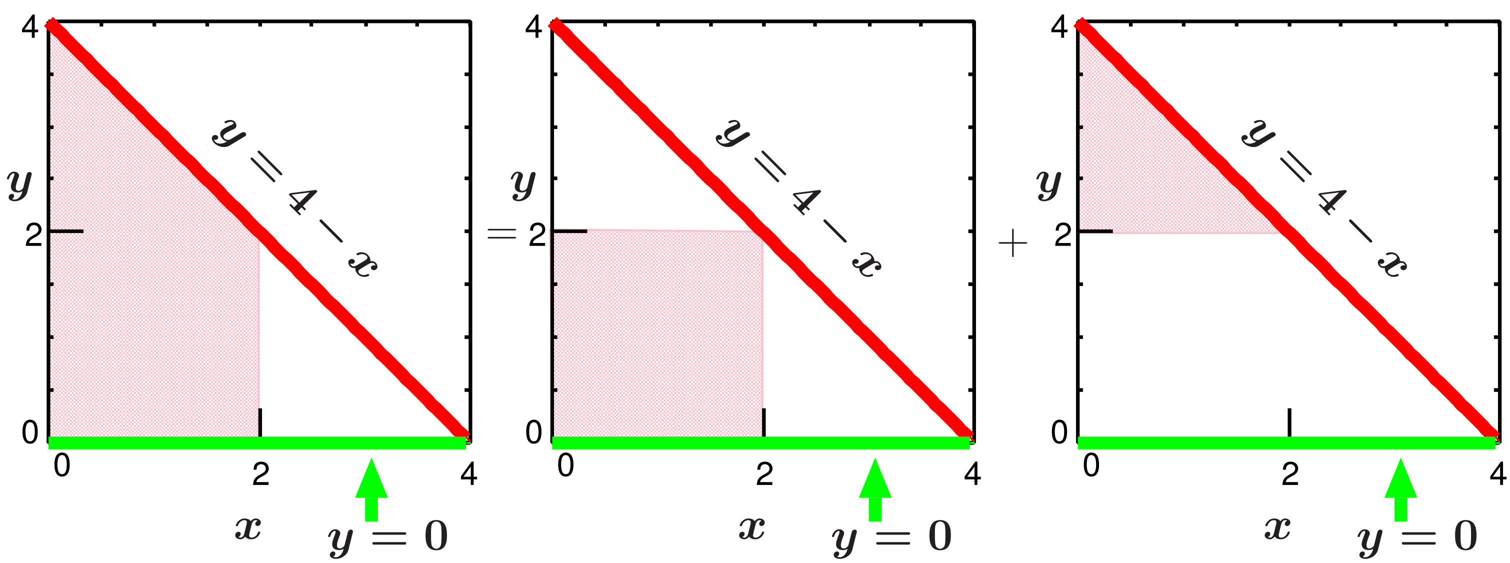
Using ④ and ⑦, we can re-write ③ as

$$\int_1^{\mathfrak{e}}\int_1^2 ydxdy + \int_{\mathfrak{e}}^{\mathfrak{e}^2}\int_{\ln(y)}^{2} ydxdy = \frac{\mathfrak{e}^2-1}{2} + \frac{\mathfrak{e}^4-3\mathfrak{e}^2}{4} = \frac{2\mathfrak{e}^2-2+\mathfrak{e}^4-3\mathfrak{e}^2}{4} = \frac{\mathfrak{e}^4-\mathfrak{e}^2-2}{4}$$

31) Sketch the region of integration of the double integral of

$$\int_0^2\int_0^{4-x} dydx$$

and rewrite the integral with the order of integration reversed and evaluate the integral
The region of integral is




From the figure, we can write down the integral region in the following ways:

$0 \le x \le 2$ for $0 \le y \le 2$ and $0 \le x \le 4-y$ for $2 \le y \le 4$ ①
or $0 \le y \le 4-x$ for $0 \le x \le 2$ ②

② is the original integral range. ① is the order of integration reversed. Since $y$ 's limit is fixed(=without any variables), integration for $y$ becomes the second integral and thus integration for $x$ becomes the first integral. The integral can be re-written as

$$\int_0^2\int_0^2 dxdy + \int_2^4\int_0^{4-y} dxdy$$

The first term is

$$\int_0^2\int_0^2 dxdy = \int_0^2 [x]_0^2 dy = \int_0^2 2dy = [2y]_0^2 = 4$$

The second term is

$$\int_2^4\int_0^{4-y} dxdy = \int_2^4 [x]_0^{4-y} dy = \int_2^4 (4-y)dy = \left[4y - \frac{y^2}{2}\right]_2^4 = 16 - 8 - (8-2) = 2$$

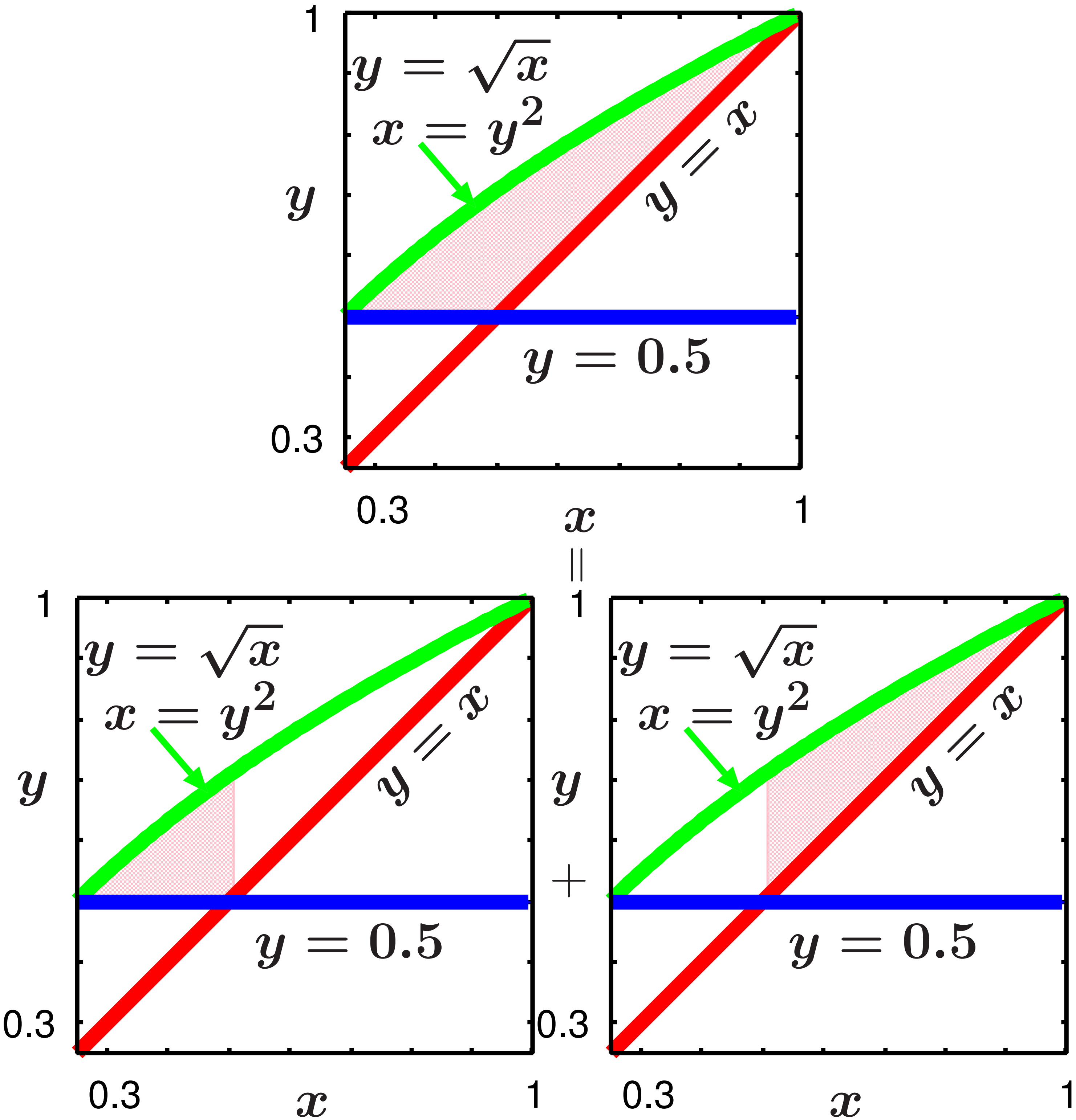
When we add these two terms,

$$\int_0^2\int_0^2 dxdy + \int_2^4\int_0^{4-y} dxdy = 4 + 2 = 6$$

32) Sketch the region of integration of the double integral of

$$\int_{\frac{1}{2}}^1\int_{y^2}^{y} dxdy$$

and rewrite the integral with the order of integration reversed and evaluate the integral
The region of integral is




From the figure, we can write down the integral region in the following ways:

$$0.5 \leq y \leq \sqrt{x} \text{ for } 0.25 \leq x \leq 0.5 \text{ and } x \leq y \leq \sqrt{x} \text{ for } 0.5 \leq x \leq 1 \qquad ①$$

$$\text{or } y^2 \leq x \leq y \text{ for } \frac{1}{2} \leq y \leq 1 \qquad ②$$

② is the original integral range. ① is the order of integration reversed. Since $x$ 's limit is fixed(=without any variables), integration for $x$ becomes the second integral and thus integration for $y$ becomes the first integral. The integral can be re-written as

$$\int_{0.25}^{0.5}\int_{0.5}^{\sqrt{x}} dydx + \int_{0.5}^{1}\int_{x}^{\sqrt{x}} dydx$$

The first term is

$$\int_{0.25}^{0.5}\int_{0.5}^{\sqrt{x}} dydx = \int_{0.25}^{0.5} [y]_{0.5}^{\sqrt{x}} dx = \int_{0.25}^{0.5} (\sqrt{x} - 0.5)dx = \left[\frac{x^{1.5}}{1.5} - 0.5x\right]_{0.25}^{0.5}$$

$$= \frac{0.5^{1.5}}{1.5} - 0.5 \cdot 0.5 - (\frac{0.25^{1.5}}{1.5} - 0.5 \cdot 0.25) = \frac{0.5^{1.5}}{1.5} - 0.5 \cdot 0.5 - \frac{0.25^{1.5}}{1.5} + 0.5 \cdot 0.25$$

The second term is

$$\int_{0.5}^{1}\int_{x}^{\sqrt{x}} dydx = \int_{0.5}^{1} [y]_{x}^{\sqrt{x}} dx = \int_{0.5}^{1} \left(\sqrt{x} - x\right) dx$$

$$= \left[\frac{x^{1.5}}{1.5} - \frac{x^2}{2}\right]_{0.5}^{1} = \frac{1^{1.5}}{1.5} - \frac{1^2}{2} - \left(\frac{0.5^{1.5}}{1.5} - \frac{0.5^2}{2}\right) = \frac{1}{1.5} - \frac{1}{2} - \frac{0.5^{1.5}}{1.5} + \frac{0.5^2}{2}$$

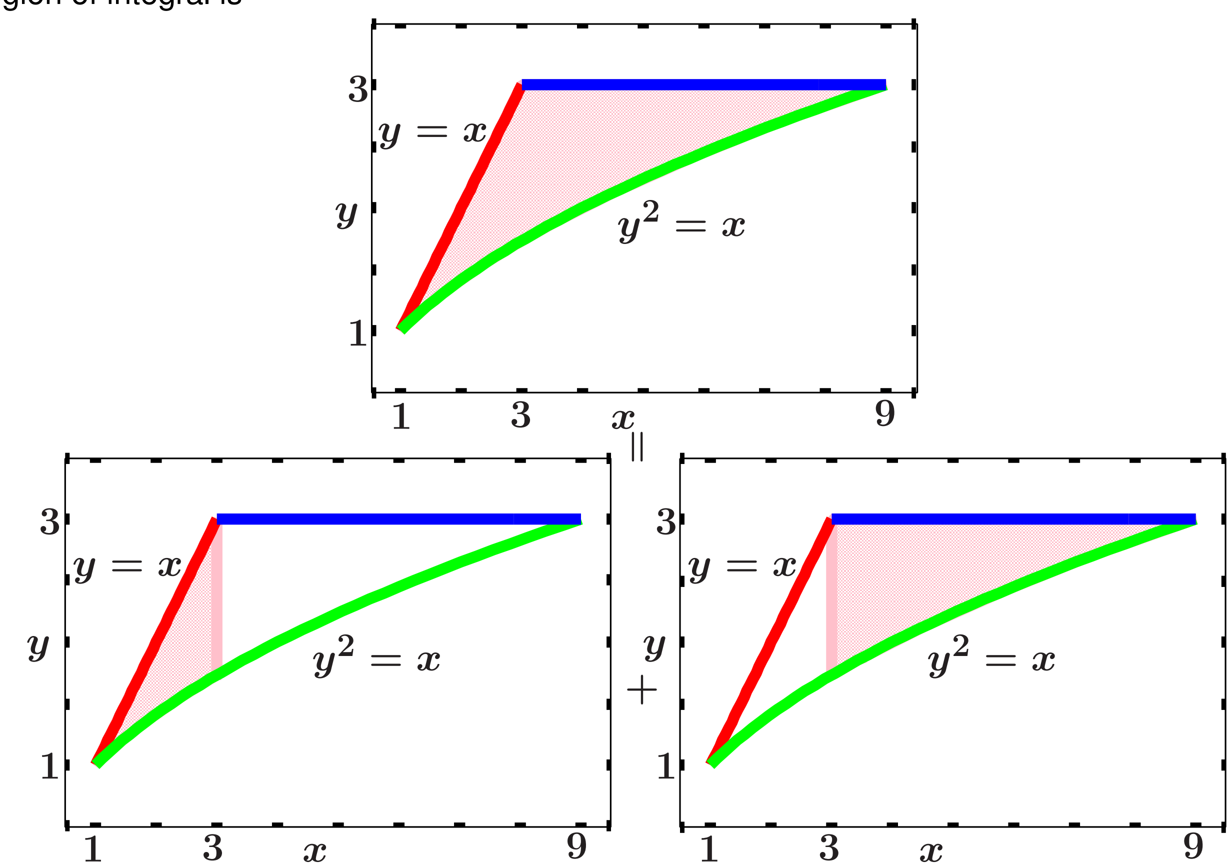
When we add these two terms,

$$\int_{0.25}^{0.5}\int_{0.5}^{\sqrt{x}} dydx + \int_{0.5}^{1}\int_{x}^{\sqrt{x}} dydx = \frac{0.5^{1.5}}{1.5} - 0.5 \cdot 0.5 - \frac{0.25^{1.5}}{1.5} + 0.5 \cdot 0.25 + \frac{1}{1.5} - \frac{1}{2} - \frac{0.5^{1.5}}{1.5} + \frac{0.5^2}{2}$$

$$= -0.5 - \frac{0.25^{1.5}}{1.5} + \frac{1}{1.5} = \frac{1}{12}$$

33) Find the double integral of $\iint \frac{x}{y} dxdy$ between $1 \le y \le 3$ and $y \le x \le y^2$.

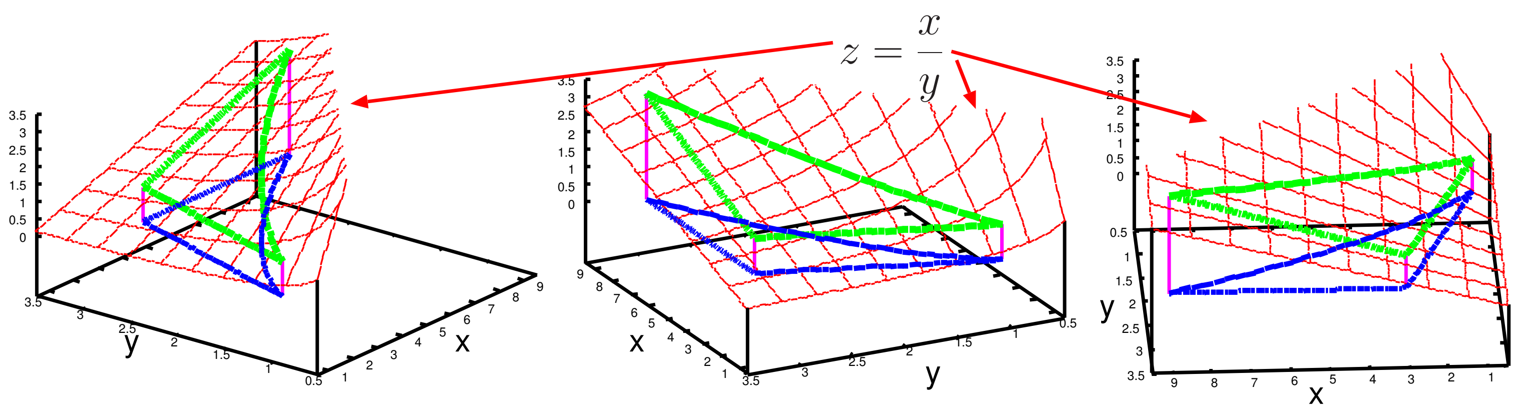
The region of integral is



From the figure, we can write down the integral region in the following ways:

$$\sqrt{x} \leq y \leq x \text{ for } 1 \leq x \leq 3 \text{ and } \sqrt{x} \leq y \leq 3 \text{ for } 3 \leq x \leq 9 \quad ①$$
$$\text{or } y \leq x \leq y^2 \text{ for } 1 \leq y \leq 3 \quad ②$$

As ② is simpler than ①, we use ② for integration. Since $y$ 's limit is fixed(=without any variables), integration for $y$ becomes the second integral and thus integration for $x$ becomes the first integral.




If we take the first integral

$$\int_y^{y^2} \frac{x}{y} dx = \frac{1}{y}\left[\frac{1}{2}x^2\right]_y^{y^2} = \frac{1}{2y}\left[x^2\right]_y^{y^2} = \frac{1}{2y}\left[y^4 - y^2\right]$$

Now substitute this back in to second integral.

$$\int_1^3 \frac{1}{2y}(y^4 - y^2)dy = \int_1^3 \frac{(y^4 - y^2)}{2y}dy = \int_1^3 \frac{(y^3 - y)}{2}dy = \frac{1}{2}\int_1^3 (y^3 - y)dy = \frac{1}{2}\left[\frac{1}{4}y^4 - \frac{1}{2}y^2\right]_1^3$$
$$= \frac{1}{2}\left[\frac{1}{4}(3^4 - 1^4) - \frac{1}{2}(3^2 - 1^2)\right] = \frac{1}{2}\left[\frac{80}{4} - \frac{8}{2}\right] = \frac{1}{2}\left[20 - 4\right] = \frac{16}{2} = 8$$

Therefore the answer to this double integral is $8$.

34) Find the $\int \cos 2x dx$.

$$\int \cos 2x dx = \frac{1}{2}\sin 2x + c$$

35) Find the $\int \pi dx$.

$$\int \pi dx = \pi x + c$$

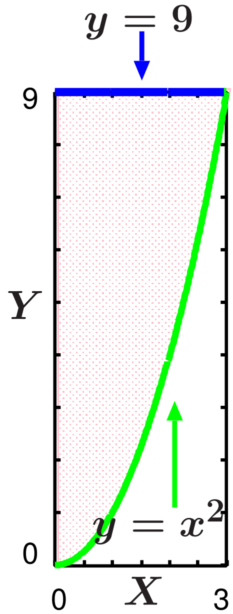
36) What is the value of $\sqrt{x^2 + 4y^2 + z}$ for $x = 3$ and $y = 1$, $z = 2$.

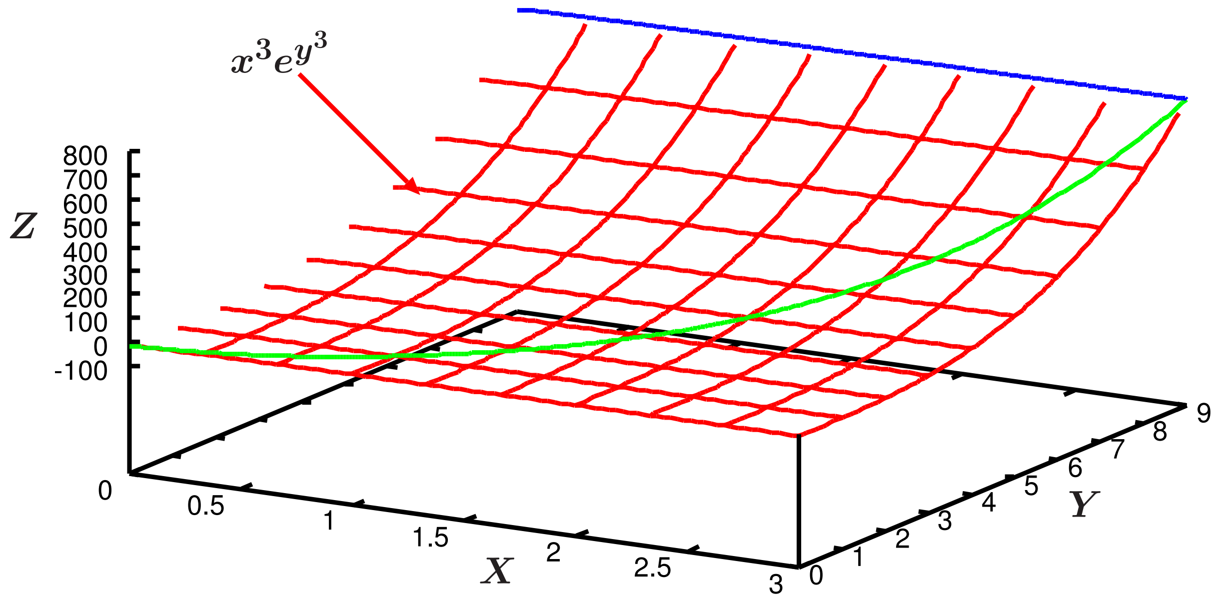
$$\sqrt{x^2 + 4y^2 + z} = \sqrt{3^2 + 4 \cdot 1^2 + 2} = \sqrt{9 + 4 + 2} = \sqrt{15}$$

37) Evaluate the following integral by first reversing the order of integration.

$$\int_0^3 \int_{x^2}^9 x^3 e^{y^3} dy dx$$

The region of integral is the coloured area in the left hand-side figure.




The original limits of the integration is

$$0 \leq x \leq 3, x^2 \leq y \leq 9$$

We now have to change these ranges so that the $y$ has limits made of the constant numbers and $x$ has the limits expressed using $y$. $0 \leq x \leq 3$ means $0 \leq x^2 \leq 3^2 = 9$ Therefore $0 \leq (x^2 \leq) y \leq 9$ As for the $x$ limits, $x^2 \leq y$ is equivalent to $x \leq \sqrt{y}$ Thus we obtain $0 \leq x \leq \sqrt{y}$ In this way, the original integral is the same as

$$\int_0^9 \int_0^{\sqrt{y}} x^3 \mathfrak{e}^{y^3} dx dy = \int_0^9 \left[\frac{1}{4} x^4 \mathfrak{e}^{y^3}\right]_0^{\sqrt{y}} dy = \int_0^9 \frac{1}{4} (\sqrt{y})^4 \mathfrak{e}^{y^3} dy = \int_0^9 \frac{1}{4} y^2 \mathfrak{e}^{y^3} dy$$

Here when we introduce a variable $t \triangleq y^3$, we obtain $dt = 3y^2 dy$.

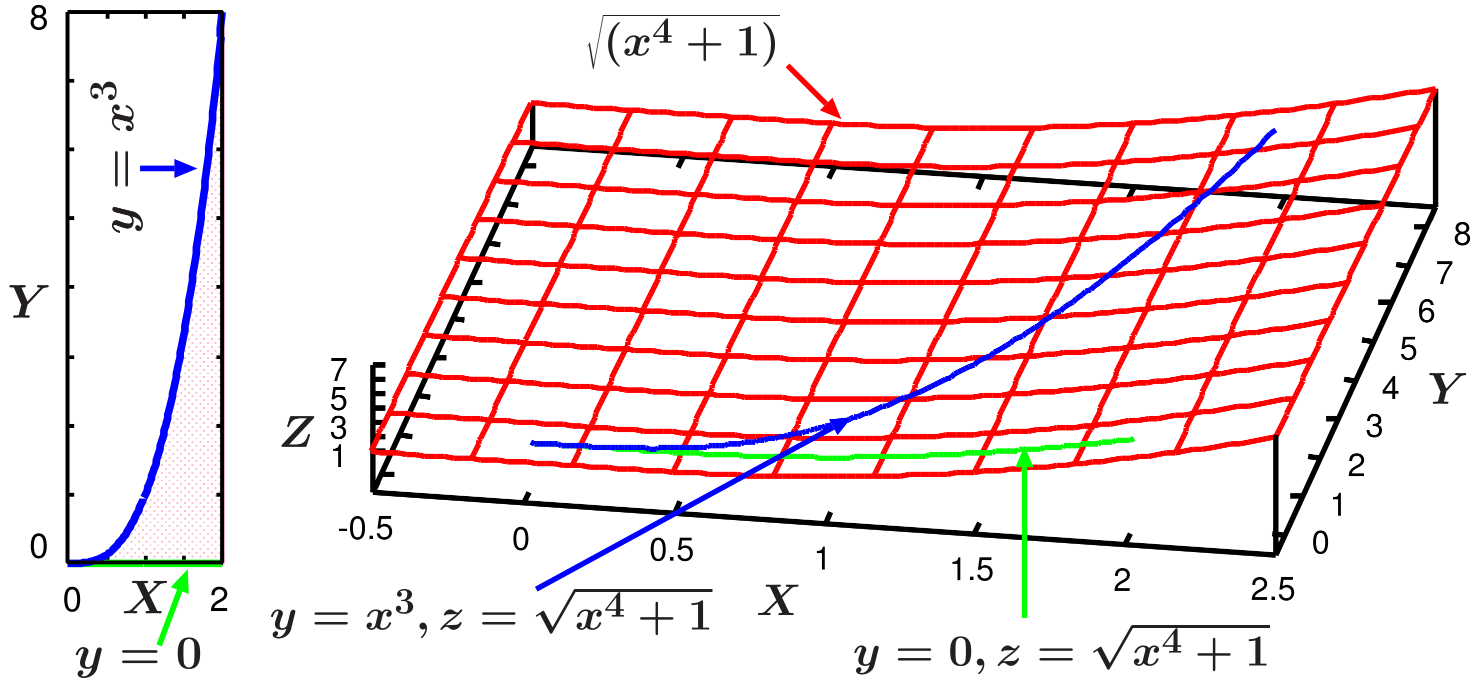
| | | |
|---|---|---|
| $y$ | $0$ | $9$ |
| $t$ | $0$ | $9^3$ |

$$\int_0^9 \frac{1}{4} y^2 \mathfrak{e}^{y^3} dy = \int_0^{9^3} \frac{1}{4} \cdot \frac{1}{3} \mathfrak{e}^t dt = \frac{1}{4} \cdot \frac{1}{3} [\mathfrak{e}^t]_0^{9^3} = \frac{\mathfrak{e}^{729} - 1}{12}$$

38) Evaluate the following integral by first reversing the order of integration.

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4 + 1} dx dy$$

The region of integral is the coloured area in the left hand-side figure.




The original limits of the integration is

$$0 \le y \le 8, \sqrt[3]{y} \le x \le 2$$

We now have to change these ranges so that the $x$ has limits made of the constant numbers and $y$ has the limits expressed using $x$. The area of integral can also be expressed as

$$0 \le x \le 2, 0 \le y \le x^3$$

In this way, the original integral is the same as

$$\int_0^2 \int_0^{x^3} \sqrt{x^4+1}dydx = \int_0^2 \sqrt{x^4+1}[y]_0^{x^3}dx = \int_0^2 \sqrt{x^4+1}x^3dx$$

Here when we introduce a variable $t \triangleq x^4+1$, we obtain $dt = 4x^3dx$.

| | | |
|---|---|---|
| $x$ | 0 | 2 |
| $t$ | 1 | $2^4+1$ |

$$\int_0^2 \sqrt{x^4+1}x^3dx = \int_1^{2^4+1} \frac{1}{4}\sqrt{t}dt = \frac{1}{4}\left[\frac{1}{1.5}t^{1.5}\right]_1^{17} = \frac{1}{4} \cdot \frac{17^{1.5}-1}{1.5} = \frac{17^{1.5}-1}{6}$$

**DAY4**

39) Using a parameter $t$, express the position vector of a point on the curve $\mathcal{C}$ where $\mathcal{C}$ is the segment of a parabola $y = x^2+4x-1$ joining the points $(-3,-4)$ and $(0,-1)$. Set the range of $t$ appropriately. When $x$ changes, $y$ changes on $y = x^2+4x-1$. The $x$ coordinate of two points is $-3$ and $0$. Therefore $-3 \leq x \leq 0$. When we set $x = t$, the range of $t$ is $-3 \leq t \leq 0$. As $y = x^2+4x-1$, $y$ can be expressed as $y = t^2+4t-1(\because t = x)$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t^2+4t-1 \end{pmatrix}$$

The range of $t$ is $-3 \leq t \leq 0$.

40) Using a parameter $t$, express the position vector of a point on the curve $\mathcal{C}$ where $\mathcal{C}$ is the left half of the circle $x^2+y^2 = 64$. The point traverses in the counter clockwise direction. Set the range of $t$ appropriately. As the curve is the circle with the radius of $8$ and its center is $(x,y) = (0,0)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8\cos t \\ 8\sin t \end{pmatrix}$$

The right half of the circle can be expressed as

$$\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$$

41) Using a parameter $t$, express the position vector of a point on the curve $\mathcal{C}$ where $\mathcal{C}$ is the line segment from $A(-2,-8)$ to $B(1,1)$. Set the range of $t$ appropriately. As $\mathcal{C}$ is a line on $xy$-plane, in general, a line can be expressed as $y = mx+b$. Since this line goes through $(x,y) = (-2,-8)$ and $(x,y) = (1,1)$

$$\begin{aligned} -8 &= -2m+b \qquad ① \\ 1 &= m+b \qquad ② \end{aligned}$$

①-② gives us $-9 = -3m$,*i.e.,* $m = 3$. When we put $m = 3$ into ②, we obtain $b = -2$. Thus the line is expressed as $y = 3x-2$. The $x$ coordinate of $A$ and $B$ are $-2$ and $1$, respectively which can be written as $-2 \leq x \leq 1$. When we set $x = t$, the range of $t$ is $-2 \leq t \leq 1$. As $y = 3x-2$, $y$ can be expressed as $y = 3t-2$ ($\because x = t$)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ 3x-2 \end{pmatrix}$$

As $x$ changes from $-2$ to $1$,

$$-2 \leq t \leq 1$$

42) Using a parameter $t$, express the position vector of a point on the line $\mathcal{C}$ where $\mathcal{C}$ is a straight line joining the points $A(0,0)$ and $B(2,4)$. Set the range of $t$ appropriately. The line which joins $A(0,0)$ and $B(2,4)$ can be written as $y = 2x$. The $x$ coordinate of $A$ and $B$ is $0$ and $2$, respectively. Thus when we set $x = t$, then the range of $t$ is $0 \leq t \leq 2$. As $y = 2x$, $y$ can be expressed as $y = 2t(\because t = x)$. Thus

$$x = t; y = 2t; 0 \leq t \leq 2$$

43) Using a parameter $t$, express the position vector of a point on the curve $\mathcal{C}$ where $\mathcal{C}$ is a the segment of a parabola $y = x^2$ joining the points $A(0,0)$ and $B(2,4)$. Set the range of $t$ appropriately. The curve which joins $A(0,0)$ and $B(2,4)$ is given as $y = x^2$. The $x$ coordinate of $A$ and $B$ is $0$ and $2$, respectively. Thus when we set $x = t$, then the range of $t$ is $0 \leq t \leq 2$. As $y = x^2$, $y$ can be expressed as $y = t^2(\because t = x)$. Thus

$$x = t; y = t^2; 0 \leq t \leq 2$$

44) Using a parameter $t$, express the position vector of a point on the curve $\mathcal{C}$ where $\mathcal{C}$ is a circle $(x-3)^2+(y-4)^2 = 25$. The point traverses in the clockwise direction. Set the range of $t$ appropriately.

$$x-3 = 5\cos t \ ; \ y-4 = 5\sin t \ ; \ \therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3+5\cos t \\ 4+5\sin t \end{pmatrix}$$

Since $t$ is measured from $x$-positive axis anti-clock wise $(0 \to 2\pi)$, in order to move clockwise, we need to set the movement of $t$ as $0 \le t \le -2\pi$.

45) Using a parameter $t$, express the position , express the position vector of a point on the curve $\mathcal{C}$ where $\mathcal{C}$ is a broken line passing through the points $A(0,0)$, $B(0,1)$, and $C(1,1)$ in the order of $A \longrightarrow B \longrightarrow C$. Set the range of $t$ appropriately.

From $A \to B$, $x$ is constant and $y$ changes. Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix}$$

The range of $t$ is $0 \le t \le 1$. From $B \to C$, $y$ is constant and $x$ changes. Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix}$$

The range of $t$ is $0 \le t \le 1$.

46) Using a parameter $t$, express the position vector of a point on the curve $\mathcal{C}$ where $\mathcal{C}$ is the line segment on $y = x^2$ for $-2 \le x \le 3$. Set the range of $t$ appropriately. The curve is given as $y = x^2$. The range of $x$ is also given as $-2 \le x \le 3$. Thus when we set $x = t$, then the range of $t$ is $-2 \le t \le 3$. As $y = x^2$, $y$ can be expressed as $y = t^2 (\because t = x)$. Thus

$$x = t; y = t^2; -2 \le t \le 3$$

**DAY5**

47) Using a parameter $t$, express the position vector of a point on the line $\mathcal{C}$ where $\mathcal{C}$ is a straight line joining the points $A(0,0)$ and $B(1,1)$. Set the range of $t$ appropriately. The line which joins $A(0,0)$ and $B(1,1)$ can be written as $y = x$. The $x$ coordinate of $A$ and $B$ is $0$ and $1$, respectively. Thus when we set $x = t$, then the range of $t$ is $0 \le t \le 1$. As $y = x$, $y$ can be expressed as $y = t (\because t = x)$. Thus

$$x = t; y = t; 0 \le t \le 1$$

48) Using a parameter $t$, express the position vector of a point on the curve $\mathcal{C}$ where $\mathcal{C}$ is a the segment of a parabola $y = x^2$ joining the points $A(0,0)$ and $B(1,1)$. Set the range of $t$ appropriately. The curve which joins $A(0,0)$ and $B(1,1)$ is given as $y = x^2$. The $x$ coordinate of $A$ and $B$ is $0$ and $1$, respectively. Thus when we set $x = t$, then the range of $t$ is $0 \le t \le 1$. As $y = x^2$, $y$ can be expressed as $y = t^2 (\because t = x)$. Thus

$$x = t; y = t^2; 0 \le t \le 1$$

49) Using a parameter $t$, express the position vector of a point on the curve $\mathcal{C}$ where $\mathcal{C}$ is a circle $(x-1)^2+(y-1)^2 = 4$. The point traverses in the clockwise direction. Set the range of $t$ appropriately.

$$x - 1 = 2\cos t \ ; \ y - 1 = 2\sin t \ ; \ \therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + 2\cos t \\ 1 + 2\sin t \end{pmatrix}$$

Since $t$ is measured from $x$-positive axis anti-clock wise ($0 \to 2\pi$), in order to move clockwise, we need to set the movement of $t$ as $0 \le t \le -2\pi$.

50) Using a parameter $t$, express the position , express the position vector of a point on the curve $\mathcal{C}$ where $\mathcal{C}$ is a broken line passing through the points $A(0,0)$, $B(1,0)$, and $C(1,1)$ in the order of $A \longrightarrow B \longrightarrow C$. Set the range of $t$ appropriately.
From $A \to B$, $y$ is constant and $x$ changes. Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix}$$

The range of $t$ is $0 \le t \le 1$. From $B \to C$, $x$ is constant and $y$ changes. Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix}$$

The range of $t$ is $0 \le t \le 1$.

51) Using a parameter $t$, express the position vector of a point on the curve $\mathcal{C}$ where $\mathcal{C}$ is the line segment on $y = x^2$ for $-1 \le x \le 1$. Set the range of $t$ appropriately. The curve is given as $y = x^2$. The range of $x$ is also given as $-1 \le x \le 1$. Thus when we set $x = t$, then the range of $t$ is $-1 \le t \le 1$. As $y = x^2$, $y$ can be expressed as $y = t^2 (\because t = x)$. Thus

$$x = t; y = t^2; -1 \le t \le 1$$

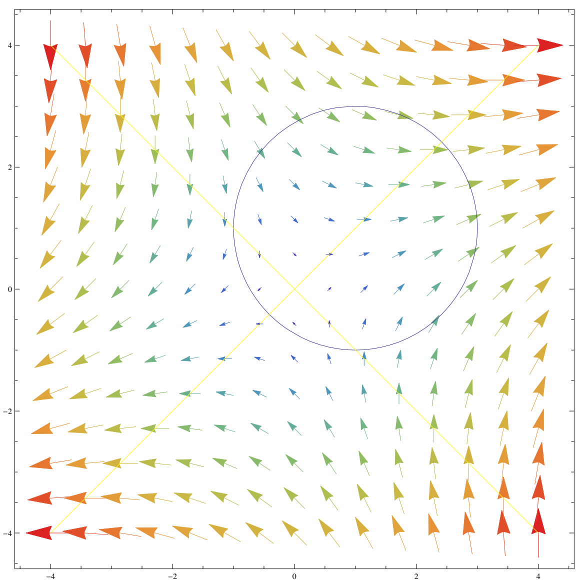
52) Find

$$\oint_C [(x + y)dx + (x - y)dy]$$

where $C$ is a circle

$$(x - 1)^2 + (y - 1)^2 = 4$$

traversed in the clockwise direction.



a) Express $x, y, z$ using $t$

$$x - 1 = 2\cos t \;\; ; \quad ; \;\; y - 1 = 2\sin t \;\; ; \;\; 0 \le t \le -2\pi$$

Since $t$ is measured from $x$-positive axis anti-clock wise ($0 \to 2\pi$), in order to move clockwise, we need to set the movement of $t$ as $0 \le t \le -2\pi$.

b) Express $\mathbf{F}$ as the function of $t$ Using $x = 2\cos t + 1$ and $y = 2\sin t + 1$

$$\mathbf{F} = \begin{pmatrix} x + y \\ x - y \end{pmatrix} = \begin{pmatrix} 2\cos t + 2\sin t + 2 \\ 2\cos t - 2\sin t \end{pmatrix}$$

c) Express $\dfrac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \dfrac{d\{x\}}{dt} \\ \dfrac{d\{y\}}{dt} \end{pmatrix}$ using $t$

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \dfrac{d\{2\cos t + 1\}}{dt} \\ \dfrac{d\{2\sin t + 1\}}{dt} \end{pmatrix} = \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix}$$

d) Put all of them into $\int \mathbf{F} \cdot \dfrac{d\{\boldsymbol{r}\}}{dt} dt$

$$\int_0^{-2\pi} \begin{pmatrix} 2\cos t + 2\sin t + 2 \\ 2\cos t - 2\sin t \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix} dt$$

$$= \int_0^{-2\pi} (-4\cos t \sin t - 4\sin^2 t - 4\sin t + 4\cos^2 t - 4\cos t \sin t) dt$$

$$= \int_0^{-2\pi} (-8\cos t \sin t - 4\sin^2 t + 4\cos^2 t - 4\sin t)dt$$

$$= \int_0^{-2\pi} (-4\sin 2t - 4\frac{1-\cos 2t}{2} + 4\frac{1+\cos 2t}{2} - 4\sin t)dt$$

$$= \int_0^{-2\pi} (-4\sin 2t - 2 + 2\cos 2t + 2 + 2\cos 2t - 4\sin t)dt$$

$$= \int_0^{-2\pi} (-4\sin 2t + 4\cos 2t - 4\sin t)dt = [2\cos 2t + 2\sin 2t + 4\cos t]_0^{-2\pi} = 0$$

53) Solve the equation $\log(x) - \log(x^2-1) = -2\log(x-1)$
We work under the condition of $x > 0$, $x^2 - 1 > 0$, and $x - 1 > 0$.

$$\log(x) - \log(x^2-1) = -2\log(x-1) \ ; \quad \therefore \log(x) - \log(x^2-1) + 2\log(x-1) = 0$$

$$\therefore \log(x) - \log(x^2-1) + \log(x-1)^2 = 0 \ ; \quad \therefore \log\frac{x(x-1)^2}{x^2-1} = \log 1$$

$$\therefore \frac{x(x-1)^2}{x^2-1} = 1 \ ; \quad \therefore x(x-1)^2 = x^2 - 1$$

$$\therefore x(x^2-2x+1) = x^2 - 1 \ ; \quad \therefore x^3 - 2x^2 + x = x^2 - 1$$

$$\therefore x^3 - 3x^2 + x + 1 = 0$$

Since we can easily tell that $x = 1$ is one of the three answers, wen can factorise the equation as

$$x^3 - 3x^2 + x + 1 = 0 \ ; \quad \therefore (x-1)(x^2-2x-1) = 0 \ ; \quad \therefore x^2 - 2x - 1 = 0 (\because x - 1 > 0)$$

$$\therefore x = \frac{2 \pm \sqrt{2^2+4}}{2} \ ; \quad \therefore x = 1 \pm \sqrt{2} \ ; \quad \therefore x = 1 + \sqrt{2} (\because x > 1)$$

54) Solve the equation $\log_2(2x-9) = 2 - \log_2(x-1)$
We work under the condition of $2x - 9 > 0$ and $x - 1 > 0$

$$\log_2(2x-9) = 2 - \log_2(x-1) = 2\log_2 2 - \log_2(x-1) = \log_2 2^2 - \log_2(x-1) = \log_2\frac{2^2}{x-1}$$

$$\therefore 2x - 9 = \frac{2^2}{x-1} \ ; \quad \therefore (2x-9)(x-1) = 2^2 \ ; \quad \therefore 2x^2 - 2x - 9x + 9 = 4 \ ; \quad \therefore 2x^2 - 11x + 5 = 0$$

$$\therefore x = \frac{11 \pm \sqrt{11^2 - 4\cdot 2\cdot 5}}{2\cdot 2} \ ; \quad \therefore x = \frac{11 \pm \sqrt{81}}{4} \ ; \quad \therefore x = 5 (\because 2x - 9 > 0)$$

55) Solve the equation

$$\log_x\left(\frac{1}{9}\right) = 2$$

We work under the condition of $x > 0$.

$$\log_x\left(\frac{1}{9}\right) = 2 = 2\log_x x = \log_x x^2$$

$$\therefore \frac{1}{9} = x^2 \ ; \quad \therefore x = \pm\frac{1}{3} \ ; \quad \therefore x = \frac{1}{3} (\because x > 0)$$

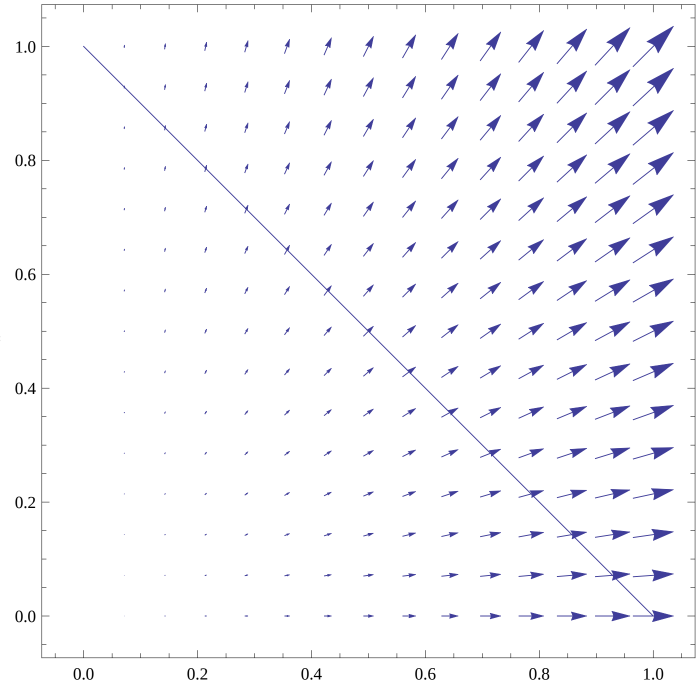
56) Evaluate

$$\int_C (x^2 dx + xy dy)$$

over the following path

$$x = 1 - t$$
$$y = t$$
$$0 \le t \le 1$$

Out[16]=




a) Express $x, y, z$ using $t$

$$x = 1 - t \ \ ; \ \ y = t$$

b) Express $\mathbf{F}$ as the function of $t$

$$\mathbf{F} = \begin{pmatrix} x^2 \\ xy \end{pmatrix} = \begin{pmatrix} (1-t)^2 \\ t(1-t) \end{pmatrix}$$

c) Express $\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \end{pmatrix}$ using $t$

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{1-t\}}{dt} \\ \frac{d\{t\}}{dt} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

d) Put all of them into $\int \mathbf{F} \cdot \frac{d\{\boldsymbol{r}\}}{dt} dt$

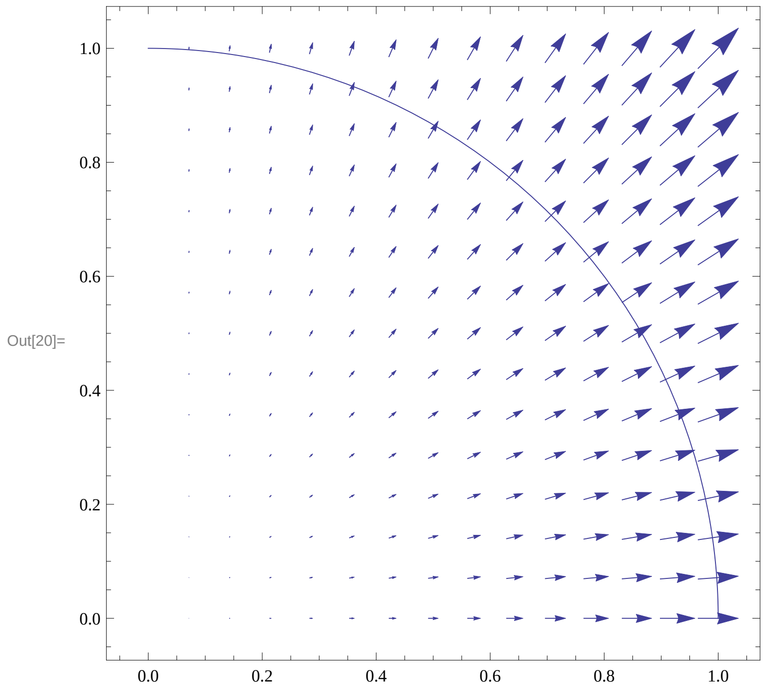
$$\int_0^1 \begin{pmatrix} (1-t)^2 \\ t(1-t) \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} dt = \int_0^1 \left(-(1-t)^2 + t(1-t)\right) dt = \int_0^1 \left(-(1-t)(1-t) + t(1-t)\right) dt$$
$$= \int_0^1 \left(-(1-2t+t^2) + t - t^2\right) dt = \int_0^1 \left(-1 + 2t - t^2 + t - t^2\right) dt = \int_0^1 \left(-1 + 3t - 2t^2\right) dt$$
$$= \left[\frac{-2t^3}{3} - t + \frac{3t^2}{2}\right]_0^1 = \frac{-2}{3} - 1 + \frac{3}{2} = \frac{-4}{6} - \frac{6}{6} + \frac{9}{6} = \frac{-1}{6}$$

57) Evaluate

$$\int_C (x^2 dx + xydy)$$

over the following path

$$x = \cos t \quad ; \quad y = \sin t \quad ; \quad 0 \le t \le \pi/2$$




a) Express $x, y, z$ using $t$

$$x = \cos t \quad ; \quad y = \sin t$$

b) Express $\mathbf{F}$ as the function of $t$

$$\mathbf{F} = \begin{pmatrix} x^2 \\ xy \end{pmatrix} = \begin{pmatrix} \cos^2 t \\ \sin(t)\cos(t) \end{pmatrix}$$

c) Express $\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \end{pmatrix}$ using $t$

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{\cos t\}}{dt} \\ \frac{d\{\sin t\}}{dt} \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

d) Put all of them into $\int \mathbf{F} \cdot \frac{d\{\boldsymbol{r}\}}{dt} dt$

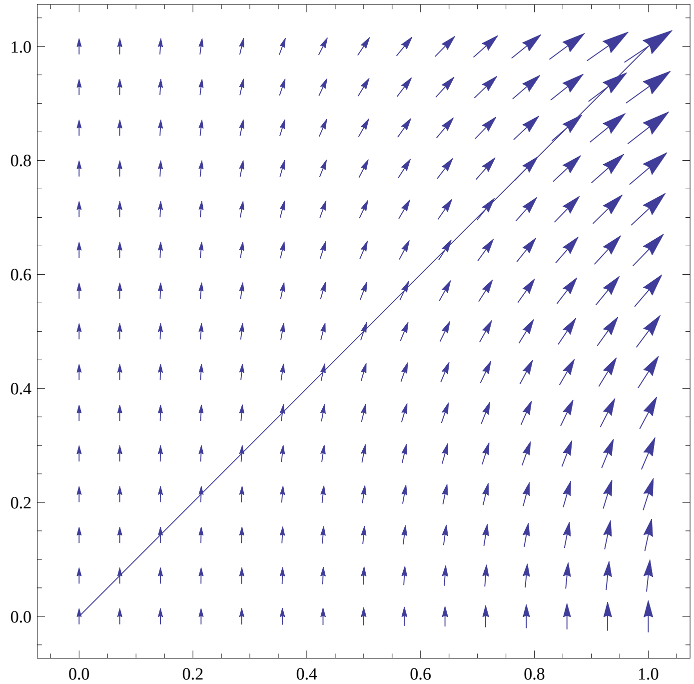
$$\int_0^{\pi/2} \begin{pmatrix} \cos^2 t \\ \sin(t)\cos(t) \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} dt = \int_0^{\pi/2} -\sin t \cos^2 t + \sin(t)\cos^2(t) dt = \int_0^{\pi/2} 0 dt = 0$$

58) Find

$$\int_C [3x^2 y dx + (x^3 + 1) dy]$$

over the following path: the segment of a straight line joining the points $(0,0)$ and $(1,1)$

Out[24]=




a) Express $x, y, z$ using $t$

$$x = t \ ; \ y = t \ ; \ 0 \leq t \leq 1$$

b) Express $\mathbf{F}$ as the function of $t$ Using $x = t$ and $y = t$

$$\mathbf{F} = \begin{pmatrix} 3x^2y \\ x^3+1 \end{pmatrix} = \begin{pmatrix} 3t^3 \\ t^3+1 \end{pmatrix}$$

c) Express $\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \end{pmatrix}$ using $t$

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{t\}}{dt} \\ \frac{d\{t\}}{dt} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

d) Put all of them into $\int \mathbf{F} \cdot \frac{d\{\boldsymbol{r}\}}{dt} dt$

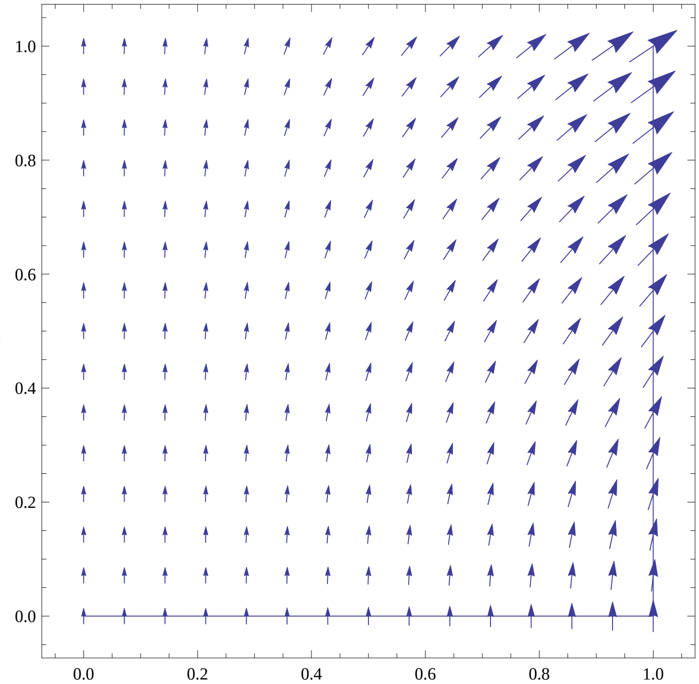
$$\int_0^1 \begin{pmatrix} 3t^3 \\ t^3+1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} dt = \int_0^1 (3t^3 + t^3 + 1) dt = \int_0^1 (4t^3+1) dt = [t^4 + t]_0^1 = 2$$

59) Find

$$\int_C [3x^2y dx + (x^3+1) dy]$$

over the following path: a broken line passing through the points $A(0,0)$, $B(1,0)$, and $C(1,1)$

Out[29]=




a) Express $x, y, z$ using $t$

$$x = t \ ; \ y = 0 \ ; \ 0 \leq t \leq 1$$

for $A \to B$ and

$$x = 1 \ ; \ y = t \ ; \ 0 \leq t \leq 1$$

for $B \to C$.

b) Express $\mathbf{F}$ as the function of $t$ For $A \to B$, using $x = t$ and $y = 0$

$$\mathbf{F} = \begin{pmatrix} 3x^2y \\ x^3 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ t^3 + 1 \end{pmatrix}$$

For $B \to C$, using $x = 1$ and $y = t$

$$\mathbf{F} = \begin{pmatrix} 3x^2y \\ x^3 + 1 \end{pmatrix} = \begin{pmatrix} 3t \\ 2 \end{pmatrix}$$

c) Express $\dfrac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \dfrac{d\{x\}}{dt} \\ \dfrac{d\{y\}}{dt} \end{pmatrix}$ using $t$

For $A \to B$

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \dfrac{d\{t\}}{dt} \\ \dfrac{d\{0\}}{dt} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For $B \to C$

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \dfrac{d\{1\}}{dt} \\ \dfrac{d\{t\}}{dt} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

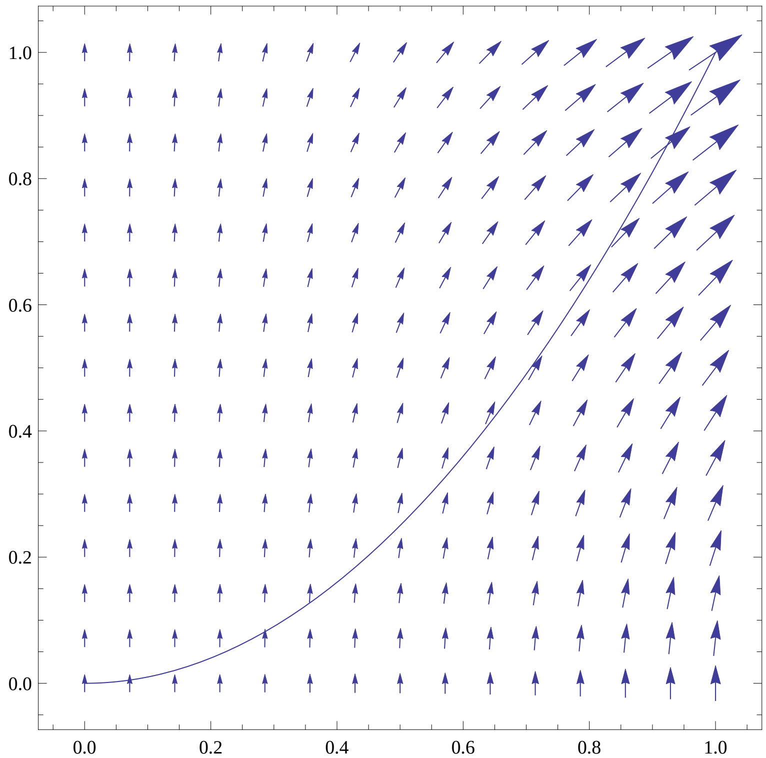
d) Put all of them into $\int \mathbf{F} \cdot \dfrac{d\{\boldsymbol{r}\}}{dt} dt$

$$\int_0^1 \begin{pmatrix} 0 \\ t^3 + 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \int_0^1 \begin{pmatrix} 3t \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = \int_0^1 2dt = [2t]_0^1 = 2$$

60) Find

$$\int_C [3x^2 y dx + (x^3+1)dy]$$

over the following path: the segment of a parabola $y = x^2$ joining the points $(0,0)$ and $(1,1)$




a) Express $x, y$ using $t$

$$x = t \;\; ; \;\; y = t^2 \;\; ; \;\; 0 \le t \le 1$$

b) Express $\mathbf{F}$ as the function of $t$ Using $x = t$ and $y = t^2$

$$\mathbf{F} = \begin{pmatrix} 3x^2 y \\ x^3 + 1 \end{pmatrix} = \begin{pmatrix} 3t^4 \\ t^3 + 1 \end{pmatrix}$$

c) Express $\dfrac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \dfrac{d\{x\}}{dt} \\ \dfrac{d\{y\}}{dt} \end{pmatrix}$ using $t$

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \dfrac{d\{t\}}{dt} \\ \dfrac{d\{t^2\}}{dt} \end{pmatrix} = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

d) Put all of them into $\int \mathbf{F} \cdot \dfrac{d\{\boldsymbol{r}\}}{dt} dt$

$$\int_0^1 \begin{pmatrix} 3t^4 \\ t^3 + 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt = \int_0^1 (3t^4 + 2t(t^3+1))dt$$

$$= \int_0^1 (3t^4 + 2t^4 + 2t)dt = \int_0^1 (5t^4 + 2t)dt = [t^5 + t^2]_0^1 = 2$$

61) Evaluate

$$\int_C (x^3 dx + xydy)$$

over the following path

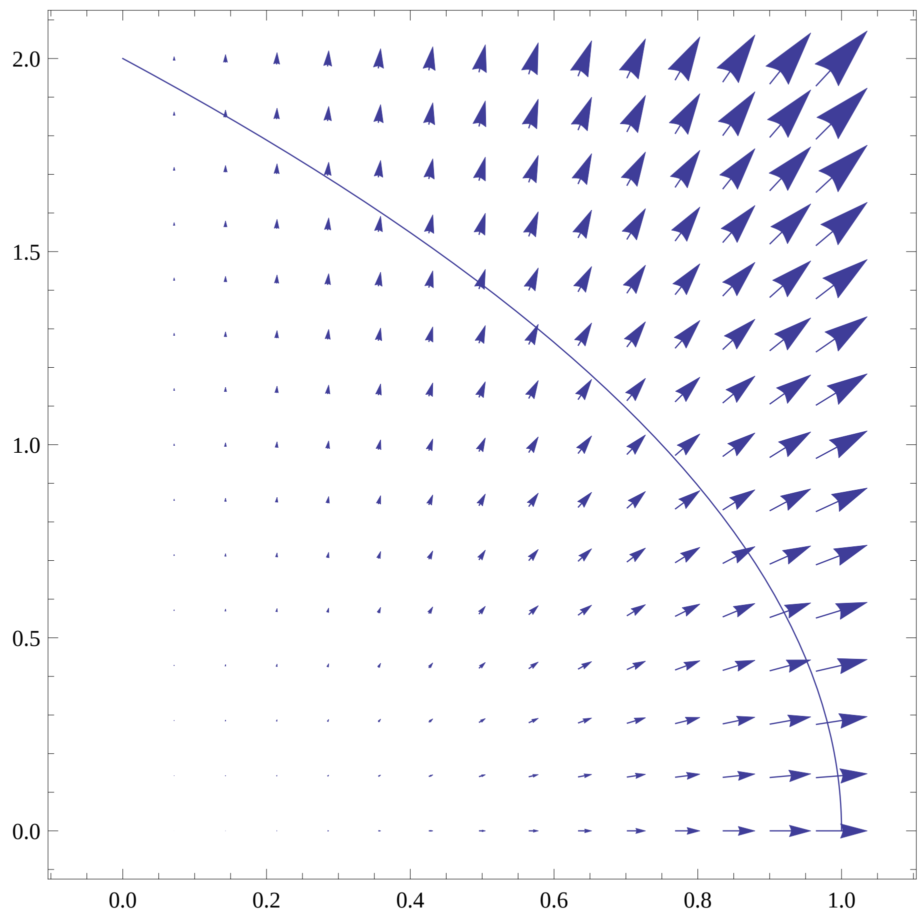
$$x = 1 - t^2 \quad ; \quad y = 2t \quad ; \quad 0 \le t \le 1$$

a) Express $x, y, z$ using $t$
To help us solve this, first we must express $x$ and $y$ in terms of $t$.

$$x = 1 - t^2 \;\; ; \;\; y = 2t$$

b) Express **F** as the function of $t$
Now we need to express **F** as a function of $t$, using vector form.

$$\mathbf{F} = \begin{pmatrix} x^3 \\ xy \end{pmatrix} = \begin{pmatrix} (1-t^2)^3 \\ (1-t^2)(2t) \end{pmatrix}$$



c) Express $\frac{d\{\mathbf{r}\}}{dt}$ as a function of $\begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \end{pmatrix}$ using $t$

$$\frac{d\{\mathbf{r}\}}{dt} = \begin{pmatrix} \frac{d\{1-t^2\}}{dt} \\ \frac{d\{2t\}}{dt} \end{pmatrix} = \begin{pmatrix} -2t \\ 2 \end{pmatrix}$$

d) Put all of them into $\int \mathbf{F} \cdot \frac{d\{\mathbf{r}\}}{dt} dt$

Using the Formula $\int \mathbf{F} \cdot \frac{d\{\mathbf{r}\}}{dt} dt$

$$\int_0^1 \begin{pmatrix} (1-t^2)^3 \\ (1-t^2)(2t) \end{pmatrix} \cdot \begin{pmatrix} -2t \\ 2 \end{pmatrix} dt = \int_0^1 (-2t(1-t^2)^3) + 2(1-t^2)(2t)dt$$

Split the integral up

$$\int_0^1 (-2t(1-t^2)^3)dt + \int_0^1 2(1-t^2)(2t)dt$$

Let $u = 1 - t^2$

| $t$ | $0$ | $\rightarrow$ | $1$ |
|---|---|---|---|
| $u = 1 - t^2$ | $1$ | $\rightarrow$ | $0$ |

Therefore $\frac{d\{u\}}{dt} = -2t$

The first term can be handled as follows:

$$\int_0^1 (-2t(1-t^2)^3)dt = \int_1^0 (-2t(1-t^2)^3)dt = \int_{t=0}^{t=1} \frac{-2tu^3}{-2t} du = \int_1^0 u^3 du = \left[\frac{1}{4}u^4\right]_1^0 = \frac{-1}{4}$$

The second term can be handled as follows:

$$\int_0^1 2(1-t^2)(2t)dt = \int_0^1 4t(1-t^2)dt = \int_1^0 \frac{4tu}{-2t} du = -\int_1^0 -2udu = \left[-u^2\right]_1^0 = 1$$

Therefore the answer is

$$\int_0^1 (-2t(1-t^2)^3)dt + \int_0^1 2(1-t^2)(2t)dt = \frac{-1}{4} + 1 = \frac{3}{4}$$

**DAY6**

62) Using a parameter $t$, express the position vector of a point on the curve $\mathcal{C}$ where $\mathcal{C}$ is the segment of a parabola $y = x^2$ joining the points $(0,0)$ and $(2,4)$. Set the range of $t$ appropriately. When $x$ changes, $y$ changes on $y = x^2$. Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

As $x$ changes from 0 to 2 and $x = t$, the range of $t$ is $0 \le t \le 2$.

63) Using a parameter $t$, express the position vector of a point on the curve $\mathcal{C}$ where $\mathcal{C}$ is the right half of the circle $x^2 + y^2 = 16$. The point traverses in the counter clockwise direction. Set the range of $t$ appropriately. As the curve is the circle with the radius of $4$ and its center is $(x, y) = (0, 0)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4\cos t \\ 4\sin t \end{pmatrix}$$

The right half of the circle can be expressed as

$$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

64) Using a parameter $t$, express the position vector of a point on the curve $\mathcal{C}$ where $\mathcal{C}$ is the line segment from $A(-2,-1)$ to $B(1,2)$. Set the range of $t$ appropriately. As $\mathcal{C}$ is a line on $xy$-plane, in general, a line can be expressed as $y = mx + b$. Since this line goes through $(x, y) = (-2, -1)$ and $(x, y) = (1, 2)$

$$\begin{aligned} -1 &= -2m + b \qquad ① \\ 2 &= m + b \qquad ② \end{aligned}$$

①-② gives us $-3 = -3m$,*i.e.,* $m = 1$. When we put $m = 1$ into ②, we obtain $b = 1$. Thus the line is expressed as $y = x + 1$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t+1 \end{pmatrix}$$

As $x$ changes from $-2$ to $1$,

$$-2 \le t \le 1$$

Note:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2+t \\ -1+t \end{pmatrix}$$

with $0 \le t \le 3$ is also correct.

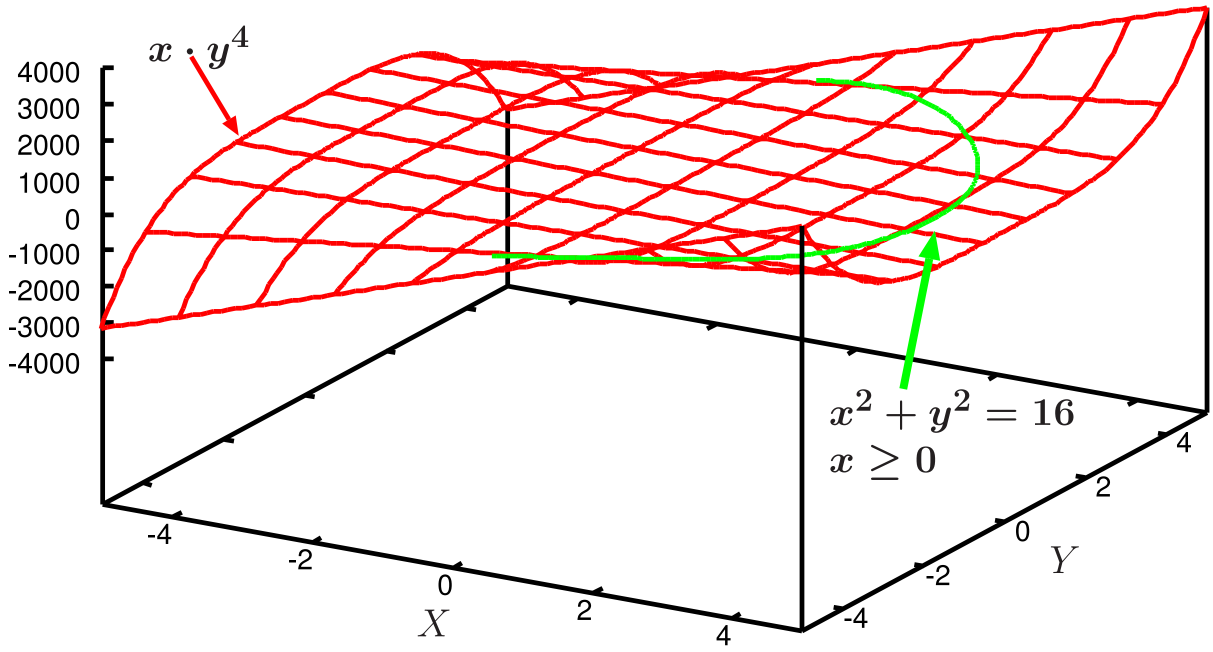
65) Evaluate

$$\int_C xy^4 ds$$

where $C$ is the right half of the circle

$$x^2 + y^2 = 16$$

rotated in the counter clockwise direction.




a) Express $x, y, z$ using $t$

$$x = 4\cos t$$
$$y = 4\sin t$$
$$-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$
because of half a circle

which satisfies $x^2 + y^2 = 16$.

b) Express $f(x, y)$ as the function of $t$ Using $x = 4\cos t$ and $y = 4\sin t$

$$f(x, y) = xy^4 = 4\cos t \cdot (4\sin t)^4 = 4^5 \cos t \sin^4 t$$

c) Express $\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \end{pmatrix}$ using $t$

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{4\cos t\}}{dt} \\ \frac{d\{4\sin t\}}{dt} \end{pmatrix} = \begin{pmatrix} -4\sin t \\ 4\cos t \end{pmatrix}$$

d) Put all of them into

$$\int_{t=a}^{t=b} f(x, y, z) \cdot \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2} \cdot dt$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^5 \cos t \sin^4 t \sqrt{(-4\sin t)^2 + (4\cos t)^2} dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^5 \cos t \sin^4 t \sqrt{16\sin^2 t + 16\cos^2 t} dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^5 \cos t \sin^4 t \sqrt{16(\sin^2 t + \cos^2 t)} dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^5 \cos t \sin^4 t \sqrt{16 \cdot 1} dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^5 \cos t \sin^4 t \cdot 4 dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^6 \cos t \sin^4 t dt$$

When we define $\theta = \sin^5 t$, we obtain $d\theta = 5\sin^4 t \cos t dt$.

| $t$ | $-\frac{\pi}{2}$ | $\frac{\pi}{2}$ |
|---|---|---|
| $(\sin t)^5$ | $(-1)^5$ | $(1)^5$ |
| $\theta$ | -1 | 1 |

Thus

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4^6 \cos t \sin^4 t dt = 4^6 \int_{-1}^{1} \frac{d\theta}{5} = \frac{4^6}{5}[\theta]_{-1}^{1} = \frac{4^6}{5} \cdot 2 = \frac{8192}{5}$$

66) Simplify $(-8a^2b^{-3})^2$

$$(-8a^2b^{-3})^2 = (-8)^2 a^{2\times 2} b^{-3\times 2} = 64a^4b^{-6}$$

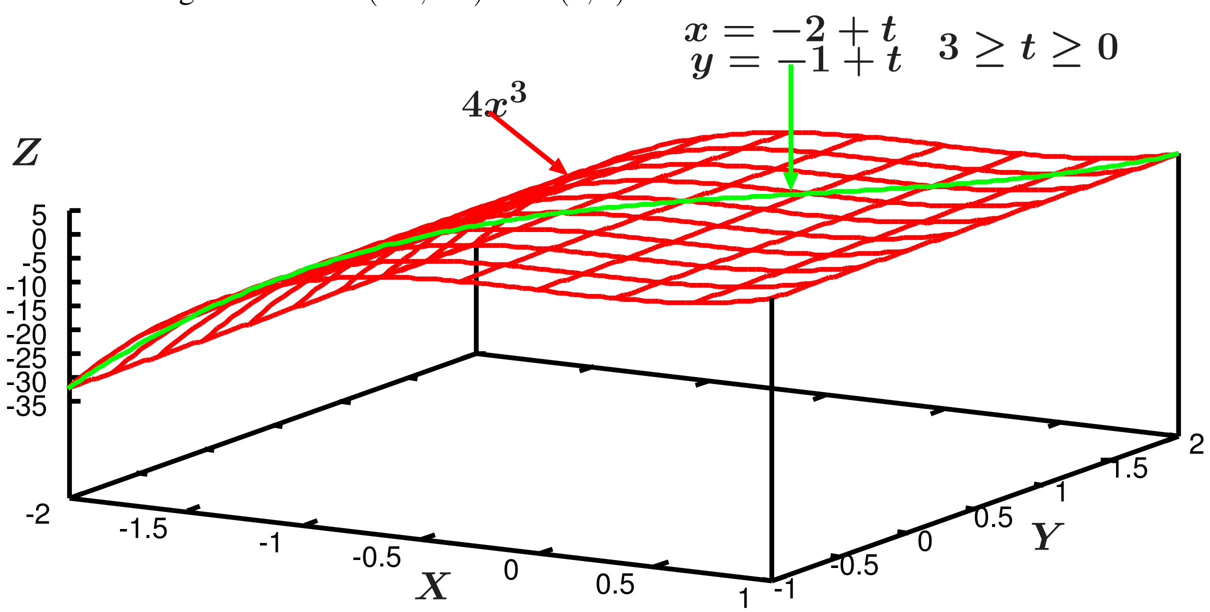
67) Find the $\int 0dt$.

$$\int 0dt = c \because \frac{d\{c\}}{dx} = 0$$

68) Evaluate

$$\int_C 4x^3 ds$$

where $C$ is the line segment from $A(-2,-1)$ to $B(1,2)$




Solution 1

a) Express $x, y, z$ using $t$
Since $x$, $y$ are both increasing 3 from the point $A$ to $B$, we can write

$$x = t \;\; ; \;\; y = t + 1 \;\; ; \;\; -2 \leq t \leq 1$$

b) Express $f(x,y)$ as the function of $t$ Using $x = t$ and $y = t + 1$

$$f(x,y) = 4x^3 = 4 \cdot t^3$$

c) Express $\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \end{pmatrix}$ using $t$

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{t\}}{dt} \\ \frac{d\{t+1\}}{dt} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

d) Put all of them into

$$\int_{t=a}^{t=b} f(x,y,z)\cdot\sqrt{\left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2}\cdot dt$$

$$\int_{-2}^{1} 4\cdot t^3\sqrt{1^2+1^2}dt=\int_{-2}^{1} 4\cdot t^3\sqrt{2}dt=[\sqrt{2}t^4]_{-2}^{1}=\sqrt{2}(1-(-2)^4)=-15\sqrt{2}$$

Solition 2

a) Express $x,y,z$ using $t$
Since $x$, $y$ are both increasing 3 from the point $A$ to $B$, we can write

$$x=-2+t\ \ ;\ \ y=-1+t\ \ ;\ \ 0\le t\le 3$$

b) Express $f(x,y)$ as the function of $t$ Using $x=-2+t$ and $y=-1+t$

$$f(x,y)=4x^3=4\cdot(t-2)^3$$

c) Express $\frac{d\{\boldsymbol{r}\}}{dt}=\begin{pmatrix}\frac{d\{x\}}{dt}\\ \frac{d\{y\}}{dt}\end{pmatrix}$ using $t$

$$\frac{d\{\boldsymbol{r}\}}{dt}=\begin{pmatrix}\frac{d\{-2+t\}}{dt}\\ \frac{d\{-1+t\}}{dt}\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

d) Put all of them into

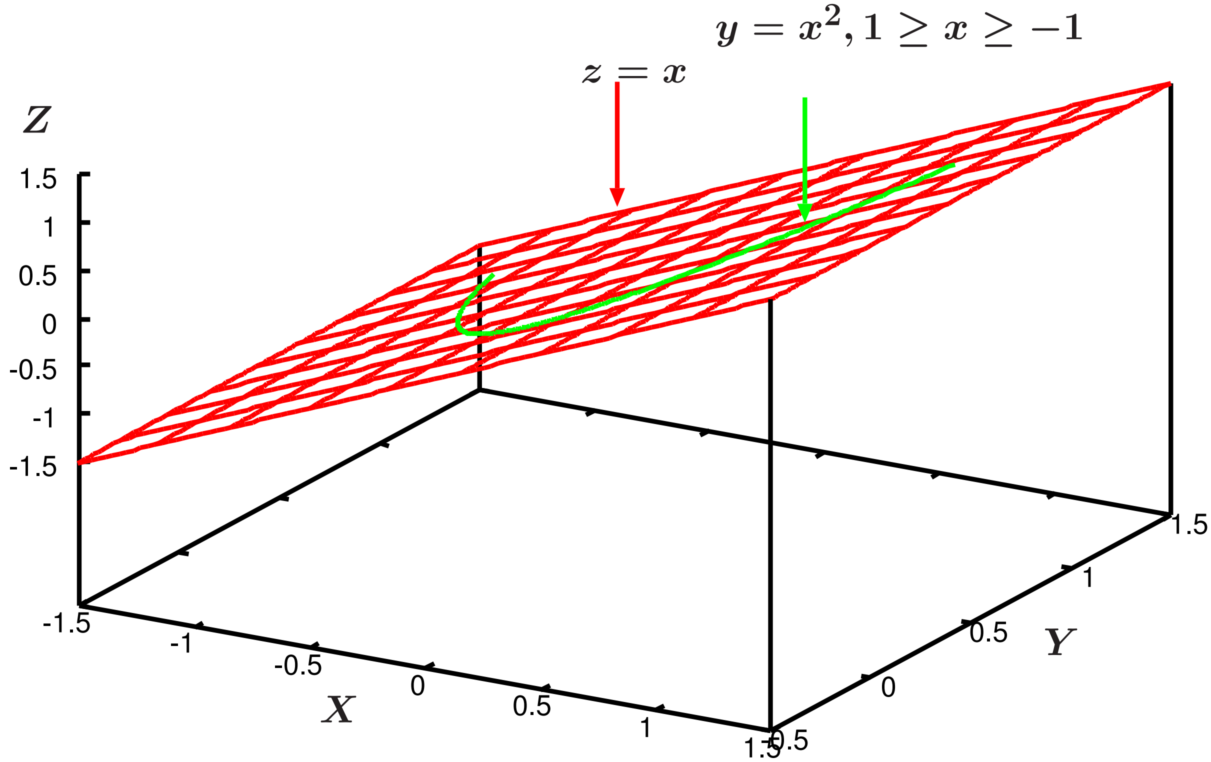
$$\int_{t=a}^{t=b} f(x,y,z)\cdot\sqrt{\left(\frac{\partial x}{\partial t}\right)^2+\left(\frac{\partial y}{\partial t}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2}\cdot dt$$

$$\int_{0}^{3} 4\cdot(t-2)^3\sqrt{1^2+1^2}dt=\int_{0}^{3} 4\cdot(t-2)^3\sqrt{2}dt=[\sqrt{2}(t-2)^4]_{0}^{3}=\sqrt{2}(1-2^4)=-15\sqrt{2}$$

69) Evaluate

$$\int_C xds$$

where $C$ is the line segment on $y=x^2$ for $-1\le x\le 1$




a) Express $x, y, z$ using $t$

$$x = t \;\; ; \;\; y = t^2 \;\; ; \;\; -1 \leq t \leq 1$$

b) Express $f(x, y)$ as the function of $t$ Using $x = t$ and $y = t^2$

$$f(x, y) = x = t$$

c) Express $\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \end{pmatrix}$ using $t$

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{t\}}{dt} \\ \frac{d\{t^2\}}{dt} \end{pmatrix} = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$$

d) Put all of them into

$$\int_{t=a}^{t=b} f(x, y, z) \cdot \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2} \cdot dt$$

$$\int_{-1}^{1} t\sqrt{1 + (2t)^2} dt = \int_{-1}^{1} t\sqrt{4t^2 + 1} dt$$

When $\theta \triangleq 4t^2 + 1$, we obtain $d\theta = 8tdt$.

| $t$ | $-1$ | $1$ |
|---|---|---|
| $4t^2 + 1$ | $4 \cdot (-1)^2 + 1$ | $4 \cdot (1)^2 + 1$ |
| $\theta$ | 5 | 5 |

Thus

$$\int_{-1}^{1} t\sqrt{4t^2 + 1} dt = \int_{5}^{5} \theta^{0.5} \frac{d\theta}{8} = \left[\frac{1}{1.5 \cdot 8} \theta^{1.5}\right]_5^5 = 0$$

70) Evaluate

$$\int_C xyzds$$

where $C$ is the helix given by $\begin{pmatrix} \cos t \\ \sin t \\ 3t \end{pmatrix}$ with $0 \le t \le 4\pi$.

a) Express $x, y, z$ using $t$

$$\begin{aligned} x &= \cos t \\ y &= \sin t \\ z &= 3t \\ 0 \le t &\le 4\pi \end{aligned}$$

b) Express $f(x, y, z)$ as the function of $t$ Using $x = \cos t$ and $y = \sin t$ and $z = 3t$,

$$f(x, y) = xyz = 3t \cos t \sin t = 1.5t \sin 2t$$

c) Express $\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \\ \frac{d\{z\}}{dt} \end{pmatrix}$ using $t$

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{\cos t\}}{dt} \\ \frac{d\{\sin t\}}{dt} \\ \frac{d\{3t\}}{dt} \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \\ 3 \end{pmatrix}$$

d) Put all of them into

$$\int_{t=a}^{t=b} f(x, y, z) \cdot \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2} \cdot dt$$

$$\int_0^{4\pi} 1.5t \sin 2t \sqrt{(-\sin t)^2 + (\cos t)^2 + 9} dt = 1.5\sqrt{10} \int_0^{4\pi} t \sin 2t dt$$

Now we need to evaluate $\int t \sin 2t dt$. Use by parts

$$\int t \sin 2t dt = t \int \sin 2t dt - \int 1 \cdot \left(\int \sin 2t dt\right) dt = t \frac{-\cos 2t}{2} - \int \left(\frac{-\cos 2t}{2}\right) dt$$
$$= \frac{-t \cos 2t}{2} - \frac{-\sin 2t}{2 \cdot 2} = \frac{-t \cos 2t}{2} + \frac{\sin 2t}{2 \cdot 2} = \frac{-2t \cos 2t + \sin 2t}{4}$$

Putting this result into the original integral, we get

$$1.5\sqrt{10} \int_0^{4\pi} t \sin 2t dt = \frac{1.5\sqrt{10}}{4} [-2t \cos 2t + \sin 2t]_0^{4\pi} = \frac{1.5\sqrt{10}}{4} \cdot (-8\pi) = -3\sqrt{10}\pi$$

**DAY7**

71) A loudspeaker cone is generated by rotating the curve $y = \cosh x - 1$ about the $x-$ axis through $2\pi$ radians from $x = 0$ to $x = 1$. Calculate the surface area of the cone excluding the two ends.

[5 marks]

Since $\cosh x = \frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}$, $\frac{d\{\cosh x\}}{dx} = \frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2}$. Surface area is

$$\int_0^1 (2\pi y)\sqrt{dx^2 + dy^2} = \int_0^1 (2\pi y)\sqrt{1 + \left(\frac{d\{y\}}{dx}\right)^2}dx = \int_0^1 (2\pi y)\sqrt{1 + \left(\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2}\right)^2}dx$$

$$= \int_0^1 (2\pi y)\sqrt{1 + \frac{\mathfrak{e}^{2x} + \mathfrak{e}^{-2x} - 2}{4}}dx = \int_0^1 (2\pi y)\sqrt{\frac{4 + \mathfrak{e}^{2x} + \mathfrak{e}^{-2x} - 2}{4}}dx = \int_0^1 (2\pi y)\sqrt{\frac{\mathfrak{e}^{2x} + \mathfrak{e}^{-2x} + 2}{4}}dx$$

$$= \int_0^1 (2\pi y)\sqrt{\frac{\mathfrak{e}^{2x} + \mathfrak{e}^{-2x} + 2}{4}}dx = \int_0^1 (2\pi y)\sqrt{\left(\frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}\right)^2}dx = \int_0^1 (2\pi y) \times \frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}dx$$

$$= 0.5\pi \int_0^1 (\mathfrak{e}^x + \mathfrak{e}^{-x} - 2) \times (\mathfrak{e}^x + \mathfrak{e}^{-x})dx = 0.5\pi \int_0^1 (\mathfrak{e}^{2x} + \mathfrak{e}^{-2x} + 2 - 2\mathfrak{e}^x - 2\mathfrak{e}^{-x})dx$$

$$= 0.5\pi[\frac{1}{2}\mathfrak{e}^{2x} - \frac{1}{2}\mathfrak{e}^{-2x} + 2x - 2\mathfrak{e}^x + 2\mathfrak{e}^{-x}]_0^1 = 0.5\pi(\frac{1}{2}\mathfrak{e}^2 - \frac{1}{2}\mathfrak{e}^{-2} + 2 - 2\mathfrak{e} + 2\mathfrak{e}^{-1})$$

72) For the force

$$\mathbf{F} = (y + 3x^2z^2)\mathbf{i} + (x - z)\mathbf{j} + (2x^3z - y)\mathbf{k}$$

find the potential $\phi$ such that $\mathbf{F} = \nabla\phi$. Hence evaluate

$$\int_{(0,0,0)}^{(1,2,3)} (y + 3x^2z^2)dx + (x - z)dy + (2x^3z - y)dz$$

$$\mathbf{F} = \nabla\phi = \frac{d\{\phi\}}{dx}\mathbf{i} + \frac{d\{\phi\}}{dy}\mathbf{j} + \frac{d\{\phi\}}{dz}\mathbf{k}$$
$$\equiv (y + 3x^2z^2)\mathbf{i} + (x - z)\mathbf{j} + (2x^3z - y)\mathbf{k}$$

Therefore

$$\frac{d\{\phi\}}{dx} = y + 3x^2z^2 \ ; \ \frac{d\{\phi\}}{dy} = x - z \ ; \ \frac{d\{\phi\}}{dz} = 2x^3z - y$$

This is written as

$$\partial\phi = (y + 3x^2z^2)\partial x \ ; \ \partial\phi = (x - z)\partial y \ ; \ \partial\phi = (2x^3z - y)\partial z$$

Thus

$$\int \partial\phi = \int (y + 3x^2z^2)\partial x \ ; \ \therefore \phi = xy + x^3z^2 + c_\alpha(y, z)$$
$$\int \partial\phi = \int (x - z)\partial y \ ; \ \therefore \phi = xy - yz + c_\beta(x, z)$$
$$\int \partial\phi = \int (2x^3z - y)\partial z \ ; \ \therefore \phi = x^3z^2 - yz + c_\gamma(x, y)$$

Thus we can tell that $\phi = xy - yz + x^3z^2$ When we define $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$, $(y + 3x^2z^2)dx + (x - z)dy + (2x^3z - y)dz = F \cdot d\mathbf{r}$. Since $\mathbf{F} = \nabla\phi$, the integral in question is manipulated as

$$\int_{(0,0,0)}^{(1,2,3)} \mathbf{F} \cdot d\mathbf{r} = \int_{(0,0,0)}^{(1,2,3)} \nabla\phi \cdot d\mathbf{r} = \int_{(0,0,0)}^{(1,2,3)} \frac{\partial\{\phi\}}{\partial\{\mathbf{r}\}} \cdot d\mathbf{r} = \int_{(0,0,0)}^{(1,2,3)} d\phi$$
$$= [\phi]_{(0,0,0)}^{(1,2,3)} = \left[xy - yz + x^3z^2\right]_{(0,0,0)}^{(1,2,3)} = 1 \cdot 2 - 2 \cdot 3 + 1^3 \cdot 3^2 = 5$$

Alternatively

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ 3t \end{pmatrix}$$

where $0 \le t \le 1$.

$$\boldsymbol{F} = \begin{pmatrix} y + 3x^2 z^2 \\ x - z \\ 2x^3 z - y \end{pmatrix} = \begin{pmatrix} 2t + 3t^2(3t)^2 \\ t - 3t \\ 2t^3(3t) - 2t \end{pmatrix} = \begin{pmatrix} 2t + 27t^4 \\ -2t \\ 6t^4 - 2t \end{pmatrix}$$

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Thus the integration in question can be re-written as

$$\int_0^1 \boldsymbol{F} \cdot \frac{d\{\boldsymbol{r}\}}{dt} dt$$

$$= \int_0^1 2t + 27t^4 - 2t \cdot 2 + (6t^4 - 2t) \cdot 3dt = \int_0^1 2t + 27t^4 - 4t + 18t^4 - 6tdt$$

$$= \int_0^1 45t^4 - 8tdt = [9t^5 - 4t^2]_0^1 = 9 - 4 = 5$$

73) For the double integral

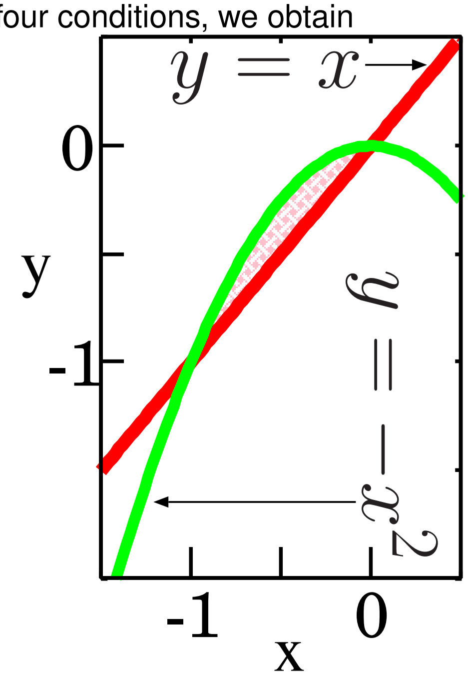
$$\int_{-1}^{0} \int_{x}^{-x^2} 12xydydx$$

draw a clear, labelled sketch of the region of integration and evaluate the integral using any suitable method.

From the given equation we get the range of $x$ and $y$ as follows

$$x \le y \le -x^2 \qquad -1 \le x \le 0$$

From these four conditions, we obtain




$$\int_{-1}^{0} \int_{x}^{-x^2} 12xydydx = \int_{-1}^{0} \left[12xy^2\right]_x^{-x^2} dx = \int_{-1}^{0} 12x(x^4 - x^2)dx$$

$$= \int_{-1}^{0} 12(x^5 - x^3)dx = 12\left[\frac{x^6}{6} - \frac{x^4}{4}\right]_{-1}^{0} = \left[2x^6 - 3x^4\right]_{-1}^{0} = -(2-3) = 1$$

74) Find the work

$$W = \int_C \mathbf{F} \cdot d\boldsymbol{r}$$

done by the force $\mathbf{F} = x^2\boldsymbol{i} + xy\boldsymbol{j}$ in moving a particle along the curve given parametrically by

$$x(t) = 1 - t$$

and

$$y(t) = t$$

where $0 \le t \le 1$.

a) Express $x, y, z$ on the curve $C$ using $t$ and set the range of $t$

$$\begin{aligned} x(t) &= 1 - t \\ y(t) &= t \\ 0 \le t &\le 1 \end{aligned}$$

b) Express $\mathbf{F}$ as the function of $t$

$$\mathbf{F} = \begin{pmatrix} x^2 \\ xy \end{pmatrix} = \begin{pmatrix} (1-t)^2 \\ (1-t)t \end{pmatrix} = \begin{pmatrix} 1 + t^2 - 2t \\ t - t^2 \end{pmatrix}$$

c) Express $\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \\ \frac{d\{z\}}{dt} \end{pmatrix}$ using $t$

$$\frac{d\{\boldsymbol{r}\}}{dt} = \begin{pmatrix} \frac{d\{x\}}{dt} \\ \frac{d\{y\}}{dt} \end{pmatrix} = \begin{pmatrix} \frac{d\{1-t\}}{dt} \\ \frac{d\{t\}}{dt} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

d) Put all of them into $\int \mathbf{F} \cdot \frac{d\{\boldsymbol{r}\}}{dt} dt$

$$\int \mathbf{F} \cdot \frac{d\{\boldsymbol{r}\}}{dt} dt = \int_0^1 \begin{pmatrix} 1 + t^2 - 2t \\ t - t^2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} dt = \int_0^1 -1 - t^2 + 2t + t - t^2 dt$$

$$= \int_0^1 -1 - 2t^2 + 3t dt = \left[-t - \frac{2}{3}t^3 + \frac{3}{2}t^2\right]_{t=0}^{t=1} = -1 - \frac{2}{3} + \frac{3}{2} = \frac{-1}{6}$$

1) **DAY1**

2) When $x(t) = c\mathfrak{e}^{-3t}$ and $x(0) = 3$, find $c$.

$$(x,t) = (3,0) \ ; \quad \therefore 3 = c\mathfrak{e}^{-3\times 0} \ ; \quad \therefore 3 = c\mathfrak{e}^{0} \ ; \quad \therefore c = 3 (\because \mathfrak{e}^0 = 1)$$

3) Simplify$(-2x^3)^2$.

$$(-2x^3)^2 = (-2)^2 x^{3\times 2} = 4x^6$$

4) If $y(t) = \ln|t^3 + c|$ and $y(0) = 1$, then find $c$.

$$(y,t) = (1,0) \ ; \quad \therefore 1 = \ln|0^3 + c| \ ; \quad \therefore 1 = \ln|c| \ ; \quad \therefore \mathfrak{e}^1 = c \ ; \quad \therefore c = \mathfrak{e}$$

5) Simplify $\frac{1}{2}\left(-\ln|1-t| + \ln|1+t|\right)$.

$$\frac{1}{2}\left(-\ln|1-t| + \ln|1+t|\right) = \frac{1}{2}\left[\ln|1+t| - \ln|1-t|\right] = \frac{1}{2}\left[\ln|\frac{1+t}{1-t}|\right]$$

6) Find the general solution of

$$x\sqrt{1+y^2} + y\sqrt{1+x^2}\frac{d\{y\}}{dx} = 0$$

and then find the particular solution that satisfies $y(7) = 1$.
The given equation is manipulated as

$$x\sqrt{1+y^2} + y\sqrt{1+x^2}\frac{d\{y\}}{dx} = 0 \ ; \quad \therefore y\sqrt{1+x^2}\frac{d\{y\}}{dx} = -x\sqrt{1+y^2}$$

$$\therefore \frac{y}{\sqrt{1+y^2}}dy = -\frac{x}{\sqrt{1+x^2}}dx \ ; \quad \therefore \int \frac{y}{\sqrt{1+y^2}}dy = -\int \frac{x}{\sqrt{1+x^2}}dx$$

When we set $z = 1 + y^2$, we get

$$z = 1 + y^2 \ ; \quad \therefore \frac{d\{z\}}{dy} = \frac{d\{1+y^2\}}{dy} \ ; \quad \therefore \frac{d\{z\}}{dy} = 2y \ ; \quad \therefore dz = 2ydy \ ; \quad \therefore \frac{1}{2}dz = ydy$$

Using this,

$$\int \frac{y}{\sqrt{1+y^2}}dy = \int \frac{ydy}{\sqrt{1+y^2}} = \int \frac{\frac{1}{2}dz}{\sqrt{z}} = \frac{1}{2}\int z^{-\frac{1}{2}}dz = \frac{1}{2}\cdot\frac{1}{-\frac{1}{2}+1}z^{-\frac{1}{2}+1} = \frac{1}{2}\cdot\frac{1}{\frac{1}{2}}z^{\frac{1}{2}} = z^{\frac{1}{2}} = (1+y^2)^{\frac{1}{2}}$$

In the similar manner we obtain

$$\int \frac{x}{\sqrt{1+x^2}}dx = (1+x^2)^{\frac{1}{2}}$$

Using these results, the equation is furthermore manipulated as

$$\int \frac{y}{\sqrt{1+y^2}}dy = -\int \frac{x}{\sqrt{1+x^2}}dx \ ; \quad \therefore (1+y^2)^{\frac{1}{2}} = -(1+x^2)^{\frac{1}{2}} + c \ ; \quad \therefore (1+y^2)^{\frac{1}{2}} + (1+x^2)^{\frac{1}{2}} = c \qquad ①$$

which is the general solution. Now we find the value of $c$ from the given condition of $y(7) = 1$. By substituting $(x,y) = (7,1)$ into ①,

$$(1+1^2)^{\frac{1}{2}} + (1+7^2)^{\frac{1}{2}} = c \ ; \quad \therefore 2^{\frac{1}{2}} + 50^{\frac{1}{2}} = c \ ; \quad \therefore 2^{\frac{1}{2}} + 5\cdot 2^{\frac{1}{2}} = c \ ; \quad \therefore 6\cdot 2^{\frac{1}{2}} = c$$

Thus we obtain the particular solution of

$$(1+y^2)^{\frac{1}{2}} + (1+x^2)^{\frac{1}{2}} = 6\sqrt{2}$$

7) Find the general solution of

$$x^2\left(\frac{d\{y\}}{dx}\right)^2+3xy\frac{d\{y\}}{dx}+2y^2=0$$

The roots of

$$az^2+bz^1+c=0$$

are

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

The equation in question is identical to

$$az^2+bz^1+c=0$$

when

$$a=x^2\ \ ;\ \ b=3xy\ \ ;\ \ c=2y^2\ \ ;\ \ z=\frac{d\{y\}}{dx}$$

Thus

$$\frac{d\{y\}}{dx}=\frac{-3xy\pm\sqrt{(3xy)^2-4\cdot x^2\cdot(2y^2)}}{2x^2}=\frac{-3xy\pm\sqrt{(xy)^2}}{2x^2}=\frac{-3xy\pm xy}{2x^2}=\frac{-4xy}{2x^2},\frac{-2xy}{2x^2}=\frac{-2y}{x},\frac{-y}{x}$$

From the first equation we get

$$\frac{d\{y\}}{dx}=\frac{-2y}{x}\ \ ;\ \ \therefore\frac{1}{y}dy=\frac{-2}{x}dx\ \ ;\ \ \therefore\int\frac{1}{y}dy=\int\frac{-2}{x}dx\ \ ;\ \ \therefore\ln y=-2\int\frac{1}{x}dx$$
$$\therefore\ln y=-2\ln x+c\ \ ;\ \ \therefore\ln y=\ln x^{-2}+\ln c=\ln(c\cdot x^{-2})\ \ ;\ \ \therefore y=c\cdot x^{-2}$$

From the second equation we get

$$\frac{d\{y\}}{dx}=\frac{-y}{x}\ \ ;\ \ \therefore\frac{1}{y}dy=\frac{-1}{x}dx\ \ ;\ \ \therefore\int\frac{1}{y}dy=\int\frac{-1}{x}dx$$
$$\therefore\ln y=-\int\frac{1}{x}dx\ \ ;\ \ \therefore\ln y=-\ln x+c\ \ ;\ \ \therefore\ln y=\ln x^{-1}+\ln c=\ln(c\cdot x^{-1})\ \ ;\ \ \therefore y=c\cdot x^{-1}$$

Thus we obtain the general solution of

$$y=c\cdot x^{-1},y=c\cdot x^{-2}$$

8) Find the $\int\frac{1}{x}dx$.

$$\int\frac{1}{x}dx=\ln|x|+c$$

9) Solve the equation of $|2x+3|=5$

$|2x+3|=5\ \ ;\ \ \therefore 2x+3=\pm5\ \ ;\ \ \therefore 2x=\pm5-3\ \ ;\ \ \therefore 2x=5-3,-5-3\ \ ;\ \ \therefore 2x=2,-8\ \ ;\ \ \therefore x=1,-4$

10) Find the $\int kdt$.

$$\int kdt=kt+c$$

11) Simplify $y = \ln \mathfrak{e}^{kt+c}$ where $k$ and $c$ are constant.

$$\log_{\mathfrak{e}} \mathfrak{e} = 1 \;\; ; \quad \therefore y = kt + c$$

12) Find the general solution of

$$\frac{d\{x\}}{dt} = 3t^2(x^2 - 1)$$

and then find the particular solution that satisfies $x(0) = 3$.
To solve this we must rearrange it so all the $x$ terms are on the left hand side and all the $t$ terms are on the right hand side. Then we integrate both sides.

$$\frac{d\{x\}}{dt} = 3t^2(x^2 - 1) \;\; ; \quad \frac{1}{(x^2-1)}\frac{d\{x\}}{dt} = 3t^2 \;\; ; \quad \frac{1}{(x^2-1)}\partial x = 3t^2 \partial t$$

Factorising the fraction gives

$$\frac{1}{(x+1)(x-1)}\partial x = 3t^2 \partial t$$

Now we simplify the fraction at the left hand side as follows:

$$\frac{1}{(x+1)(x-1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} \;\; ; \quad \therefore 1 = \frac{A(x+1)(x-1)}{(x-1)} + \frac{B(x+1)(x-1)}{(x+1)}$$
$$\therefore 1 = A(x+1) + B(x-1)$$

When we substitute $x = -1$ in $1 = A(x+1) + B(x-1)$, we get

$$1 = A(x+1) + B(x-1) \;\; ; \quad \therefore 1 = A(-1+1) + B(-1-1) \;\; ; \quad \therefore 1 = B(-2) \therefore B = -\frac{1}{2}$$

When we substitute $x = 1$ in $1 = A(x+1) + B(x-1)$, we get

$$1 = A(x+1) + B(x-1) \;\; ; \quad \therefore 1 = A(1+1) + B(1-1) \;\; ; \quad \therefore 1 = A(2) \;\; ; \quad \therefore A = \frac{1}{2}$$

Thus the fraction $\frac{1}{(x+1)(x-1)}$ can be written as

$$\frac{1}{(x+1)(x-1)} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

Thus $\frac{1}{(x^2-1)}\partial x = 3t^2\partial t$ can be re-written as

$$\left(\frac{1}{2(x-1)} - \frac{1}{2(x+1)}\right)\partial x = 3t^2 \partial t$$

and now we can integrate both sides

$$\int \left(\frac{1}{2(x-1)} - \frac{1}{2(x+1)}\right)\partial x = \int 3t^2 \partial t \;\; ; \quad \therefore \int \frac{1}{2(x-1)}dx - \int \frac{1}{2(x+1)}dx = \int 3t^2 dt$$
$$\therefore \frac{1}{2}\int \frac{1}{(x-1)}dx - \frac{1}{2}\int \frac{1}{(x+1)}dx = \int 3t^2 dt \;\; ; \quad \therefore \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| = \frac{3}{3}t^3 + c$$
$$\therefore \frac{1}{2}(\ln|x-1| - \ln|x+1|) = t^3 + c \;\; ; \quad \therefore \frac{1}{2}\left(\ln|\frac{x-1}{x+1}|\right) = t^3 + c$$
$$\therefore \ln|\frac{x-1}{x+1}| = 2t^3 + c \;\; ; \quad \therefore \left|\frac{x-1}{x+1}\right| = \mathfrak{e}^{2t^3+c} \;\; ; \quad \therefore \frac{x-1}{x+1} = \pm\mathfrak{e}^{2t^3}\mathfrak{e}^{c}$$

Let $K \triangleq \pm\mathfrak{e}^c$,

$$\frac{x-1}{x+1} = K\mathfrak{e}^{2t^3} \quad ; \quad \therefore x = \frac{1+K\mathfrak{e}^{2t^3}}{1-K\mathfrak{e}^{2t^3}}$$

To find the particular solution we substitute in $x(0) = 3$ which means $x = 3$ when $t = 0$.

$$\frac{x-1}{x+1}|_{x=3} = K\mathfrak{e}^{2t^3}|_{t=0} \quad ; \quad \therefore \frac{3-1}{3+1} = K\mathfrak{e}^{2\cdot 0^3} \quad ; \quad \therefore \frac{2}{4} = K\mathfrak{e}^0 \quad ; \quad \therefore \frac{1}{2} = K$$

Therefore the particular solution is

$$x = \frac{1+K\mathfrak{e}^{2t^3}}{1-K\mathfrak{e}^{2t^3}} = \frac{\frac{1}{K}+\mathfrak{e}^{2t^3}}{\frac{1}{K}-\mathfrak{e}^{2t^3}} = \frac{2+\mathfrak{e}^{2t^3}}{2-\mathfrak{e}^{2t^3}}$$

13) Solve the following equation under the condition $x(0) = 3$

$$\frac{d\{x\}}{dt} = kx$$

$\frac{d\{x\}}{dt} = kx$ can be written as

$$\frac{1}{x}dx = kdt \quad ; \quad \therefore \int \frac{1}{x}dx = \int kdt \quad ; \quad \therefore \ln x = kt + c = \ln \mathfrak{e}^{kt+c} \quad ; \quad \therefore x = \mathfrak{e}^{kt+c} = \mathfrak{e}^c\mathfrak{e}^{kt} = C\mathfrak{e}^{kt}$$

In order to find out $C$ under the condition $x(0) = 3$, we put $(t, x) = (0, 3)$ into $x = C\mathfrak{e}^{kt}$. Thus $3 = C\mathfrak{e}^0 = C$. Therefore, the answer is $x = 3\mathfrak{e}^{kt}$.

14) Solve the equation

$$y\frac{d\{y\}}{dx} - \mathfrak{e}^{-x} = 0$$

$\frac{d\{y\}}{dx} = \frac{\mathfrak{e}^{-x}}{y}$ can be written as

$$ydy = \mathfrak{e}^{-x}dx \quad ; \quad \therefore \int ydy = \int \mathfrak{e}^{-x}dx \quad ; \quad \therefore \frac{1}{2}y^2 = -\mathfrak{e}^{-x} + c \quad ; \quad \therefore y^2 = C - 2\mathfrak{e}^{-x} \quad ; \quad \therefore y = \pm\sqrt{C - 2\mathfrak{e}^{-x}}$$

15) Solve the following equation subject to the condition $y(0) = 1$

$$\frac{d\{y\}}{dx} = 3x^2\mathfrak{e}^{-y}$$

$\frac{d\{y\}}{dx} = 3x^2\mathfrak{e}^{-y}$ can be written as

$$\mathfrak{e}^y dy = 3x^2 dx \quad ; \quad \therefore \int \mathfrak{e}^y dy = \int 3x^2 dx \quad ; \quad \therefore \mathfrak{e}^y = x^3 + c \quad ; \quad \therefore \ln \mathfrak{e}^y = \ln(x^3 + c) \quad ; \quad \therefore y = \ln(x^3 + c)$$

In order to satisfy

$$y(0) = 1$$

we perform

$$y(0) = \ln(0^3 + c) = \ln c = 1 = \ln \mathfrak{e}$$

Thus we now know $c = \mathfrak{e}$. Thus the answer is $y = \ln(x^3 + \mathfrak{e})$.

16) Solve the equation

$$\frac{d\{y\}}{dx} = \frac{6\sin x}{y}$$

$\frac{d\{y\}}{dx} = \frac{6\sin x}{y}$ can be written as

$$ydy = 6\sin x dx \;\; ; \;\; \therefore \int ydy = \int 6\sin x dx \;\; ; \;\; \therefore \frac{1}{2}y^2 = -6\cos x + c$$
$$\therefore y^2 = -12\cos x + C \;\; ; \;\; \therefore y = \pm\sqrt{-12\cos x + C}$$

17) Solve the equation

$$\frac{d\{x\}}{dt} = t(x-2)$$

$$\frac{1}{x-2}dx = tdt \;\; ; \;\; \therefore \int \frac{1}{x-2}dx = \int tdt \;\; ; \;\; \therefore \ln|x-2| = \frac{1}{2}t^2 + c \;\; ; \;\; \therefore \ln|x-2| = \ln \mathfrak{e}^{\frac{1}{2}t^2+c} (\because \ln \mathfrak{e}^A = A)$$
$$\therefore |x-2| = \mathfrak{e}^{\frac{1}{2}t^2+c} \;\; ; \;\; \therefore x-2 = \pm\mathfrak{e}^c\mathfrak{e}^{\frac{1}{2}t^2} \;\; ; \;\; \therefore x-2 = C\mathfrak{e}^{\frac{1}{2}t^2} \;\; ; \;\; \therefore x = C\mathfrak{e}^{\frac{1}{2}t^2} + 2$$

18) Solve the following equation

$$\left(\frac{d\{y\}}{dx}\right)^2 - \frac{d\{y\}}{dx} = 0$$

The given equation is factorised as

$$\left(\frac{d\{y\}}{dx} - 1\right)\frac{d\{y\}}{dx} = 0$$

Thus we obtain

$$\frac{d\{y\}}{dx} = 0 \;\; ; \;\; \frac{d\{y\}}{dx} - 1 = 0$$

From the first equation of $\frac{d\{y\}}{dx} = 0$, we get

$$\frac{d\{y\}}{dx} = 0 \;\; ; \;\; \therefore dy = 0 \cdot dx \;\; ; \;\; \therefore \int dy = \int 0 \cdot dx \;\; ; \;\; \therefore y = 0 + c \;\; ; \;\; \therefore y = c$$

From the second equation, we get

$$\frac{d\{y\}}{dx} = 1 \;\; ; \;\; \therefore dy = dx \;\; ; \;\; \therefore \int dy = \int dx \;\; ; \;\; \therefore y = x + c$$

Thus the solution is

$$y = c, y = x + c$$

19) Solve the following equation

$$\frac{d\{y\}}{dx} + \frac{1-y^2}{1-x^2} = 0$$

The given equation can be manipulated as follows:

$$\frac{d\{y\}}{dx}+\frac{1-y^2}{1-x^2}=0 \;\; ; \quad \therefore \frac{d\{y\}}{dx}=-\frac{1-y^2}{1-x^2} \;\; ; \quad \therefore \frac{1}{1-y^2}\frac{d\{y\}}{dx}=-\frac{1}{1-x^2}$$

$$\therefore \frac{1}{1-y^2}dy=-\frac{1}{1-x^2}dx \;\; ; \quad \therefore \int \frac{1}{1-y^2}dy=-\int \frac{1}{1-x^2}dx$$

Since $\frac{1}{1-y^2}=\frac{1}{(1-y)(1+y)}$ we can express $\frac{1}{(1-y)(1+y)}$ in the form of $\frac{A}{1-y}+\frac{B}{1+y}$ where $A$ and $B$ are constants which are found in the following procedure.

$$\frac{1}{(1-y)(1+y)}=\frac{A}{1-y}+\frac{B}{1+y} \;\; ; \quad \therefore 1=A(1+y)+B(1-y)$$

By substituting $y=1$ into the above equation,

$$1=A(1+1)+B(1-1) \;\; ; \quad \therefore 1=2A \;\; ; \quad \therefore \frac{1}{2}=A$$

By substituting $y=-1$ into the above equation,

$$1=A(1-1)+B(1-(-1)) \;\; ; \quad \therefore 1=2B \;\; ; \quad \therefore \frac{1}{2}=B=A$$

Thus

$$\frac{1}{(1-y)(1+y)}=\frac{A}{1-y}+\frac{B}{1+y}=\frac{A}{1-y}+\frac{A}{1+y}=A(\frac{1}{1-y}+\frac{1}{1+y})=\frac{1}{2}(\frac{1}{1-y}+\frac{1}{1+y})$$

In the same way,

$$\frac{1}{(1-x)(1+x)}=\frac{1}{2}(\frac{1}{1-x}+\frac{1}{1+x})$$

Therefore, the original equation can be further manipulated as

$$\int \frac{1}{1-y^2}dy=-\int \frac{1}{1-x^2}dx \;\; ; \quad \therefore \int \frac{1}{2}(\frac{1}{1-y}+\frac{1}{1+y})dy=-\int \frac{1}{2}(\frac{1}{1-x}+\frac{1}{1+x})dx$$

$$\therefore \int (\frac{1}{1-y}+\frac{1}{1+y})dy=-\int (\frac{1}{1-x}+\frac{1}{1+x})dx \;\; ; \quad \therefore \int (-\frac{1}{y-1}+\frac{1}{1+y})dy=-\int (-\frac{1}{x-1}+\frac{1}{1+x})dx$$

$$\therefore -\ln(y-1)+\ln(y+1)=-\left(-\ln(x-1)+\ln(x+1)\right)+c=\ln(x-1)-\ln(x+1)+c$$

$$\therefore \ln\frac{y+1}{y-1}=\ln\left(\frac{x-1}{x+1}\right)+c \;\; ; \quad \therefore \ln\frac{y+1}{y-1}=\ln\left(\frac{x-1}{x+1}\right)+\ln(c)=\ln\left(c\frac{x-1}{x+1}\right) \;\; ; \quad \therefore \frac{y+1}{y-1}=c\frac{x-1}{x+1}$$

$$\therefore y+1=c\frac{x-1}{x+1}(y-1) \;\; ; \quad \therefore y=c\frac{x-1}{x+1}y-c\frac{x-1}{x+1}-1$$

$$\therefore y(1-c\frac{x-1}{x+1})=-c\frac{x-1}{x+1}-1 \;\; ; \quad \therefore y=\frac{c\frac{x-1}{x+1}+1}{c\frac{x-1}{x+1}-1} \;\; ; \quad \therefore y=\frac{c(x-1)+x+1}{c(x-1)-x-1}$$

**DAY2**

20) Find the solution to the differential equation

$$\frac{d\{y\}}{dx} = -2y + 20\sin(4x) - 10\cos(4x)$$

satisfying the condition $(x, y) = (0, 2)$

a) **Allocate** $P(x)$ **and** $Q(x)$
When we compare the equation with Equation (85), we obtain

$$P(x) = 2$$

and

$$Q(x) = 20\sin(4x) - 10\cos(4x).$$

b) **Calculate** $A = \int P(x)dx$

$$A = \int P(x)dx = \int 2dx = 2x$$

c) **Obtain** $\Phi(x) = \mathfrak{e}^A$
From Equation (86),

$$\Phi(x) = \mathfrak{e}^A = \mathfrak{e}^{2x}$$

d) **Calculate** $B = \int \Phi(x)Q(x)dx$

$$B = \int \Phi(x)Q(x)dx = \int \mathfrak{e}^{2x}(20\sin(4x) - 10\cos(4x))dx = 20\int \mathfrak{e}^{2x}\sin(4x)dx - 10\int \mathfrak{e}^{2x}\cos(4x)dx \quad ①$$

Now we need to find out $\int \mathfrak{e}^{2x}\sin(4x)dx$ and $\int \mathfrak{e}^{2x}\cos(4x)dx$. First we perform the following two differentiation:

$$\frac{d\{\mathfrak{e}^{2x}\sin(4x)\}}{dx} = \frac{d\{\mathfrak{e}^{2x}\}}{dx}\sin(4x) + \mathfrak{e}^{2x}\frac{d\{\sin(4x)\}}{dx} = 2\mathfrak{e}^{2x}\sin(4x) + 4\mathfrak{e}^{2x}\cos(4x) \quad ②$$

$$\frac{d\{\mathfrak{e}^{2x}\cos(4x)\}}{dx} = \frac{d\{\mathfrak{e}^{2x}\}}{dx}\cos(4x) + \mathfrak{e}^{2x}\frac{d\{\cos(4x)\}}{dx} = 2\mathfrak{e}^{2x}\cos(4x) - 4\mathfrak{e}^{2x}\sin(4x) \quad ③$$

②×2+③ gives us

$$2\frac{d\{\mathfrak{e}^{2x}\sin(4x)\}}{dx} + \frac{d\{\mathfrak{e}^{2x}\cos(4x)\}}{dx} = 10\mathfrak{e}^{2x}\cos(4x)$$

$$\therefore \int 2\frac{d\{\mathfrak{e}^{2x}\sin(4x)\}}{dx} + \frac{d\{\mathfrak{e}^{2x}\cos(4x)\}}{dx}dx = \int 10\mathfrak{e}^{2x}\cos(4x)dx$$

$$\therefore \int 2\partial(\mathfrak{e}^{2x}\sin(4x)) + \partial(\mathfrak{e}^{2x}\cos(4x)) = \int 10\mathfrak{e}^{2x}\cos(4x)dx$$

$$\therefore 2\mathfrak{e}^{2x}\sin(4x) + \mathfrak{e}^{2x}\cos(4x) = \int 10\mathfrak{e}^{2x}\cos(4x)dx$$

$$\therefore \frac{2\mathfrak{e}^{2x}\sin(4x) + \mathfrak{e}^{2x}\cos(4x)}{10} = \int \mathfrak{e}^{2x}\cos(4x)dx \quad ④$$

②-③×2 gives us

$$\frac{d\{\mathfrak{e}^{2x}\sin(4x)\}}{dx} - 2\frac{d\{\mathfrak{e}^{2x}\cos(4x)\}}{dx} = 10\mathfrak{e}^{2x}\sin(4x)$$

$$\therefore \int \frac{d\{\mathfrak{e}^{2x}\sin(4x)\}}{dx} - 2\frac{d\{\mathfrak{e}^{2x}\cos(4x)\}}{dx}dx = \int 10\mathfrak{e}^{2x}\sin(4x)dx$$

$$\therefore \int \partial(\mathfrak{e}^{2x}\sin(4x)) - 2\partial(\mathfrak{e}^{2x}\cos(4x)) = \int 10\mathfrak{e}^{2x}\sin(4x)dx$$

$$\therefore \mathfrak{e}^{2x}\sin(4x) - 2\mathfrak{e}^{2x}\cos(4x) = \int 10\mathfrak{e}^{2x}\sin(4x)dx$$

$$\therefore \frac{\mathfrak{e}^{2x}\sin(4x) - 2\mathfrak{e}^{2x}\cos(4x)}{10} = \int \mathfrak{e}^{2x}\sin(4x)dx \qquad ⑤$$

By putting ④ and ⑤ into ①, we get

$$B = 20\int \mathfrak{e}^{2x}\sin(4x)dx - 10\int \mathfrak{e}^{2x}\cos(4x)dx$$

$$= 20\frac{\mathfrak{e}^{2x}\sin(4x) - 2\mathfrak{e}^{2x}\cos(4x)}{10} - 10\frac{2\mathfrak{e}^{2x}\sin(4x) + \mathfrak{e}^{2x}\cos(4x)}{10}$$

$$= 2\mathfrak{e}^{2x}\sin(4x) - 4\mathfrak{e}^{2x}\cos(4x) - 2\mathfrak{e}^{2x}\sin(4x) - \mathfrak{e}^{2x}\cos(4x) = -5\mathfrak{e}^{2x}\cos(4x)$$

e) **Obtain the general solution** $y = \frac{1}{\Phi(x)}[B + c]$

$$y = \frac{1}{\Phi(x)}[B + c] = \frac{1}{\mathfrak{e}^{2x}}\left[-5\mathfrak{e}^{2x}\cos(4x) + c\right]$$

f) **Apply the condition to** $y = \frac{1}{\Phi(x)}[B + c]$ **in order to find out** $c$ **and thus the particular solution**

As the condition $(x, y) = (0, 2)$

$$2 = \frac{1}{\mathfrak{e}^{0}}\left[-5\mathfrak{e}^{0}\cos(4 \cdot 0) + c\right] = -5 + c \ ; \quad \therefore c = 7$$

Thus the particular solution is $y = \frac{1}{\mathfrak{e}^{2x}}\left[-5\mathfrak{e}^{2x}\cos(4x) + 7\right]$

21) Find the solution to the differential equation

$$\frac{d\{y\}}{dx} + 2y = 4\mathfrak{e}^{-2x}$$

satisfying the condition $(0, 1)$

a) **Allocate** $P(x)$ **and** $Q(x)$

When we compare the equation with Equation (85), we obtain

$$P(x) = 2$$

and

$$Q(x) = 4\mathfrak{e}^{-2x}.$$

b) **Calculate** $A = \int P(x)dx$

$$A = \int P(x)dx = \int 2dx = 2x$$

c) **Obtain** $\Phi(x) = \mathfrak{e}^{A}$
From Equation (86),

$$\Phi(x) = \mathfrak{e}^{A} = \mathfrak{e}^{2x}$$

d) **Calculate** $B = \int \Phi(x)Q(x)dx$

$$B = \int \Phi(x)Q(x)dx = \int \mathfrak{e}^{2x} \cdot (4\mathfrak{e}^{-2x})dx = \int 4dx = 4x$$

e) **Obtain the general solution** $y = \frac{1}{\Phi(x)}[B + c]$

$$y = \frac{1}{\Phi(x)}[B + c] = \frac{1}{\mathfrak{e}^{2x}}[4x + c]$$

f) **Apply the condition to** $y = \frac{1}{\Phi(x)}[B + c]$ **in order to find out** $c$ **and thus the particular solution**

$$1 = \frac{1}{\mathfrak{e}^{0}}[4 \cdot 0 + c] = c$$

Thus the particular solution is $y = \frac{1}{\mathfrak{e}^{2x}}[4x + 1]$

22) Make $y$ the subject of $2\mathfrak{e}^{9y} = x + c$.

$$2\mathfrak{e}^{9y} = x + c$$
$$\mathfrak{e}^{9y} = \frac{1}{2}(x + c) \ln \mathfrak{e}^{9y} = \ln|\frac{1}{2}(x + c)| 9y = \ln|\frac{1}{2}(x + c)| y = \frac{1}{9}\ln|\frac{1}{2}(x + c)|$$

23) Simplify $2(x + 2) - x + 3$

$$2(x + 2) - x + 3 = 2x + 4 - x + 3 = x + 7$$

24) Find $D$ from $\frac{y^4}{x^4} = \ln|Dx^4|$ when $(x, y) = (1, 1)$.

$$\frac{y^4}{x^4} = \ln|Dx^4| \therefore \mathfrak{e}^{\frac{y^4}{x^4}} = \mathfrak{e}^{\ln|Dx^4|} \therefore \mathfrak{e}^{\frac{y^4}{x^4}} = Dx^4 \therefore \frac{1}{x^4}\mathfrak{e}^{\frac{y^4}{x^4}} = D \therefore \frac{1}{(1)^4}\mathfrak{e}^{\frac{(1)^4}{(1)^4}} = D \therefore 1 \cdot \mathfrak{e}^{1} = D \therefore D = \mathfrak{e}$$

25) Solve the following equation

$$|3x - 8| - 3 = 4$$

$$|3x - 8| - 3 = 4 \;\; ; \;\; \therefore |3x - 8| = 4 + 3 \;\; ; \;\; \therefore |3x - 8| = 7 \;\; ; \;\; \therefore 3x - 8 = \pm 7 \;\; ; \;\; \therefore 3x = \pm 7 + 8$$
$$\therefore 3x = 7 + 8, -7 + 8 \;\; ; \;\; \therefore 3x = 15, 1 \;\; ; \;\; \therefore x = 5, \frac{1}{3}$$

26) Find the solution to the differential equation

$$\frac{d\{y\}}{dx} + \frac{y}{x} + x^2 = 0$$

satisfying the condition $(x, y) = (1, \frac{3}{4})$

a) **Allocate** $P(x)$ **and** $Q(x)$
When we compare the equation with Equation (85), we obtain

$$P(x) = \frac{1}{x}$$

and

$$Q(x) = -x^2.$$

b) **Calculate** $A = \int P(x)dx$

$$A = \int P(x)dx = \int \frac{1}{x}dx = \ln x$$

c) **Obtain** $\Phi(x) = \mathfrak{e}^A$
From Equation (86),

$$\Phi(x) = \mathfrak{e}^A = \mathfrak{e}^{\ln x} = x$$

d) **Calculate** $B = \int \Phi(x)Q(x)dx$

$$B = \int \Phi(x)Q(x)dx = \int x \cdot (-x^2)dx = \int (-x^3)dx = -\frac{1}{4}x^4$$

e) **Obtain the general solution** $y = \frac{1}{\Phi(x)}[B + c]$

$$y = \frac{1}{\Phi(x)}[B + c] = \frac{1}{x}\left[-\frac{1}{4}x^4 + c\right]$$

f) **Apply the condition to** $y = \frac{1}{\Phi(x)}[B + c]$ **in order to find out** $c$ **and thus the particular solution**
As the condition is $(x, y) = (1, \frac{3}{4})$,

$$\frac{3}{4} = \left[-\frac{1}{4} + c\right] \quad ; \quad \therefore c = 1$$

Thus the particular solution is $y = \frac{1}{x}\left[-\frac{1}{4}x^4 + 1\right] = -\frac{1}{4}x^3 + \frac{1}{x}$

27) Find the solution to the differential equation

$$\frac{d\{y\}}{dx} + 2y\tan x - \sin x = 0$$

satisfying the condition $(x, y) = (0, 0)$

a) **Allocate** $P(x)$ **and** $Q(x)$
When we compare the equation with Equation (85), we obtain

$$P(x) = 2\tan x$$

and

$$Q(x) = \sin x.$$

b) **Calculate** $A = \int P(x)dx$

$$A = \int P(x)dx = \int 2\tan x dx = 2\int \tan x dx = 2(-\ln(\cos x)) = -2\ln(\cos x) = \ln(\cos x)^{-2}$$

c) **Obtain** $\Phi(x) = \mathfrak{e}^{A}$
From Equation (86),

$$\Phi(x) = \mathfrak{e}^{A} = \mathfrak{e}^{\ln(\cos x)^{-2}} = (\cos x)^{-2}$$

d) **Calculate** $B = \int \Phi(x)Q(x)dx$

$$B = \int \Phi(x)Q(x)dx = \int (\cos x)^{-2} \sin x dx = \int (\cos x)^{-2} \frac{d\{-\cos x\}}{dx} dx$$
$$= -\int (\cos x)^{-2} d(\cos x) = -\int (s)^{-2} d(s)(\because s \triangleq \cos x) = -\frac{1}{-2+1} s^{-2+1} = s^{-1} = (\cos x)^{-1}$$

e) **Obtain the general solution** $y = \frac{1}{\Phi(x)}[B + c]$

$$y = \frac{1}{\Phi(x)}[B+c] = \frac{1}{(\cos x)^{-2}}\left[(\cos x)^{-1} + c\right] = (\cos x)^{2}\left[(\cos x)^{-1} + c\right] = \cos x + c\cos^{2} x$$

f) **Apply the condition to** $y = \frac{1}{\Phi(x)}[B + c]$ **in order to find out** $c$ **and thus the particular solution**

$$0 = \cos 0 + c\cos^{2} 0 = 1 + c \ ; \quad \therefore c = -1$$

Thus the particular solution is $y = \cos x - \cos^{2} x$

28) Solve the following inequality

$$9 < 3 - 2(x-5) \leq 21$$

$$9 < 3 - 2(x-5) \leq 21 \ ; \quad \therefore 9 < 3 - 2x + 10 \leq 21 \ ; \quad \therefore 9 < -2x + 13 \leq 21 \ ; \quad \therefore 9 - 13 < -2x \leq 21 - 13$$
$$\therefore -4 < -2x \leq 8 \ ; \quad \therefore \frac{-4}{-2} > x \geq \frac{8}{-2} \ ; \quad \therefore 2 > x \geq -4$$

29) Solve the equation

$$3^{x} = 9^{x^2 - x}$$

$$3^{x} = 9^{x^2-x} \ ; \quad \therefore 3^{x} = (3^{2})^{x^2-x} \ ; \quad \therefore 3^{x} = 3^{2(x^2-x)} \ ; \quad \therefore x = 2(x^2 - x) \ ; \quad \therefore x = 2x^2 - 2x$$
$$\therefore 0 = 2x^2 - 2x - x \ ; \quad \therefore 0 = 2x^2 - 3x \ ; \quad \therefore 0 = (2x-3)x \ ; \quad \therefore x = 0, \frac{3}{2}$$

30) Solve the equation

$$\sqrt{16^{x}} = \left(\frac{1}{2}\right)^{x^2-1}$$

$$\sqrt{16^{x}} = \left(\frac{1}{2}\right)^{x^2-1} \ ; \quad \therefore \sqrt{(2^{4})^{x}} = \left(2^{-1}\right)^{x^2-1} \ ; \quad \therefore \sqrt{2^{4x}} = 2^{-(x^2-1)}$$
$$\therefore 2^{4x \times \frac{1}{2}} = 2^{-(x^2-1)} \ ; \quad \therefore 2x = -(x^2 - 1) \ ; \quad \therefore 2x = -x^2 + 1$$
$$\therefore x^2 + 2x - 1 = 0 \ ; \quad \therefore x = \frac{-2 \pm \sqrt{2^2 + 4}}{2} = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$$

31) Solve the equation

$$16^x = \left(\frac{1}{2}\right)^2 8^{2x-1}$$

$$16^x = \left(\frac{1}{2}\right)^2 8^{2x-1} \;\; ; \quad \therefore \left(2^4\right)^x = \left(2^{-1}\right)^2 \left(2^3\right)^{2x-1} \;\; ; \quad \therefore 2^{4x} = \left(2^{-2}\right)\left(2^{3(2x-1)}\right)$$
$$\therefore 2^{4x} = 2^{-2+3(2x-1)} \;\; ; \quad \therefore 4x = -2 + 3(2x-1) \;\; ; \quad \therefore 4x = -2 + 6x - 3$$
$$\therefore 4x - 6x = -5 \;\; ; \quad \therefore -2x = -5 \;\; ; \quad \therefore x = 2.5$$

32) Find the solution to the differential equation

$$\frac{d\left\{y\right\}}{dx} = \frac{3y}{x+4} + x + 4$$

satisfying the condition $(x, y) = (-5, -2)$

a) **Allocate** $P(x)$ **and** $Q(x)$
When we compare the equation with Equation (85), we obtain

$$P(x) = -\frac{3}{x+4}$$

and

$$Q(x) = x + 4.$$

b) **Calculate** $A = \int P(x)dx$

$$A = \int P(x)dx = \int -\frac{3}{x+4}dx = -3\int \frac{1}{x+4}dx = {}^{-3}\ln|x+4| = \ln|x+4|^{-3}$$

c) **Obtain** $\Phi(x) = \mathfrak{e}^A$
From Equation (86),

$$\Phi(x) = \mathfrak{e}^A = \mathfrak{e}^{\ln|x+4|^{-3}} = |x+4|^{-3} (\because \mathfrak{e}^{\log_{\mathfrak{e}} r} = r)$$

d) **Calculate** $B = \int \Phi(x)Q(x)dx$

$$B = \int \Phi(x)Q(x)dx = \int |x+4|^{-3} \cdot (x+4)dx = \int |x+4|^{-2}dx = \frac{1}{-2+1}(x+4)^{-2+1} = -(x+4)^{-1}$$

e) **Obtain the general solution** $y = \frac{1}{\Phi(x)}\left[B + c\right]$

$$y = \frac{1}{\Phi(x)}\left[B + c\right] = \frac{1}{(x+4)^{-3}}\left[-(x+4)^{-1} + c\right] = (x+4)^3\left[-(x+4)^{-1} + c\right] = -(x+4)^2 + c(x+4)^3$$

f) **Apply the condition to** $y = \frac{1}{\Phi(x)}\left[B + c\right]$ **in order to find out** $c$ **and thus the particular solution**
As the condition $(x, y) = (-5, -2)$,

$$-2 = -(-5+4)^2 + c(-5+4)^3 = -1 - c \;\; ; \quad \therefore c = 1$$

Thus the particular solution is $y = -(x+4)^2 + (x+4)^3$

33) Find the solution to the differential equation

$$\frac{d\left\{y\right\}}{dx} = x + 2y$$

satisfying the condition $(x, y) = \left(0, \frac{3}{4}\right)$

a) **Allocate** $P(x)$ **and** $Q(x)$
When we compare the equation with Equation (85), we obtain $P(x) = -2$ and $Q(x) = x$.

b) **Calculate** $A = \int P(x)dx$

$$A = \int P(x)dx = \int -2dx = -2x$$

c) **Obtain** $\Phi(x) = \mathfrak{e}^A$
From Equation (86),

$$\Phi(x) = \mathfrak{e}^A = \mathfrak{e}^{-2x}$$

d) **Calculate** $B = \int \Phi(x)Q(x)dx$

$$B = \int \Phi(x)Q(x)dx = \int \mathfrak{e}^{-2x}xdx == x \cdot \int \mathfrak{e}^{-2x}dx - \int \left(\frac{d\{x\}}{dx}\int \mathfrak{e}^{-2x}dx\right)dx$$
$$= x \cdot \frac{1}{-2}\mathfrak{e}^{-2x} - \int \frac{1}{-2}\mathfrak{e}^{-2x}dx = x \cdot \frac{1}{-2}\mathfrak{e}^{-2x} - \frac{1}{-2} \cdot \frac{1}{-2} \cdot \mathfrak{e}^{-2x} = \frac{x\mathfrak{e}^{-2x}}{-2} - \frac{\mathfrak{e}^{-2x}}{4}$$

e) **Obtain the general solution** $y = \frac{1}{\Phi(x)}[B + c]$

$$y = \frac{1}{\Phi(x)}[B + c] = \frac{1}{\mathfrak{e}^{-2x}}\left[\frac{x\mathfrak{e}^{-2x}}{-2} - \frac{\mathfrak{e}^{-2x}}{4} + c\right] = \frac{x}{-2} - \frac{1}{4} + c\mathfrak{e}^{2x}$$

f) **Apply the condition to** $y = \frac{1}{\Phi(x)}[B + c]$ **in order to find out** $c$ **and thus the particular solution**

As the condition $(x, y) = \left(0, \frac{3}{4}\right)$,

$$\frac{3}{4} = -\frac{1}{4} + c\mathfrak{e}^0 \quad ; \quad \therefore c = 1$$

Therefore the particular solution is $y = \frac{x}{-2} - \frac{1}{4} + \mathfrak{e}^{2x}$

**DAY3**

34) Solve the equation

$$2^{\sqrt{x}} = \left(\frac{1}{\left(\frac{1}{2}\right)^{-2}}\right)^{-x}$$

$$2^{\sqrt{x}} = \left(\frac{1}{\left(\frac{1}{2}\right)^{-2}}\right)^{-x} \;;\; \therefore 2^{\sqrt{x}} = \left(\frac{1}{(2^{-1})^{-2}}\right)^{-x} \;;\; \therefore 2^{\sqrt{x}} = \left(\frac{1}{2^2}\right)^{-x} \;;\; \therefore 2^{\sqrt{x}} = \frac{1}{2^{-2x}} \;;\; \therefore 2^{\sqrt{x}} = 2^{2x}$$

$$\therefore \sqrt{x} = 2x \;;\; \therefore 2x - \sqrt{x} = 0 \;;\; \therefore x^{0.5}\left(2x^{0.5} - 1\right) = 0$$

$$\therefore x^{0.5} = 0, 0.5 \;;\; \therefore \left(x^{0.5}\right)^2 = 0^2, 0.5^2 \;;\; \therefore x = 0, 0.25$$

35) Solve the equation

$$5^{x+1} = 100 \cdot 3^x$$

$$5^{x+1} = 100 \cdot 3^x \;;\; \therefore 5^{x+1} = 2^2 \cdot 5^2 \cdot 3^x$$

$$\therefore 5^{x+1} = 2^2 \cdot 5^2 \cdot 3^x \;;\; \therefore 5^{x+1-2} = 2^2 \cdot 3^x \;;\; \therefore \log_5 5^{x-1} = \log_5(2^2) + \log_5(3^x)$$

$$\therefore x - 1 = \log_5(2^2) + x\log_5(3) \;;\; \therefore x - x\log_5(3) = \log_5(2^2) + 1$$

$$\therefore x(1 - \log_5(3)) = 2\log_5 2 + 1 \;;\; \therefore x = \frac{2\log_5 2 + 1}{1 - \log_5(3)}$$

36) If $Cx = z$ and $z = \frac{y}{x}$ find $C$ in terms of $x$ and $y$.

$$Cx = \frac{y}{x} \;;\; \therefore C = \frac{y}{x^2}$$

37) Find $\frac{d\{y\}}{dx}$ of $y = \frac{1}{x}$.

$$y = x^{-1} \;;\; \therefore \frac{d\{y\}}{dx} = -x^{-2} \;;\; \therefore \frac{d\{y\}}{dx} = -\frac{1}{x^2}$$

38) Obtain the particular solution of the equation

$$\frac{d^2y}{dx^2} - 2\frac{d\{y\}}{dx} - 3y = (4x+2)\mathfrak{e}^{3x}$$

satisfying $y(0) = 0$ and $\left.\frac{d\{y\}}{dx}\right|_{x=0} = -15/4$

When the auxiliary equation

$$\lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3) = 0$$

is solved, $\lambda = 3, -1 \equiv \alpha, \beta$ is obtained. From Equation (105), we set

$$Y_1(x) = a\mathfrak{e}^{3x} + b\mathfrak{e}^{-x}$$

because $\alpha \neq \beta$ in Equation (105). Since $r(x)$ has the factor of $\mathfrak{e}^{3x}$, we tell that $c = 3$ which is the same as $\alpha$, *i.e.* , $\alpha = c$. Since $r(x)$ has the factor of $4x+2$, we tell that $n = 1$. Thus from Equation (113), we set

$$Y_2(x) = \mathfrak{e}^{3x}x^k\left(\sum_{m=0}^{1} g_m x^m\right) = \mathfrak{e}^{3x}x(g_1x + g_0)$$

with $k = 1$ because $Y_1(x)$ does not have the term $\mathfrak{e}^{3x}x$.

$$\frac{d\{Y_2(x)\}}{dx} = \frac{d\{\mathfrak{e}^{3x}(g_1x^2 + g_0x)\}}{dx} = \mathfrak{e}^{3x}\left(3g_1x^2 + (3g_0 + 2g_1)x + g_0\right)$$

$$\frac{d^2Y_2(x)}{dx^2} = \frac{d\{\mathfrak{e}^{3x}\left(3g_1x^2 + (3g_0 + 2g_1)x + g_0\right)\}}{dx} = \mathfrak{e}^{3x}\Bigg(3(3g_1x^2 + (3g_0 + 2g_1)x + g_0) + (6g_1x + (3g_0 + 2g_1))\Bigg)$$
$$= \mathfrak{e}^{3x}\left(9g_1x^2 + (9g_0 + 12g_1)x + (6g_0 + 2g_1)\right)$$

When we put

$$Y_2(x) = \mathfrak{e}^{3x}x(g_1x + g_0)$$
$$\frac{d\{Y_2(x)\}}{dx} = \mathfrak{e}^{3x}\left(3g_1x^2 + (3g_0 + 2g_1)x + g_0\right)$$
$$\frac{d^2Y_2(x)}{dx^2} = \mathfrak{e}^{3x}\left(9g_1x^2 + (9g_0 + 12g_1)x + (6g_0 + 2g_1)\right)$$

into Equation (118), we get

$$\mathfrak{e}^{3x}\left(9g_1x^2 + (9g_0 + 12g_1)x + (6g_0 + 2g_1)\right) - 2\mathfrak{e}^{3x}\left(3g_1x^2 + (3g_0 + 2g_1)x + g_0\right) - 3\mathfrak{e}^{3x}(g_1x^2 + g_0x) = (4x + 2)\mathfrak{e}^{3x}$$
$$\therefore \mathfrak{e}^{3x}\left(8g_1x + (4g_0 + 2g_1)\right) = (4x + 2)\mathfrak{e}^{3x}$$

By equating coefficients of $x$ and $x^0$, we obtain

$$8g_1 = 4$$
$$4g_0 + 2g_1 = 2$$

From these two equations, we obtain $g_1 = 1/2$ and $g_0 = 1/4$. Thus the general solution

$$y = a\mathfrak{e}^{3x} + b\mathfrak{e}^{-x} + \mathfrak{e}^{3x}x(\frac{1}{2}x + \frac{1}{4})$$

is obtained. In order to use the initial condition, we produce $\frac{d\{y\}}{dx}$ from the general solution and we get

$$\frac{d\{y\}}{dx} = 3a\mathfrak{e}^{3x} - b\mathfrak{e}^{-x} + 3\mathfrak{e}^{3x}(\frac{1}{2}x^2 + \frac{1}{4}x) + \mathfrak{e}^{3x}(x + \frac{1}{4}).$$

By using the initial condition

$$y(0) = a + b = 0$$

and

$$\left.\frac{d\{y\}}{dx}\right|_{x=0} = 3a - b + \frac{1}{4} = -\frac{15}{4},$$

we get $a = -1$ and $b = 1$. Thus the particular solution is

$$y = -\mathfrak{e}^{3x} + \mathfrak{e}^{-x} + \mathfrak{e}^{3x}x(\frac{1}{2}x + \frac{1}{4})$$

39) Obtain the particular solution of the equation

$$\frac{d^2y}{dx^2} - 4\frac{d\{y\}}{dx} + 4y = 8x + 4$$

satisfying $y(0) = 4$ and $\left.\frac{d\{y\}}{dx}\right|_{x=0} = 2$

When the auxiliary equation

$$\lambda^2 - 4\lambda + 4$$

$$= (\lambda - 2)^2 = 0$$

is solved, $\lambda = 2 = \alpha = \beta$ is obtained. From Equation (106), we set

$$Y_1(x) = a\mathfrak{e}^{2x} + bx\mathfrak{e}^{2x}$$

because $\alpha = 2$ in Equation (106). From Equation (110), we set

$$Y_2(x) = \sum_{m=0}^{1} g_m x^m = g_0 + g_1 x$$

$$\frac{d\{Y_2(x)\}}{dx} = \frac{d\{g_1 x + g_0\}}{dx} = g_1$$

$$\frac{d^2 Y_2(x)}{dx^2} = \frac{d\{g_1\}}{dx} = 0$$

When we put

$$Y_2(x) = g_1 x + g_0$$
$$\frac{d\{Y_2(x)\}}{dx} = g_1$$
$$\frac{d^2 Y_2(x)}{dx^2} = 0$$

into Equation (118), we get

$$-4g_1 + 4(g_1 x + g_0) = 8x + 4 \ ; \quad \therefore 4g_1 x + 4g_0 - 4g_1 = 8x + 4$$

By equating coefficients of $x$ and $x^0$ we obtain

$$4g_1 = 8 \qquad ①$$
$$4g_0 - 4g_1 = 4 \qquad ②$$

From the equation ①, we obtain $g_1 = 2$ and from the equation ② we get $g_0 = 3$. Thus the general solution

$$y(x) = a\mathfrak{e}^{2x} + bx\mathfrak{e}^{2x} + 2x + 3$$

is obtained. In order to use the initial condition, we produce $\frac{d\{y\}}{dx}$ from the general solution and we get

$$\frac{d\{y\}}{dx} = 2a\mathfrak{e}^{2x} + b\mathfrak{e}^{2x} + 2bx\mathfrak{e}^{2x} + 2.$$

By using the initial condition

$$y(0) = a + 3 = 4$$

and

$$\left.\frac{d\{y\}}{dx}\right|_{x=0} = 2a + b + 2 = 2,$$

we get $a = 1$ and $b = -2$. Thus the particular solution is

$$y(x) = \mathfrak{e}^{2x} - 2x\mathfrak{e}^{2x} + 2x + 3$$

40) Find the roots of the quadratic equation $\lambda^2 - 3\lambda + 2 = 0$.

$$\lambda^2 - 3\lambda + 2 = 0 \ ; \quad \therefore (\lambda - 2)(\lambda - 1) = 0 \ ; \quad \therefore \lambda = 2, \lambda = 1$$

41) Find the roots of the quadratic equation $\lambda^2 - 4\lambda + 13 = 0$.
When we compare

$$\lambda^2 - 4\lambda + 13 = 0$$

with $ax^2 + bx + c = 0$, we obtain

$$a = 1 \ ; \ b = -4 \ ; \ c = 13$$

Thus we can obtain the answer from

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \ ; \ \therefore \lambda = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 13}}{2} \ ; \ \therefore \lambda = \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$\therefore \lambda = \frac{4}{2} \pm \frac{\sqrt{-36}}{2} \ ; \ \therefore \lambda = 2 \pm \frac{6\jmath}{2} \ ; \ \therefore \lambda = 2 \pm 3\jmath$$

42) Find the roots of the quadratic equation $\lambda^2 - 2\lambda + 1 = 0$

$$\lambda^2 - 2\lambda + 1 = 0 \ ; \ \therefore (\lambda - 1)(\lambda - 1) = 0 \ ; \ \therefore (\lambda - 1)^2 = 0 \ ; \ \therefore \lambda = 1$$

43) Find the roots of the quadratic equation $\lambda^2 - 4\lambda + 4 = 0$

$$\lambda^2 - 4\lambda + 4 = 0 \ ; \ \therefore (\lambda - 2)(\lambda - 2) = 0 \ ; \ \therefore (\lambda - 2)^2 = 0 \ ; \ \therefore \lambda = 2$$

44) Find the general solution of

$$2\frac{\partial^2 y}{\partial t^2} + 4\frac{d\{y\}}{dt} - 6y = \mathfrak{e}^{2t}$$

and then find the particular solution which satisfies $y(0) = 6$ and $\left.\frac{d\{y\}}{dt}\right|_{t=0} = -\frac{4}{5}$.
To solve the following we must first find the complementary function $Y_1(t)$ and and then the particular integral $Y_2(t)$ . The final answer will be the sum of both:

$$y(t) = Y_1(t) + Y_2(t)$$

In order to find out $Y_1(t)$, substituting Equation (103) into $2\frac{\partial^2 y}{\partial t^2} + 4\frac{d\{y\}}{dt} - 6y = 0$, we produce auxiliary equation:

$$2\lambda^2 + 4\lambda - 6 = 0 \ ; \ \therefore 2(\lambda + 3)(\lambda - 1) = 0 \ ; \ \therefore \lambda = -3, 1 \equiv \alpha, \beta$$

Since $\alpha$ and $\beta$ are real and $\alpha \neq \beta$, the complementary function $Y_1(t)$ is

$$Y_1(t) = a\mathfrak{e}^{-3t} + b\mathfrak{e}^{t}$$

Now to find the particular solution, you need to know the correct substitution. $r(x) = \mathfrak{e}^{2t}$ means the coefficient of $t$, *i.e.*, $c = 2$. Since $c \neq \alpha, \beta$, the particular integral is Equation (108)

$$Y_2(t) = g\mathfrak{e}^{2t} \ ; \ \therefore \frac{d\{Y_2(t)\}}{dt} = 2g\mathfrak{e}^{2t} \ ; \ \therefore \frac{\partial^2 Y_2(t)}{\partial t^2} = 4g\mathfrak{e}^{2t}$$

Substituting these into the original ODE

$$2\frac{\partial^2 y}{\partial t^2} + 4\frac{d\{y\}}{dt} - 6y = \mathfrak{e}^{2t} \ ; \ \therefore 2 \cdot 4g\mathfrak{e}^{2t} + 4 \cdot 2g\mathfrak{e}^{2t} - 6g\mathfrak{e}^{2t} = \mathfrak{e}^{2t} \ ; \ \therefore 8g\mathfrak{e}^{2t} + 8g\mathfrak{e}^{2t} - 6g\mathfrak{e}^{2t} = \mathfrak{e}^{2t}$$

$$\therefore 10g\mathfrak{e}^{2t} = \mathfrak{e}^{2t} \ ; \ \therefore 10g = 1 \ ; \ \therefore g = \frac{1}{10}$$

Thus the particular integral is

$$Y_2(t) = \frac{1}{10}\mathfrak{e}^{2t}$$

Therefore the general solution for this ODE is

$$y(t) = Y_1(t) + Y_2(t) = a\mathfrak{e}^{-3t} + b\mathfrak{e}^{t} + \frac{1}{10}\mathfrak{e}^{2t}$$

To find $a$ and $b$ we need to differentiate $y(t)$.

$$y(t) = a\mathfrak{e}^{-3t} + b\mathfrak{e}^{t} + \frac{1}{10}\mathfrak{e}^{2t} \;\; ; \;\; \therefore \frac{d\left\{y(t)\right\}}{dt} = -3a\mathfrak{e}^{-3t} + b\mathfrak{e}^{t} + \frac{2}{10}\mathfrak{e}^{2t} \;\; ; \;\; \therefore \frac{d\left\{y(t)\right\}}{dt} = -3a\mathfrak{e}^{-3t} + b\mathfrak{e}^{t} + \frac{1}{5}\mathfrak{e}^{2t}$$

Since $\frac{d\left\{y\right\}}{dt} = -\frac{4}{5}$ at $t = 0$, we put $(t, \frac{d\left\{y\right\}}{dt}) = (0, -\frac{4}{5})$ into the equation of $\frac{d\left\{y(t)\right\}}{dt}$ as follows:

$$-\frac{4}{5} = -3a\mathfrak{e}^{0} + b\mathfrak{e}^{0} + \frac{1}{5}\mathfrak{e}^{0} \;\; ; \;\; \therefore -\frac{4}{5} = -3a + b + \frac{1}{5} \;\; ; \;\; \therefore -\frac{4}{5} - \frac{1}{5} = -3a + b \;\; ; \;\; \therefore -1 = -3a + b \qquad ①$$

Since $y(0) = 6$, *i.e.*, $(t, y) = (0, 6)$, we get

$$y(t)|_{y=6} = a\mathfrak{e}^{-3t} + b\mathfrak{e}^{t} + \frac{1}{10}\mathfrak{e}^{2t}|_{t=0} \;\; ; \;\; \therefore 6 = a\mathfrak{e}^{0} + b\mathfrak{e}^{0} + \frac{1}{10}\mathfrak{e}^{0}$$

$$\therefore 6 = a + b + \frac{1}{10} \;\; ; \;\; \therefore 6 - \frac{1}{10} = a + b \;\; ; \;\; \therefore \frac{59}{10} = a + b \qquad ②$$

② - ① is

$$\frac{59}{10} + 1 = a + 3a \;\; ; \;\; \therefore \frac{59}{10} + \frac{10}{10} = 4a \;\; ; \;\; \therefore \frac{69}{10} = 4a \;\; ; \;\; \therefore \frac{69}{40} = a$$

By putting $a = \frac{69}{40}$ into ②, we get

$$a + b = \frac{59}{10} \;\; ; \;\; \therefore \frac{69}{40} + b = \frac{59}{10} \;\; ; \;\; \therefore b = \frac{59}{10} - \frac{69}{40} \;\; ; \;\; \therefore b = \frac{59 \cdot 4}{40} - \frac{69}{40} \;\; ; \;\; \therefore b = \frac{236}{40} - \frac{69}{40} \;\; ; \;\; \therefore b = \frac{167}{40}$$

Therefore the particular solution to the ode is

$$y(t) = \frac{69}{40}\mathfrak{e}^{-3t} + \frac{167}{40}\mathfrak{e}^{t} + \frac{1}{10}\mathfrak{e}^{2t}$$

45) Obtain the particular solution of the equation

$$\frac{d^2y}{dx^2} - 2\frac{d\left\{y\right\}}{dx} - 3y = 3x\mathfrak{e}^{2x}$$

satisfying $y(0) = 4/3$ and $\left.\frac{d\left\{y\right\}}{dx}\right|_{x=0} = -25/3$

When the auxiliary equation

$$\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$$

is solved, $\lambda = -1, 3 \equiv \alpha, \beta$ is obtained. From Equation (105), we set

$$Y_1(x) = a\mathfrak{e}^{-x} + b\mathfrak{e}^{3x}$$

because $\alpha = -1, \beta = 3$ in Equation (105). From Equation (112), we set

$$Y_2(x) = \mathfrak{e}^{2x}\left(\sum_{m=0}^{1} g_m x^m\right) = \mathfrak{e}^{2x}(g_0 + g_1 x)$$

because $c = 2$ in Equation (112).

$$\begin{aligned}\frac{d\left\{Y_2(x)\right\}}{dx} &= \frac{d\left\{\mathfrak{e}^{2x}(g_1 x + g_0)\right\}}{dx} \\ &= 2\mathfrak{e}^{2x}(g_1 x + g_0) + g_1\mathfrak{e}^{2x} \\ &= \mathfrak{e}^{2x}(2g_1 x + 2g_0 + g_1)\end{aligned}$$

$$\frac{d^2Y_2(x)}{dx^2} = \frac{d\left\{\mathfrak{e}^{2x}(2g_1x+2g_0+g_1)\right\}}{dx} = 2\mathfrak{e}^{2x}(2g_1x+2g_0+g_1)+\mathfrak{e}^{2x}(2g_1) = \mathfrak{e}^{2x}(4g_1x+4g_0+4g_1)$$

When we put

$$Y_2(x) = \mathfrak{e}^{2x}(g_1x+g_0)$$
$$\frac{d\left\{Y_2(x)\right\}}{dx} = \mathfrak{e}^{2x}(2g_1x+2g_0+g_1)$$
$$\frac{d^2Y_2(x)}{dx^2} = \mathfrak{e}^{2x}(4g_1x+4g_0+4g_1)$$

into Equation (118), we get

$$\mathfrak{e}^{2x}(4g_1x+4g_0+4g_1)-2\mathfrak{e}^{2x}(2g_1x+2g_0+g_1) \;\; ; \quad \therefore -3\mathfrak{e}^{2x}(g_1x+g_0) = 3x\mathfrak{e}^{2x}$$
$$\therefore \mathfrak{e}^{2x}(4g_1x+4g_0+4g_1-4g_1x-4g_0-2g_1-3g_1x-3g_0) = 3x\mathfrak{e}^{2x} \;\; ; \quad \therefore \mathfrak{e}^{2x}(2g_1-3g_0-3g_1x) = 3x\mathfrak{e}^{2x}$$

By equating coefficients of $x$ and $x^0$ we obtain

$$2g_1-3g_0 = 0$$
$$-3g_1 = 3$$

From these equations, we obtain $g_1 = -1$ and $g_0 = -2/3$. Thus the general solution

$$y = a\mathfrak{e}^{-x}+b\mathfrak{e}^{3x}+\mathfrak{e}^{2x}(-x-2/3)$$

is obtained. In order to use the initial condition, we produce $\frac{d\left\{y\right\}}{dx}$ from the general solution and we get

$$\frac{d\left\{y\right\}}{dx} = -a\mathfrak{e}^{-x}+3b\mathfrak{e}^{3x}+2\mathfrak{e}^{2x}(-x-2/3)-\mathfrak{e}^{2x}.$$

By using the initial condition

$$y(0) = a+b-2/3 = 4/3$$

and

$$\left.\frac{d\left\{y\right\}}{dx}\right|_{x=0} = -a+3b-7/3 = -25/3,$$

we get $a = 3$ and $b = -1$. Thus the particular solution is

$$y = 3\mathfrak{e}^{-x}-\mathfrak{e}^{3x}+\mathfrak{e}^{2x}(-x-2/3)$$

46) Obtain the particular solution of the equation

$$\frac{d^2y}{dx^2}+4\frac{d\left\{y\right\}}{dx}+3y = 3x^2+2x$$

satisfying $y(0) = 1$ and $\left.\frac{d\left\{y\right\}}{dx}\right|_{x=0} = 1$

When the auxiliary equation

$$\lambda^2+4\lambda+3 = (\lambda+1)(\lambda+3) = 0$$

is solved, $\lambda = -1, -3$ is obtained. From Equation (105), we set

$$Y_1(x) = a\mathfrak{e}^{-x}+b\mathfrak{e}^{-3x}$$

because $\alpha \neq \beta$ in Equation (105). From Equation (110), we set

$$Y_2(x) = \sum_{m=0}^{2} g_m x^m = g_2x^2+g_1x+g_0$$

$$\frac{d\{Y_2(x)\}}{dx} = \frac{d\{g_2x^2 + g_1x + g_0\}}{dx} = 2g_2x + g_1$$

$$\frac{d^2Y_2(x)}{dx^2} = \frac{d\{2g_2x + g_1\}}{dx} = 2g_2$$

When we put

$$Y_2(x) = g_2x^2 + g_1x + g_0$$
$$\frac{d\{Y_2(x)\}}{dx} = 2g_2x + g_1$$
$$\frac{d^2Y_2(x)}{dx^2} = 2g_2$$

into Equation (118), we get

$$2g_2 + 4(2g_2x + g_1) + 3(g_2x^2 + g_1x + g_0) = 3x^2 + 2x \;\; ; \quad \therefore 3g_2x^2 + (8g_2 + 3g_1)x + 2g_2 + 4g_1 + 3g_0 = 3x^2 + 2x$$

By equating coefficients of $x^2$, $x$ and $x^0$, we obtain

$$3g_2 = 3$$
$$8g_2 + 3g_1 = 2$$
$$2g_2 + 4g_1 + 3g_0 = 0$$

From these three equations, we obtain $g_2 = 1$, $g_1 = -2$ and $g_0 = 2$. Thus the general solution

$$y = a\mathfrak{e}^{-x} + b\mathfrak{e}^{-3x} + x^2 - 2x + 2$$

is obtained. In order to use the initial condition, we produce $\frac{d\{y\}}{dx}$ from the general solution and we get

$$\frac{d\{y\}}{dx} = -a\mathfrak{e}^{x} - 3b\mathfrak{e}^{-3x} + 2x - 2.$$

By using the initial condition

$$y(0) = a + b + 2 = 1$$

and

$$\left.\frac{d\{y\}}{dx}\right|_{x=0} = -a - 3b - 2 = 1,$$

we get $a = 0$ and $b = -1$. Thus the particular solution is

$$y = -\mathfrak{e}^{-3x} + x^2 - 2x + 2.$$

47) Obtain the particular solution of the equation

$$\frac{d^2y}{dx^2} + \frac{d\{y\}}{dx} = 2x + 4$$

satisfying $y(0) = 8$ and $\left.\frac{d\{y\}}{dx}\right|_{x=0} = -3$

When the auxiliary equation

$$\lambda^2 + \lambda = \lambda(\lambda + 1) = 0$$

is solved, $\lambda = -1, 0$ is obtained. From Equation (105), we set

$$Y_1(x) = a + b\mathfrak{e}^{-x}$$

because $\alpha = 0, \beta = -1$ in Equation (105). From Equation (111), we set

$$Y_2(x) = x^k \left( \sum_{m=0}^{1} g_m x^m \right) = x(g_0 + g_1 x) = g_0 x + g_1 x^2$$

with $k = 1$ because $Y_1(x)$ does not have $x$ or $x^2$.

$$\frac{d\{Y_2(x)\}}{dx} = \frac{d\{g_1 x^2 + g_0 x\}}{dx} = 2g_1 x + g_0$$

$$\frac{d^2 Y_2(x)}{dx^2} = \frac{d\{2g_1 x + g_0\}}{dx} = 2g_1$$

When we put

$$Y_2(x) = g_1 x^2 + g_0 x$$
$$\frac{d\{Y_2(x)\}}{dx} = 2g_1 x + g_0$$
$$\frac{d^2 Y_2(x)}{dx^2} = 2g_1$$

into Equation (118), we get

$$2g_1 + 2g_1 x + g_0 = 2x + 4$$

By equating coefficients of $x$ and $x^0$ we obtain

$$2g_1 = 2$$
$$2g_1 + g_0 = 4$$

From these equations, we obtain $g_1 = 1$ and $g_0 = 2$. Thus the general solution

$$y = a + b\mathfrak{e}^{-x} + x^2 + 2x$$

is obtained. In order to use the initial condition, we produce $\frac{d\{y\}}{dx}$ from the general solution and we get

$$\frac{d\{y\}}{dx} = -b\mathfrak{e}^{-x} + 2x + 2.$$

By using the initial condition

$$y(0) = a + b = 8$$

and

$$\left.\frac{d\{y\}}{dx}\right|_{x=0} = -b + 2 = -3,$$

we get $a = 3$ and $b = 5$. Thus the particular solution is $y = 3 + 5\mathfrak{e}^{-x} + x^2 + 2x$

48) Obtain the particular solution of the equation

$$\frac{d^2 y}{dx^2} - 3\frac{d\{y\}}{dx} + 2y = \mathfrak{e}^x$$

satisfying $y(0) = 1$ and $\left.\frac{d\{y\}}{dx}\right|_{x=0} = 2$

When the auxiliary equation

$$\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0$$

is solved,

$$\lambda = 1, 2 \triangleq \alpha, \beta$$

is obtained. From Equation (105), we set

$$Y_1(x) = a\mathfrak{e}^{x} + b\mathfrak{e}^{2x}$$

because

$$\alpha \neq \beta$$

in Equation (105).
Since

$$c = 1$$

and

$$\alpha = 1,$$

from Equation (109) we set

$$Y_2(x) = gx^k\mathfrak{e}^{x}.$$

Since $Y_1(x)$ does not have a term of $x\mathfrak{e}^x$, we try to set $k = 1$. As $Y_2(x)$ is in the same form as $Y_1(x)$ we need to add an $x$ to make it a unique solution.
Now we need to find out $g$ by using Equation (118). Thus we need the first and second derivative of $Y_2(x)$.

$$\frac{d\left\{Y_2(x)\right\}}{dx} = \frac{d\left\{gx\mathfrak{e}^x\right\}}{dx} = g\frac{\partial\{x\mathfrak{e}^x\}}{\partial x} = g(x\mathfrak{e}^x + \mathfrak{e}^x) = gx\mathfrak{e}^x + g\mathfrak{e}^x$$

$$\frac{d^2Y_2(x)}{dx^2} = \frac{d\left\{gx\mathfrak{e}^x + g\mathfrak{e}^x\right\}}{dx} = g\frac{\partial(x\mathfrak{e}^x + \mathfrak{e}^x)}{\partial x} = g(\mathfrak{e}^x + x\mathfrak{e}^x + \mathfrak{e}^x) = gx\mathfrak{e}^x + 2g\mathfrak{e}^x$$

When we put

$$Y_2(x) = gx\mathfrak{e}^x$$

$$\frac{d\left\{Y_2(x)\right\}}{dx} = gx\mathfrak{e}^x + g\mathfrak{e}^x$$

$$\frac{d^2Y_2(x)}{dx^2} = gx\mathfrak{e}^x + 2g\mathfrak{e}^x$$

into Equation (118), we get

$$\frac{d^2y}{dx^2} - 3\frac{d\left\{y\right\}}{dx} + 2y = \mathfrak{e}^x \quad ; \quad \therefore gx\mathfrak{e}^x + 2g\mathfrak{e}^x - 3(gx\mathfrak{e}^x + g\mathfrak{e}^x) + 2gx\mathfrak{e}^x = \mathfrak{e}^x \quad ; \quad \therefore -g\mathfrak{e}^x = \mathfrak{e}^x \quad ; \quad \therefore g = -1$$

Thus the general solution

$$y = a\mathfrak{e}^x + b\mathfrak{e}^{2x} - x\mathfrak{e}^x$$

is obtained.
In order to use the initial condition to find out $a$ and $b$, we produce $\frac{d\left\{y\right\}}{dx}$ from the general solution and we get

$$\frac{d\left\{y\right\}}{dx} = a\mathfrak{e}^x + 2b\mathfrak{e}^{2x} - x\mathfrak{e}^x - \mathfrak{e}^x.$$

By using the initial condition

$$y(0) = a + b = 1$$

and

$$\left.\frac{d\left\{y\right\}}{dx}\right|_{x=0} = a + 2b - 1 = 2,$$

we get

$$a = -1$$

and

$$b = 2.$$

Thus the particular solution is

$$y = -\mathfrak{e}^x + 2\mathfrak{e}^{2x} - x\mathfrak{e}^x$$

**DAY4**

49) Simplify $y = \mathfrak{e}^{-3\ln|x+4|}$.

$$y = \mathfrak{e}^{-3\ln|x+4|} \;\; ; \quad \therefore y = \mathfrak{e}^{\ln|x+4|^{-3}} \;\; ; \quad \therefore \ln y = \ln\left(\mathfrak{e}^{\ln|x+4|^{-3}}\right) \;\; ;$$
$$\therefore \ln y = \ln|x+4|^{-3}\ln(\mathfrak{e}) \therefore \ln y = \ln|x+4|^{-3} \;\; ; \quad \therefore y = |x+4|^{-3}$$

50) Simplify $\ln y = -\ln(x+c)$.

$$\ln y = -\ln(x+c) \;\; ; \quad \therefore \ln y = \ln(x+c)^{-1} \;\; ; \quad \therefore y = (x+c)^{-1} \;\; ; \quad \therefore y = \frac{1}{x+c}$$

51) Make $a$ the subject of $0 = 2x - \frac{1}{4a}$.

$$\frac{1}{4a} = 2x \;\; ; \quad \therefore \frac{1}{2x} = 4a \;\; ; \quad \therefore a = \frac{1}{8x}$$

52) Solve the following equation for $x$. $x^2 - 4 = 0$

$$x^2 - 4 = 0 \;\; ; \quad \therefore x^2 = 4 \;\; ; \quad \therefore x = \pm 4^{0.5} \;\; ; \quad \therefore x = \pm 2$$

53) Find the general solution of

$$\frac{d^2y}{dx^2} - 4\frac{d\{y\}}{dx} + 13y = 0$$

and then find the particular solution that satisfies $\left.\frac{d\{y\}}{dx}\right|_{x=0} = -4$ and $y(0) = 3$.

Substituting Equation (103) into $\frac{d^2y}{dx^2} - 4\frac{d\{y\}}{dx} + 13y = 0$, we produce auxiliary equation:

$$\lambda^2 - 4\lambda + 13 = 0$$

Now we need to factorise this and work out the roots.
Because $b^2 - 4ac < 0$ there are no real roots to this quadratic. Therefore both roots will be complex. Remember $\jmath = \sqrt{-1}$.

$$\lambda^2 - 4\lambda + 13 = 0$$

is identical to $ax^2 + bx + c = 0$ when

$$a = 1 \;\; ; \;\; b = -4 \;\; ; \;\; c = 13$$

Thus, the answer is

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 13}}{2} = \frac{4 \pm \sqrt{16 - 52}}{2}$$
$$= \frac{4}{2} \pm \frac{\sqrt{-36}}{2} = 2 \pm \frac{6\jmath}{2} = 2 \pm 3\jmath \equiv p \pm \jmath q \therefore p = 2, q = 3$$

The general form of the solution of an ODE with complex roots is

$$y(x) = \mathfrak{e}^{px}\left[a\cos qx + b\sin qx\right]$$

Now by substituting $p = 2, q = 3$ into the general form, the general solution is

$$y(x) = \mathfrak{e}^{2x}\left[a\cos 3x + b\sin 3x\right]$$

In order to find the particular solution we need to find $\frac{d\{y\}}{dx}$. Let $g(x) = \mathfrak{e}^{2x}$ and $f(x) = b\sin 3x + a\cos 3x$. Using the product rule

$$\frac{d\{y\}}{dx} = g(x) \cdot \frac{d\{f(x)\}}{dx} + f(x) \cdot \frac{d\{g(x)\}}{dx}$$

with

$$\frac{d\{g(x)\}}{dx} = 2\mathfrak{e}^{2x}$$

$$\frac{d\{f(x)\}}{dx} = 3b\cos 3x - 3a\sin 3x$$

we can obtain

$$\frac{d\{y\}}{dx} = \mathfrak{e}^{2x} \cdot (3b\cos 3x - 3a\sin 3x) + 2\mathfrak{e}^{2x} \cdot (b\sin 3x + a\cos 3x)$$

Now putting $(x, \frac{d\{y\}}{dx}) = (0, -4)$ into the equation of $\frac{d\{y\}}{dx}$, we get

$$-4 = \mathfrak{e}^0 \cdot (3b\cos 0 - 3a\sin 0) + 2\mathfrak{e}^0 \cdot (b\sin 0 + a\cos 0) \quad ; \quad \therefore -4 = 3b + 2a$$

We now use the other condition $y(0) = 3$. By putting $(x, y) = (0, 3)$ into the general solution, we get

$$3 = \mathfrak{e}^0 [a\cos 0 + b\sin 0] \quad ; \quad \therefore 3 = a$$

$b$ is obtained by putting $a = 3$ into $-4 = 3b + 2a$ :

$$-4 = 3b + 2a \quad ; \quad \therefore -4 = 3b + 2 \cdot 3 \quad ; \quad \therefore -4 = 3b + 6 \quad ; \quad \therefore b = \frac{-10}{3}$$

Therefore the particular solution is

$$y(x) = \mathfrak{e}^{2x} [b\sin 3x + a\cos 3x] = \mathfrak{e}^{2x} \left[\frac{-10}{3}\sin 3x + 3\cos 3x\right]$$

54) Obtain the particular solution of the equation

$$\frac{d^2y}{dx^2} - 2\frac{d\{y\}}{dx} + y = \mathfrak{e}^x$$

satisfying $y(0) = 2$ and $\left.\frac{d\{y\}}{dx}\right|_{x=0} = 5$

When the auxiliary equation

$$\lambda^2 - 2\lambda + 1 = 0$$

is solved, $\lambda = 1 = \alpha = \beta$ is obtained. From Equation (106), we set

$$Y_1(x) = a\mathfrak{e}^x + bx\mathfrak{e}^x$$

because $\alpha = 1$ in Equation (106). Since $c = 1$ and $\alpha = 1$, from Equation (109) we set

$$Y_2(x) = gx^k\mathfrak{e}^x.$$

Since $Y_1(x)$ has already a term of $x\mathfrak{e}^x$, we try to set $k = 2$ to avoid the duplication of the like term.

$$\frac{d\{Y_2(x)\}}{dx} = \frac{d\{gx^2\mathfrak{e}^x\}}{dx} = gx^2\mathfrak{e}^x + 2gx\mathfrak{e}^x$$

$$\frac{d^2Y_2(x)}{dx^2} = \frac{d\{gx^2\mathfrak{e}^x + 2gx\mathfrak{e}^x\}}{dx} = gx^2\mathfrak{e}^x + 2gx\mathfrak{e}^x + 2gx\mathfrak{e}^x + 2g\mathfrak{e}^x$$

When we put

$$Y_2(x) = gx^2\mathfrak{e}^x$$
$$\frac{d\{Y_2(x)\}}{dx} = gx^2\mathfrak{e}^x + 2gx\mathfrak{e}^x$$
$$\frac{d^2Y_2(x)}{dx^2} = gx^2\mathfrak{e}^x + 4gx\mathfrak{e}^x + 2g\mathfrak{e}^x$$

into Equation (118), we get

$$gx^2\mathfrak{e}^x + 2gx\mathfrak{e}^x + 2g\mathfrak{e}^x - 2(gx^2\mathfrak{e}^x + 2gx\mathfrak{e}^x) + gx^2\mathfrak{e}^x = \mathfrak{e}^x \ ; \quad \therefore 2g\mathfrak{e}^x = \mathfrak{e}^x \ ; \quad \therefore g = \frac{1}{2}$$

Thus the general solution

$$y = a\mathfrak{e}^x + bx\mathfrak{e}^x + \frac{1}{2}x^2\mathfrak{e}^x$$

is obtained. In order to use the initial condition, we produce $\frac{d\{y\}}{dx}$ from the general solution and we get

$$\frac{d\{y\}}{dx} = a\mathfrak{e}^x + b\mathfrak{e}^x + bx\mathfrak{e}^x + x\mathfrak{e}^x + \frac{1}{2}x^2\mathfrak{e}^x.$$

By using the initial condition

$$y(0) = a = 2$$

and

$$\left.\frac{d\{y\}}{dx}\right|_{x=0} = a + b = 5,$$

we get $a = 2$ and $b = 3$. Thus the particular solution is

$$y = 2\mathfrak{e}^x + 3x\mathfrak{e}^x + \frac{1}{2}x^2\mathfrak{e}^x.$$

55) Solve for $x$ for the following $\frac{3x-1}{2} = x + 2$

$$\frac{3x-1}{2} = x + 2 \ ; \quad \therefore 3x - 1 = 2(x+2) \ ; \quad \therefore 3x - 1 = 2x + 4 \ ; \quad \therefore 3x - 2x = 4 + 1 \ ; \quad \therefore x = 5$$

56) Find the roots of this quadratic equation $\lambda^2 + 2\lambda + 2 = 0$.

$$\lambda^2 + 2\lambda + 2 = 0$$

is identical to $ax^2 + bx + c = 0$ when

$$a = 1 \ ; \ b = 2 \ ; \ c = 2$$

Thus, the answer is

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot 2}}{2} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2\jmath}{2} = -1 \pm \jmath$$

57) Find the roots of this quadratic equation $\lambda^2 - 2\lambda - 3 = 0$

$$\lambda^2 - 2\lambda - 3 = 0$$

is identical to $ax^2 + bx + c = 0$ when

$$a = 1 \ ; \ b = -2 \ ; \ c = -3$$

Thus the answer is

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-3)}}{2} = \frac{2 \pm \sqrt{4 + 12}}{2} = \frac{2 \pm \sqrt{16}}{2} = \frac{2 \pm 4}{2} = 1 \pm 2 = 3, -1$$

58) Find the roots of the quadratic equation $\lambda^2 + 12\lambda + 8 = 0$.

$$\lambda^2 + 12\lambda + 8 = 0$$

is identical to $ax^2 + bx + c = 0$ when

$$a = 1 \ ; \ b = 12 \ ; \ c = 8$$

Thus, the answer is

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-12 \pm \sqrt{12^2 - 4 \cdot 1 \cdot 8}}{2} = \frac{-12 \pm \sqrt{144 - 32}}{2} = \frac{-12 \pm \sqrt{112}}{2}$$
$$= \frac{-12}{2} \pm \frac{\sqrt{112}}{2} = -6 \pm \frac{4\sqrt{7}}{2} = -6 \pm 2\sqrt{7}$$

59) Obtain the particular solution of the equation

$$\frac{d^2y}{dx^2} + 2\frac{d\{y\}}{dx} + 2y = 5 \sin 2x$$

satisfying

$$y(0) = 0$$

and

$$\left.\frac{d\{y\}}{dx}\right|_{x=0} = 2$$

When the auxiliary equation

$$\lambda^2 + 2\lambda + 2 = 0$$

is solved,

$$\lambda = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm \jmath \equiv p \pm q\jmath$$

is obtained because $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ satisfies $ax^2 + bx + c = 0$. From Equation (107), we set

$$Y_1(x) = \mathfrak{e}^{-x}(a \cos x + b \sin x)$$

because $p = -1$ and $q = 1$. Since $\omega = 2$ and $-1 \pm \jmath \neq \jmath\omega$, from Equation (114) we set

$$Y_2(x) = g \cos 2x + h \sin 2x.$$

In order to make use of Equation (118), we need to find out $\frac{d\{Y_2(x)\}}{dx}$ and $\frac{d^2Y_2(x)}{dx^2}$. Thus we calculate these as follows:

$$\frac{d\{Y_2(x)\}}{dx} = \frac{d\{g \cos 2x + h \sin 2x\}}{dx} = -2g \sin 2x + 2h \cos 2x$$

$$\frac{d^2Y_2(x)}{dx^2} = \frac{d\{-2g \sin 2x + 2h \cos 2x\}}{dx} = -4g \cos 2x - 4h \sin 2x$$

When we put

$$Y_2(x) = g\cos 2x + h\sin 2x$$
$$\frac{d\{Y_2(x)\}}{dx} = -2g\sin 2x + 2h\cos 2x$$
$$\frac{d^2Y_2(x)}{dx^2} = -4g\cos 2x - 4h\sin 2x$$

into Equation (118), we get

$$-4g\cos 2x - 4h\sin 2x + 2(-2g\sin 2x + 2h\cos 2x)$$
$$+2(g\cos 2x + h\sin 2x) = 5\sin 2x \;\; ; \;\; \therefore -4g\cos 2x - 4h\sin 2x - 4g\sin 2x + 4h\cos 2x$$
$$+2g\cos 2x + 2h\sin 2x = 5\sin 2x \;\; ; \;\; \therefore (-2g+4h)\cos 2x + (-2h-4g)\sin 2x = 5\sin 2x$$

By equating coefficients of $\cos 2x$ and $\sin 2x$ in the equation, we find

$$-2g + 4h = 0$$
$$-2h - 4g = 5.$$

$-2g + 4h = 0$ is the same as $g = 2h$ and we put $g = 2h$ into $-2h - 4g = 5$ and we obtain $h = -0.5$ and thus

$$g = 2h = 2 \times (-0.5) = -1.$$

Thus the general solution

$$y = \mathfrak{e}^{-x}(a\cos x + b\sin x) - \cos 2x - 0.5\sin 2x$$

is obtained. In order to use the initial condition, we produce $\frac{d\{y\}}{dx}$ from the general solution and we get

$$\frac{d\{y\}}{dx} = -\mathfrak{e}^{-x}(a\cos x + b\sin x) + \mathfrak{e}^{-x}(-a\sin x + b\cos x) + 2\sin 2x - \cos 2x.$$

By using the initial condition

$$y(0) = a - 1 = 0$$

and

$$\left.\frac{d\{y\}}{dx}\right|_{x=0} = -a + b - 1 = 2,$$

we get $a = 1$ and $b = 4$. Thus the particular solution is

$$y = \mathfrak{e}^{-x}(\cos x + 4\sin x) - \cos 2x - 0.5\sin 2x.$$

60) Obtain the particular solution of the equation

$$\frac{d^2y}{dx^2} + y = 2\sin x + 3\cos x$$

satisfying $y(0) = 3$ and $\left.\frac{d\{y\}}{dx}\right|_{x=0} = 4$

When the auxiliary equation

$$\lambda^2 + 1 = 0$$

is solved, $\lambda = 0 \pm \jmath$ is obtained. From Equation (107), we set

$$Y_1(x) = a\cos x + b\sin x$$

because p= 0, q = 1 in Equation (107). From Equation (115), we set

$$Y_2(x) = x^k(g\cos x + h\sin x)$$

with $k = 1$ because $\omega = 1$ and $Y_1(x)$ does not have $x \cos x$ or $x \sin x$.

$$\frac{d\{Y_2(x)\}}{dx} = \frac{d\{x(g\cos x + h\sin x)\}}{dx} = (g\cos x + h\sin x) + x(-g\sin x + h\cos x)$$
$$= g\cos x + h\sin x + -gx\sin x + hx\cos x = (g + hx)\cos x + (h - gx)\sin x$$

$$\frac{d^2Y_2(x)}{dx^2} = \frac{d\{(g + hx)\cos x + (h - gx)\sin x\}}{dx} = -(g + hx)\sin x + (h - gx)\cos x + h\cos x - g\sin x = -(2g + h$$

When we put

$$Y_2(x) = gx\cos x + hx\sin x$$
$$\frac{d\{Y_2(x)\}}{dx} = (g + hx)\cos x + (h - gx)\sin x$$
$$\frac{d^2Y_2(x)}{dx^2} = -(2g + hx)\sin x + (2h - gx)\cos x$$

into Equation (118), we get

$$-(2g + hx)\sin x + (2h - gx)\cos x + gx\cos x + hx\sin x = 2\sin x + 3\cos x$$
$$\therefore -2g\sin x + 2h\cos x = 2\sin x + 3\cos x$$

By equating coefficients of $\cos x$ and $\sin x$, we obtain

$$-2g = 2$$
$$2h = 3$$

From these two equations, we obtain $g = -1$ and $h = 3/2$. Thus the general solution

$$y = a\cos x + b\sin x + x(-\cos x + \frac{3}{2}\sin x)$$

is obtained. In order to use the initial condition, we produce $\frac{d\{y\}}{dx}$ from the general solution and we get

$$\frac{d\{y\}}{dx} = -a\sin x + b\cos x + (-\cos x + \frac{3}{2}\sin x) + x(\sin x + \frac{3}{2}\cos x).$$

By using the initial condition

$$y(0) = a = 3$$

and

$$\left.\frac{d\{y\}}{dx}\right|_{x=0} = b - 1 = 4,$$

we get $a = 3$ and $b = 5$. Thus the particular solution is

$$y = 3\cos x + 5\sin x + x(-\cos x + \frac{3}{2}\sin x)$$

61) Obtain the particular solution of the equation

$$\frac{d^2y}{dx^2} - 4\frac{d\{y\}}{dx} + 5y = -3\mathfrak{e}^x\cos x$$

satisfying $y(0) = 7/5$ and $\left.\frac{d\{y\}}{dx}\right|_{x=0} = 18/5$

When the auxiliary equation

$$\lambda^2 - 4\lambda + 5 = 0$$

is solved,

$$\lambda = 2 \pm \jmath \equiv p \pm q\jmath$$

is obtained because $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ satisfies $ax^2 + bx + c = 0$. From Equation (107), we set

$$Y_1(x) = \mathfrak{e}^{2x}(a \cos x + b \sin x)$$

because $p = 2, q = 1$ in Equation (107). From Equation (116), we set

$$Y_2(x) = \mathfrak{e}^{x}(g \cos x + h \sin x)$$

because $\omega = 1$ and $c = 1$.

$$\frac{d\{Y_2(x)\}}{dx} = \frac{d\{\mathfrak{e}^{x}(g \cos x + h \sin x)\}}{dx} = \mathfrak{e}^{x}(g \cos x + h \sin x) + \mathfrak{e}^{x}(-g \sin x + h \cos x)$$
$$= \mathfrak{e}^{x}((g+h) \cos x + (h-g) \sin x)$$

$$\frac{d^2 Y_2(x)}{dx^2} = \frac{d\{\mathfrak{e}^{x}((g+h) \cos x + (h-g) \sin x)\}}{dx}$$
$$= \mathfrak{e}^{x}((g+h) \cos x + (h-g) \sin x) + \mathfrak{e}^{x}(-(g+h) \sin x + (h-g) \cos x) = \mathfrak{e}^{x}(2h \cos x - 2g \sin x)$$

When we put

$$Y_2(x) = \mathfrak{e}^{x}(g \cos x + h \sin x)$$
$$\frac{d\{Y_2(x)\}}{dx} = \mathfrak{e}^{x}((g+h) \cos x + (h-g) \sin x)$$
$$\frac{d^2 Y_2(x)}{dx^2} = \mathfrak{e}^{x}(2h \cos x - 2g \sin x)$$

into Equation (118), we get

$$\mathfrak{e}^{x}(2h \cos x - 2g \sin x) - 4\mathfrak{e}^{x}((g+h) \cos x + (h-g) \sin x) + 5\mathfrak{e}^{x}(g \cos x + h \sin x) = -3\mathfrak{e}^{x} \cos x$$
$$\therefore \mathfrak{e}^{x}((2h - 4g - 4h + 5g) \cos x + (-2g - 4h + 4g + 5h) \sin x) = -3\mathfrak{e}^{x} \cos x$$

By equating coefficients of $\cos x$ and $\sin x$, we obtain

$$\begin{aligned} -2h + g &= -3 \\ 2g + h &= 0 \end{aligned}$$

From these two equations, we obtain $g = -3/5$ and $h = 6/5$. Thus the general solution

$$y = \mathfrak{e}^{2x}(a \cos x + b \sin x) + \mathfrak{e}^{x}(\frac{-3}{5} \cos x + \frac{6}{5} \sin x)$$

is obtained. In order to use the initial condition, we produce $\dfrac{d\{y\}}{dx}$ from the general solution and we get

$$\frac{d\{y\}}{dx} = 2\mathfrak{e}^{2x}(a \cos x + b \sin x) + \mathfrak{e}^{2x}(-a \sin x + b \cos x) + \mathfrak{e}^{x}(\frac{-3}{5} \cos x + \frac{6}{5} \sin x) + \mathfrak{e}^{x}(\frac{3}{5} \sin x + \frac{6}{5} \cos x)$$

By using the initial condition

$$y(0) = a - 3/5 = 7/5$$

and

$$\left.\frac{d\{y\}}{dx}\right|_{x=0} = 2a + b + 3/5 = 18/5,$$

we get $a = 2$ and $b = -1$. Thus the particular solution is

$$y = \mathfrak{e}^{2x}(2 \cos x - \sin x) + \mathfrak{e}^{x}(\frac{-3}{5} \cos x + \frac{6}{5} \sin x)$$

62) Obtain the particular solution of the equation

$$\frac{d^2y}{dx^2} - 2\frac{d\left\{y\right\}}{dx} + 2y = 2\cos x$$

satisfying $y(0) = 3$ and $\left.\frac{d\left\{y\right\}}{dx}\right|_{x=0} = 4$

When the auxiliary equation

$$\lambda^2 - 2\lambda + 2 = 0$$

is solved, $\lambda = 1 \pm \jmath$ is obtained. From Equation (107), we set

$$Y_1(x) = \mathfrak{e}^x(a\cos x + b\sin x)$$

because $p = 1, q = 1$ in Equation (107). From Equation (114), we set

$$Y_2(x) = g\cos x + h\sin x$$

because $\omega = 1$ in Equation (114)

$$\frac{d\left\{Y_2(x)\right\}}{dx} = \frac{d\left\{g\cos x + h\sin x\right\}}{dx} = -g\sin x + h\cos x$$

$$\frac{d^2Y_2(x)}{dx^2} = \frac{d\left\{-g\sin x + h\cos x\right\}}{dx} = -g\cos x - h\sin x$$

When we put

$$\begin{aligned} Y_2(x) &= g\cos x + h\sin x \\ \frac{d\left\{Y_2(x)\right\}}{dx} &= -g\sin x + h\cos x \\ \frac{d^2Y_2(x)}{dx^2} &= -g\cos x - h\sin x \end{aligned}$$

into Equation (118), we get

$$-g\cos x - h\sin x - 2(-g\sin x + h\cos x) + 2(g\cos x + h\sin x) = 2\cos x$$
$$\therefore (h + 2g)\sin x + (g - 2h)\cos x = 2\cos x$$

By equating coefficients of $\cos x$ and $\sin x$ we obtain

$$\begin{aligned} h + 2g &= 0 \\ g - 2h &= 2 \end{aligned}$$

From those equations, we obtain $g = 2/5$ and $h = -4/5$. Thus the general solution

$$y = \mathfrak{e}^x(a\cos x + b\sin x) + \frac{2}{5}\cos x - \frac{4}{5}\sin x$$

is obtained. In order to use the initial condition, we produce $\frac{d\left\{y\right\}}{dx}$ from the general solution and we get

$$\frac{d\left\{y\right\}}{dx} = \mathfrak{e}^x(a\cos x + b\sin x) - \frac{2}{5}\sin x - \frac{4}{5}\cos x + \mathfrak{e}^x(-a\sin x + b\cos x).$$

By using the initial condition

$$y(0) = a + \frac{2}{5} = 3$$

and

$$\left.\frac{d\left\{y\right\}}{dx}\right|_{x=0} = a - \frac{4}{5} + b = 4,$$

we get $a = 13/5$ and $b = 11/5$. Thus the particular solution is

$$y = \mathfrak{e}^{x}(\frac{13}{5}\cos x + \frac{11}{5}\sin x) + \frac{2}{5}\cos x - \frac{4}{5}\sin x$$

63) Obtain the particular solution of the equation

$$\frac{d^2y}{dx^2} - 4\frac{d\{y\}}{dx} + 13y = 18\mathfrak{e}^{2x}$$

satisfying $y(0) = 5$ and $\left.\frac{d\{y\}}{dx}\right|_{x=0} = 4$

When the auxiliary equation $\lambda^2 - 4\lambda + 13 = 0$ is solved,

$$\lambda = 2 \pm 3\jmath \equiv p \pm q\jmath$$

is obtained because the solution of $ax^2 + bx + c = 0$ is $\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. From Equation (107), we set

$$Y_1(x) = \mathfrak{e}^{2x}(a\cos 3x + b\sin 3x)$$

because $p = 2, q = 3$ in Equation (107). From Equation (108) with $c = 2$, we set

$$Y_2(x) = g\mathfrak{e}^{2x}$$

$$\frac{d\{Y_2(x)\}}{dx} = \frac{d\{g\mathfrak{e}^{2x}\}}{dx} = 2g\mathfrak{e}^{2x}$$

$$\frac{d^2Y_2(x)}{dx^2} = \frac{d\{2g\mathfrak{e}^{2x}\}}{dx} = 4g\mathfrak{e}^{2x}$$

When we put

$$Y_2(x) = g\mathfrak{e}^{2x}$$
$$\frac{d\{Y_2(x)\}}{dx} = 2g\mathfrak{e}^{2x}$$
$$\frac{d^2Y_2(x)}{dx^2} = 4g\mathfrak{e}^{2x}$$

into Equation (118), we get

$$4g\mathfrak{e}^{2x} - 4(2g\mathfrak{e}^{2x}) + 13g\mathfrak{e}^{2x} = 18\mathfrak{e}^{2x} \quad ; \quad \therefore 9g\mathfrak{e}^{2x} = 18\mathfrak{e}^{2x}$$

By equating coefficients of $\mathfrak{e}^{2x}$ we obtain $9g = 18$. From these one equation, we obtain $g = 2$. Thus the general solution

$$y = \mathfrak{e}^{2x}(a\cos 3x + b\sin 3x) + 2\mathfrak{e}^{2x}$$

is obtained. In order to use the initial condition, we produce $\frac{d\{y\}}{dx}$ from the general solution and we get

$$\frac{d\{y\}}{dx} = 2\mathfrak{e}^{2x}(a\cos 3x + b\sin 3x + 2) + \mathfrak{e}^{2x}(-3a\sin 3x + 3b\cos 3x).$$

By using the initial condition

$$y(0) = a + 2 = 5$$

and

$$\left.\frac{d\{y\}}{dx}\right|_{x=0} = 2a + 3b + 4 = 4,$$

we get $a = 3$ and $b = -2$. Thus the particular solution is

$$y = \mathfrak{e}^{2x}(3\cos 3x - 2\sin 3x) + 2\mathfrak{e}^{2x}$$

**DAY5**

64) Consider the differential equation

$$\frac{d\{I\}}{dt} - I = r(t)$$

a) For the homogeneous equation with $r(t) = 0$, find the general solution.

$$\frac{d\{I\}}{dt} - I = 0 \;\; ; \;\; \therefore \frac{d\{I\}}{dt} = I \;\; ; \;\; \therefore \frac{1}{I}dI = dt \;\; ; \;\; \therefore \int \frac{1}{I}dI = \int dt \;\; ; \;\; \therefore \ln I = t + c = \ln \mathfrak{e}^{t+c}$$

$$\therefore I = \mathfrak{e}^{t+c} = D\mathfrak{e}^{t}$$

b) Find the general solution to the homogeneous equation charaterised by $r(t) = \mathfrak{e}^{-t}$ and the particular solution for $I = 0.5$ at $t = 0$ and describe the behavior for large $t$

i) **Allocate $P(t)$ and $Q(t)$**
When we compare the equation with Equation (85), we obtain $P(t) = -1$ and $Q(t) = \mathfrak{e}^{-t}$

ii) **Calculate $A = \int P(t)dt$**

$$A = \int P(t)dt = \int -1dt = -t$$

iii) **Obtain $\Phi(t) = \mathfrak{e}^{A}$**
From Equation (86),

$$\Phi(t) = \mathfrak{e}^{A} = \mathfrak{e}^{-t}$$

iv) **Calculate $B = \int \Phi(t)Q(t)dt$**

$$B = \int \Phi(t)Q(t)dt = \int \mathfrak{e}^{-t}\mathfrak{e}^{-t}dt = \int \mathfrak{e}^{-2t}dt = -\frac{1}{2}\mathfrak{e}^{-2t} \qquad ①$$

v) **Obtain the general solution $I = \frac{1}{\Phi(t)}[B + c]$**

$$I = \frac{1}{\Phi(t)}[B + c] = \frac{1}{\mathfrak{e}^{-t}}\left[-\frac{1}{2}\mathfrak{e}^{-2t} + c\right] = -\frac{1}{2}\mathfrak{e}^{-t} + c\mathfrak{e}^{t}$$

vi) **Apply the condition to $I = \frac{1}{\Phi(t)}[B + c]$ in order to find out $c$ and thus the particular solution**
As the condition $(t, I) = (0, 0.5)$

$$0.5 = -\frac{1}{2}\mathfrak{e}^{-0} + c\mathfrak{e}^{0} \;\; ; \;\; \therefore c = 1$$

Thus the particular solution is $I = -\frac{1}{2}\mathfrak{e}^{-t} + \mathfrak{e}^{t}$. For large $t$, the term of $-\frac{1}{2}\mathfrak{e}^{-t}$ goes to zero and the term of $\mathfrak{e}^{t}$ diverges and in the end $I$ goes to infinite.

65) Solve the differential equation

$$\frac{\partial^2 I}{\partial t^2} - I = 2\mathfrak{e}^{-t} - 1$$

subject to the conditions that y remains finite for large $t$ and that $I = 2$ when $t = 0$
To solve the following we must first find the complementary function $Y_1(t)$ and and then the particular integral $Y_2(t)$ . The final answer will be the sum of both:

$$I(t) = Y_1(t) + Y_2(t)$$

In order to find out $Y_1(t)$, substituting Equation (103) into $\frac{\partial^2 I}{\partial t^2} - I = 0$, we produce auxiliary equation:

$$\lambda^2 - 1 = 0 \;\; ; \quad \therefore \lambda = -1, 1 \equiv \alpha, \beta$$

Since $\alpha$ and $\beta$ are real and $\alpha \neq \beta$, the complementary function $Y_1(t)$ is

$$Y_1(t) = a\mathfrak{e}^{-t} + b\mathfrak{e}^{t}$$

Now to find the particular solution, you need to know the correct substitution. $r(t) = 2\mathfrak{e}^{-t}$ means the coefficient of $t$, *i.e.*, $c = -1$. Since $c = \alpha$, the particular integral is Equation (109). Thus taking into account of $r(t) = -1$

$$Y_2(t) = gt\mathfrak{e}^{-t} + h \;\; ; \quad \therefore \frac{d\{Y_2(t)\}}{dt} = g\mathfrak{e}^{-t} - gt\mathfrak{e}^{-t}$$

$$\therefore \frac{\partial^2 Y_2(t)}{\partial t^2} = -g\mathfrak{e}^{-t} - (g\mathfrak{e}^{-t} - gt\mathfrak{e}^{-t}) - g\mathfrak{e}^{-t} - g\mathfrak{e}^{-t} + gt\mathfrak{e}^{-t} = -2g\mathfrak{e}^{-t} + gt\mathfrak{e}^{-t}$$

Substituting these into the original ODE

$$\frac{\partial^2 I}{\partial t^2} - I = 2\mathfrak{e}^{-t} - 1$$
$$\therefore -2g\mathfrak{e}^{-t} + gt\mathfrak{e}^{-t} - gt\mathfrak{e}^{-t} - h = 2\mathfrak{e}^{-t} - 1$$
$$\therefore -2g\mathfrak{e}^{-t} - h = 2\mathfrak{e}^{-t} - 1$$
$$\therefore -2g = 2 \;\; ; \quad \therefore g = -1 \;\; ; \quad -h = -1 \;\; ; \quad \therefore h = 1$$

Thus the particular integral is

$$Y_2(t) = -t\mathfrak{e}^{-t} + 1$$

Therefore the general solution for this ODE is

$$I(t) = Y_1(t) + Y_2(t) = a\mathfrak{e}^{-t} + b\mathfrak{e}^{t} - t\mathfrak{e}^{-t} + 1$$

For the value of $I$ to be finite for large $t$, the value of $b$ must be zero. Thus

$$I(t) = a\mathfrak{e}^{-t} - t\mathfrak{e}^{-t} + 1$$

To find $a$ we need to substitute $I = 2$ and $t = 0$.

$$2 = a\mathfrak{e}^{0} + 1 \;\; ; \quad \therefore a = 1$$

Therefore the particular solution to the ode is

$$I(t) = \mathfrak{e}^{-t} - t\mathfrak{e}^{-t} + 1$$

66) The current $I(t)$ at time $t$ in an electric circuit with total resistance $R$ and self-inductance $L$ satisfies

$$L\frac{d\{I\}}{dt} + RI = \mathfrak{e}^{-Dt}$$

where $0 < D < \frac{R}{L}$.

a) Find the general solution to the problem.

i) **Allocate** $P(t)$ **and** $Q(t)$

When we compare the equation with Equation (85), we obtain $P(t) = \frac{R}{L}$ and $Q(t) = \frac{\mathfrak{e}^{-Dt}}{L}$

ii) **Calculate** $A = \int P(t)dt$

$$A = \int P(t)dt = \int \frac{R}{L}dt = \frac{R}{L}t$$

iii) **Obtain** $\Phi(t) = \mathfrak{e}^A$
From Equation (86),

$$\Phi(t) = \mathfrak{e}^A = \mathfrak{e}^{\frac{R}{L}t}$$

iv) **Calculate** $B = \int \Phi(t)Q(t)dt$

$$B = \int \Phi(t)Q(t)dt = \int \mathfrak{e}^{\frac{R}{L}t}\frac{\mathfrak{e}^{-Dt}}{L}dt = \frac{1}{L}\int \mathfrak{e}^{\frac{R}{L}t-Dt}dt$$
$$= \frac{1}{L}\int \mathfrak{e}^{(\frac{R}{L}-D)t}dt = \frac{1}{L(\frac{R}{L}-D)}\mathfrak{e}^{(\frac{R}{L}-D)t} = \frac{1}{R-LD}\mathfrak{e}^{(\frac{R}{L}-D)t} \qquad ①$$

v) **Obtain the general solution** $I = \frac{1}{\Phi(t)}[B+c]$

$$I = \frac{1}{\Phi(t)}[B+c] = \frac{1}{\mathfrak{e}^{\frac{R}{L}t}}\left[\frac{1}{R-LD}\mathfrak{e}^{(\frac{R}{L}-D)t}+c\right] = \mathfrak{e}^{-\frac{R}{L}t}\left[\frac{1}{R-LD}\mathfrak{e}^{(\frac{R}{L}-D)t}+c\right] = \frac{1}{R-LD}\mathfrak{e}^{-Dt}+c\mathfrak{e}^{-\frac{R}{L}t}$$

b) The switch is closed at $t = 0$ and the initial value of the current is $I(0) = 2$. When $R = 6, L = 1, D = 5$, find the solution to this initial value problem.

i) **Obtain the general solution** $I = \frac{1}{\Phi(t)}[B+c]$ **when**$R = 6, L = 1, D = 5$

$$I = \frac{1}{R-LD}\mathfrak{e}^{-Dt}+c\mathfrak{e}^{-\frac{R}{L}t} = \mathfrak{e}^{-5t}+c\mathfrak{e}^{-6t}$$

ii) **Apply the condition to** $I = \frac{1}{\Phi(t)}[B+c]$ **in order to find out** $c$ **and thus the particular solution**
As the condition $(t, I) = (0, 2)$

$$2 = \mathfrak{e}^0 + c\mathfrak{e}^0 \ \ ; \quad \therefore c = 2-1 = 1$$

Thus the particular solution is $I = \mathfrak{e}^{-5t} + \mathfrak{e}^{-6t}$

c) Write down the steady-state value $I_s$ of the current
The particular solution is $I = \mathfrak{e}^{-5t} + \mathfrak{e}^{-6t}$. When $t$ is infinite, both $\mathfrak{e}^{-5t}$ and $\mathfrak{e}^{-6t}$ approach zero. Thus $I = 0$ is the steady-state value.

**DAYX**

67) Solve the following equation

$$2x^2 \frac{d\{y\}}{dx} + xy + y^2 = 0$$

The original equation is manipulated as

$$\begin{aligned} \frac{d\{y\}}{dx} + \frac{xy + y^2}{2x^2} &= 0 \\ \therefore \frac{d\{y\}}{dx} &= -\frac{y}{2x} - \frac{y^2}{2x^2} \\ \therefore \frac{d\{y\}}{dx} &= -\frac{z}{2} - \frac{z^2}{2} \triangleq f(z) \end{aligned}$$

where $z \equiv \frac{y}{x}$. Using Equation (87), we obtain

$$\begin{aligned} \ln x + c &= \int \frac{dz}{f(z) - z} \\ &= \int \frac{dz}{-\frac{z}{2} - \frac{z^2}{2} - z} \\ &= \int \frac{dz}{-\frac{z}{2} - \frac{z^2}{2} - \frac{2z}{2}} \\ &= \int \frac{dz}{-\frac{3z}{2} - \frac{z^2}{2}} \\ &= -2 \int \frac{dz}{3z + z^2} \\ &= -2 \int \frac{dz}{z(z+3)} \end{aligned}$$

Now we express $\frac{1}{z(z+3)}$ as $\frac{A}{z} + \frac{B}{z+3}$ using the following procedure.

$$\begin{aligned} \frac{A}{z} + \frac{B}{z+3} &= \frac{1}{z(z+3)} \\ \therefore A(z+3) + B \cdot z &= 1 \end{aligned}$$

By substituting $z = 0$ into the above,

$$\begin{aligned} A \cdot (0+3) + B \cdot 0 &= 1 \\ \therefore 3A &= 1 \\ \therefore A &= \frac{1}{3} \end{aligned}$$

By substituting $z = -3$ into the above,

$$\begin{aligned} A \cdot (-3+3) + B \cdot (-3) &= 1 \\ \therefore -3B &= 1 \\ \therefore B = -\frac{1}{3} &= -A \end{aligned}$$

$$\frac{1}{z(z+3)} = \frac{A}{z} + \frac{B}{z+3}$$
$$= \frac{A}{z} - \frac{A}{z+3}$$
$$= A(\frac{1}{z} - \frac{1}{z+3})$$
$$= \frac{1}{3}(\frac{1}{z} - \frac{1}{z+3})$$

$$\ln x + c = -2\int \frac{dz}{z(z+3)}$$
$$= -2 \cdot \frac{1}{3}\int \frac{1}{z} - \frac{1}{z+3}dz$$
$$= \frac{-2}{3}[\ln(z) - \ln(z+3)]$$
$$= \frac{-2}{3}\ln\left(\frac{z}{z+3}\right)$$
$$\therefore \frac{-3}{2}(\ln x + c) = \ln\left(\frac{z}{z+3}\right)$$
$$\therefore \frac{-3}{2}\ln x + c = \ln\left(\frac{z}{z+3}\right)$$
$$\therefore \ln x^{\frac{-3}{2}} + c = \ln\left(\frac{z}{z+3}\right)$$
$$\therefore \ln(C \cdot x^{\frac{-3}{2}}) = \ln\left(\frac{z}{z+3}\right)$$
$$\therefore Cx^{\frac{-3}{2}} = \frac{z}{z+3}$$
$$\therefore Cx^{\frac{-3}{2}}(z+3) = z$$
$$\therefore 3Cx^{\frac{-3}{2}} = z(1 - Cx^{\frac{-3}{2}})$$
$$\therefore 3Cx^{\frac{-3}{2}} = \frac{y}{x}(1 - Cx^{\frac{-3}{2}})$$
$$\therefore 3Cx^{\frac{-3}{2}+1} = y(1 - Cx^{\frac{-3}{2}})$$
$$\therefore 3Cx^{\frac{-1}{2}} = y(1 - Cx^{\frac{-3}{2}})$$
$$\therefore \frac{3Cx^{\frac{-1}{2}}}{1 - Cx^{\frac{-3}{2}}} = y$$
$$\therefore \frac{3x^{\frac{-1}{2}}}{C - x^{\frac{-3}{2}}} = y$$

68) Solve the following equation under the condition $(x, y) = (1, 1)$

$$xy^3\frac{d\{y\}}{dx} = x^4 + y^4$$

We take both sides of

$$xy^3\frac{d\{y\}}{dx} = x^4 + y^4$$

divided by $x^4$. Then we get

$$\left(\frac{y}{x}\right)^3 \frac{d\{y\}}{dx} = 1 + \left(\frac{y}{x}\right)^4$$

Let $z = \frac{y}{x}$ ,

$$z^3 \frac{d\{y\}}{dx} = 1 + z^4$$
$$\therefore \frac{\partial y}{\partial x} = \frac{1+z^4}{z^3}$$
$$\therefore \frac{d\{y\}}{dx} = z^{-3} + z$$

When we put

$$f(z) = z^{-3} + z$$

into Equation (87),

$$\begin{aligned} &\ln x + c \\ &= \int \frac{dz}{f(z) - z} \\ &= \int \frac{dz}{z^{-3} + z - z} \\ &= \int \frac{dz}{z^{-3}} \\ &= \int z^3 dz \\ &= \frac{1}{4} z^4 \end{aligned}$$

$$\therefore \frac{1}{4} z^4 = \ln x + c = \ln Cx$$
$$\therefore z^4 = 4 \ln Cx = \ln C^4 x^4 = \ln D x^4$$

where $D$ is an unknown constant. When we put

$$z = \frac{y}{x}$$

into

$$z^4 = \ln D x^4,$$

we obtain

$$\frac{y^4}{x^4} = \ln(Dx^4)$$
$$\therefore y^4 = x^4 \ln(Dx^4)$$

We now apply the condition to find out $D$ by putting

$$(x, y) = (1, 1)$$

into

$$y^4 = x^4 \ln(Dx^4).$$

We obtain

$$1 = \ln D.$$

Since $1 = \ln \mathfrak{e}$, $D = \mathfrak{e}$. Thus the final answer is $y^4 = x^4 \ln(\mathfrak{e} \cdot x^4)$.

69) Solve the following equation under the condition $(x,y)=(1,\frac{3}{4})$ and $(x,y)=(2,\frac{15}{4})$

$$x\frac{d\left\{y\right\}}{dx}=y+\sqrt{x^2+y^2}$$

We take both sides of

$$x\frac{d\left\{y\right\}}{dx}=y+\sqrt{x^2+y^2}$$

divided by $x$. Then we get

$$\frac{d\left\{y\right\}}{dx}=\frac{y}{x}+\frac{\sqrt{x^2+y^2}}{x}$$
$$\therefore\frac{d\left\{y\right\}}{dx}=\frac{y}{x}+\sqrt{\frac{x^2+y^2}{x^2}}$$
$$\therefore\frac{d\left\{y\right\}}{dx}=\frac{y}{x}+\sqrt{\frac{x^2}{x^2}+\frac{y^2}{x^2}}$$
$$\therefore\frac{d\left\{y\right\}}{dx}=\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}$$

Let

$$z=\frac{y}{x},$$

$$\frac{d\left\{y\right\}}{dx}$$
$$=\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}$$
$$=z+\sqrt{1+z^2}\equiv f(z).$$

When we put

$$f(z)=z+\sqrt{1+z^2}$$

into Equation (87),

$$\ln x+c$$
$$=\int\frac{dz}{f(z)-z}$$
$$=\int\frac{dz}{z+\sqrt{1+z^2}-z}$$
$$\int\frac{dz}{\sqrt{1+z^2}}$$

Let $z=\tan\theta$. Then $dz=\frac{d\theta}{\cos^2\theta}$. Thus

$$\ln x+c=\int\frac{dz}{\sqrt{1+z^2}}$$
$$=\int\frac{1}{\sqrt{1+\tan^2\theta}}\frac{d\theta}{\cos^2\theta}$$
$$=\int\frac{\cos\theta d\theta}{\cos^2\theta}(\because 1+\tan^2\theta=\frac{1}{\cos^2\theta})$$

When we let

$$t = \sin\theta,$$

$$dt = \cos\theta d\theta.$$

So we put $dt = \cos\theta d\theta$ into

$$\ln x + c = \int \frac{\cos\theta d\theta}{\cos^2\theta}$$

as follows:

$$\begin{aligned}
\ln x + c &= \int \frac{\cos\theta d\theta}{\cos^2\theta} \\
&= \int \frac{dt}{\cos^2\theta} \\
&= \int \frac{dt}{1 - \sin^2\theta} \\
&= \int \frac{dt}{1 - t^2} \\
&= \int \frac{1}{2}\left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt \\
&= \int \frac{1}{2}\left(-\frac{1}{t-1} + \frac{1}{t+1}\right) dt \\
&= \frac{1}{2}\left(-\ln|t-1| + \ln|1+t|\right) dt \\
&= \frac{1}{2}\ln\left|\frac{1+t}{t-1}\right|
\end{aligned}$$

We now have to express $t$ using $z$. Since

$$t = \sin\theta$$

and

$$z = \tan\theta,$$

$$\begin{aligned}
\sin\theta &= \cos\theta\tan\theta \\
\therefore \sin^2\theta &= \cos^2\theta\tan^2\theta \\
\therefore \sin^2\theta &= (1 - \sin^2\theta)\tan^2\theta \\
\therefore t^2 &= (1 - t^2)z^2 \\
\therefore t^2 &= z^2 - t^2z^2 \\
\therefore t^2(1 + z^2) &= z^2 \\
\therefore \frac{t^2}{\theta} &= \frac{z^2}{1 + z^2} \\
\therefore t &= \frac{z}{\sqrt{1 + z^2}}
\end{aligned}$$

Thus

$$\begin{aligned}\ln x + c &= \frac{1}{2}\ln\left|\frac{1+t}{t-1}\right| \\ &= \frac{1}{2}\ln\left|\frac{1+\dfrac{z}{\sqrt{1+z^2}}}{\dfrac{z}{\sqrt{1+z^2}}-1}\right| \\ &= \frac{1}{2}\ln\left|\frac{\sqrt{1+z^2}+z}{\sqrt{1+z^2}-z}\right| \\ &= \frac{1}{2}\ln\left|\frac{\left(\sqrt{1+z^2}+z\right)^2}{(\sqrt{1+z^2}-z)(\sqrt{1+z^2}+z)}\right| \\ &= \frac{1}{2}\ln\left|\left(\sqrt{1+z^2}+z\right)^2\right| \\ &= \ln\left|\sqrt{1+z^2}+z\right|\end{aligned}$$

Therefore

$$\begin{aligned}&\ln x + c \\ &= \ln x + \ln C \\ &= \ln Cx \\ &= \ln|\sqrt{1+z^2}+z|\end{aligned}$$

Thus

$$Cx = \sqrt{1+z^2}+z$$

When we solve

$$Cx = \sqrt{1+z^2}+z$$

with respect to $z$,

$$\begin{aligned}Cx - z &= \sqrt{1+z^2} \\ \therefore (Cx-z)^2 &= 1+z^2 \\ \therefore (Cx)^2 - 2Cxz + z^2 &= 1 + z^2 \\ \therefore (Cx)^2 - 2Cxz &= 1 \\ \therefore 2Cxz &= (Cx)^2 - 1 \\ \therefore z &= \frac{(Cx)^2-1}{2Cx}\end{aligned}$$

When we put

$$z = \frac{y}{x}$$

back into

$$z = \frac{(Cx)^2-1}{2Cx}$$

we obtain

$$\begin{aligned}\frac{y}{x} &= \frac{(Cx)^2-1}{2Cx} \\ \therefore y &= \frac{(Cx)^2-1}{2C}\end{aligned}$$

We now apply the condition to find out $C$. By putting

$$(x, y) = (1, \frac{3}{4})$$

and

$$(x, y) = (2, \frac{15}{4})$$

into

$$y = \frac{(Cx)^2 - 1}{2C}.$$

From the condition $(x, y) = (1, \frac{3}{4})$,

$$\frac{3}{4} = \frac{C^2 - 1}{2C}$$

This can be written as

$$\begin{aligned}
\therefore \frac{3}{4} &= \frac{C^2 - 1}{2C} \\
\therefore 6c &= 4(C^2 - 1) \\
\therefore 3c &= 2(C^2 - 1) \\
\therefore 2C^2 - 2 - 3C &= 0 \\
\therefore (2C + 1)(C - 2) &= 0 \\
C &= 2, -\frac{1}{2}
\end{aligned}$$

From the condition $(x, y) = (2, \frac{15}{4})$,

$$\frac{15}{4} = \frac{4C^2 - 1}{2C}.$$

This can be written as

$$\begin{aligned}
15 \times 2C &= 4(4C^2 - 1) \\
\therefore 15C &= 2(4C^2 - 1) \\
\therefore 15C &= 8C^2 - 2 \\
\therefore 8C^2 - 15C - 2 &= 0 \\
\therefore (8C + 1)(C - 2) &= 0. \\
\therefore C &= 2, -\frac{1}{8}
\end{aligned}$$

Since

$$C = 2$$

satisfies both

$$2C^2 - 3C - 2 = 0$$

and

$$8C^2 - 15C - 2 = 0$$

, the final answer is

$$\begin{aligned}
y &= \left. \frac{(Cx)^2 - 1}{2C} \right|_{C=2} \\
&= \frac{(2x)^2 - 1}{4}.
\end{aligned}$$

70) Solve the following equation under the condition $(x, y) = (1, \frac{1}{2})$

$$\frac{d\{y\}}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$$

Let $z = \frac{y}{x}$

$$\frac{d\{y\}}{dx}$$
$$= \frac{y}{x} + \frac{y^2}{x^2} \equiv z + z^2$$
$$= f(z)$$

When we put $f(z) = z + z^2$ into Equation (87),

$$\ln x + c$$
$$= \int \frac{dz}{f(z) - z}$$
$$= \int \frac{dz}{z + z^2 - z}$$
$$\therefore \ln x + c = \int \frac{dz}{z^2} = \int z^{-2} dz$$
$$\therefore \ln x + c = \frac{1}{-2+1} z^{-2+1}$$
$$\therefore \ln x + c = -\frac{1}{z} = -\frac{x}{y}$$
$$\therefore y = -\frac{x}{\ln x + c}$$

In order to find out $C$ under the condition

$$(x, y) = (1, \frac{1}{2}),$$

we solve

$$\frac{1}{2} = -\frac{1}{\ln 1 + c} = -\frac{1}{c}$$

This gives us

$$c = -2.$$

Therefore the answer is

$$y = -\frac{x}{\ln x - 2}$$

71) Solve the equation

$$xy \frac{d\{y\}}{dx} + x^2 - y^2 = 0$$

We take both sides of

$$xy \frac{d\{y\}}{dx} + x^2 - y^2 = 0$$

divided by $x^2$ and obtain

$$\frac{y}{x} \frac{d\{y\}}{dx} + 1 - \left(\frac{y}{x}\right)^2 = 0$$

Let

$$z = \frac{y}{x},$$

$$\frac{y}{x}\frac{d\{y\}}{dx} + 1 - \left(\frac{y}{x}\right)^2 = 0$$

can be written as

$$z\frac{d\{y\}}{dx} + 1 - z^2 = 0$$

which is the same as

$$\frac{d\{y\}}{dx} = \frac{z^2 - 1}{z} = z - \frac{1}{z} \equiv f(z).$$

When we apply

$$f(z) = z - \frac{1}{z}$$

in Equation (87),

$$\begin{aligned}
\ln x + c &= \int \frac{dz}{z - \frac{1}{z} - z} \\
&= \int \frac{dz}{-\frac{1}{z}} \\
&= -\int z dz \\
&= -\frac{1}{2}z^2 \\
\therefore z^2 &= -2\ln x + C
\end{aligned}$$

We now put

$$z = \frac{y}{x}$$

into

$$z^2 = -2\ln x + C;$$

and solve it for y:

$$\begin{aligned}
\frac{y^2}{x^2} &= -2\ln x + C \\
\therefore y^2 &= x^2(-2\ln x + C)
\end{aligned}$$

72) Solve the equation

$$x^2\frac{d\{y\}}{dx} = y^2 + xy + x^2$$

We take the both sides of

$$x^2\frac{d\{y\}}{dx} = y^2 + xy + x^2$$

divided by $x^2$ and obtain

$$\frac{d\{y\}}{dx} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} + 1.$$

Let

$$z = \frac{y}{x},$$

this can be written as

$$\frac{d\{y\}}{dx} = z^2 + z + 1 \equiv f(z).$$

When we apply

$$z^2 + z + 1 \equiv f(z)$$

to Equation (87),

$$\ln x + c = \int \frac{dz}{z^2 + z + 1 - z}$$
$$= \int \frac{dz}{z^2 + 1}$$

Let

$$z = \tan\theta,$$

we get

$$\frac{dz}{d\theta} = \frac{1}{\cos^2\theta}.$$

Thus

$$\ln x + c = \int \frac{dz}{z^2 + 1}$$
$$= \int \frac{1}{\tan^2\theta + 1} \frac{1}{\cos^2\theta} d\theta$$
$$= \int \cos^2\theta \frac{1}{\cos^2\theta} d\theta$$
$$= \int d\theta$$
$$= \theta = \tan^{-1} z (\because 8 = \tan\theta)$$

We solve

$$\ln x + c = \tan^{-1} z$$

with respect to $z$:

$$\ln x + c = \tan^{-1} z$$
$$\therefore \ln Cx = \tan^{-1} z$$
$$\therefore \tan(\ln Cx) = z$$

We now put

$$z = \frac{y}{x}$$

into

$$\tan(\ln Cx) = z$$

and then obtain

$$\tan(\ln Cx) = \frac{y}{x},$$

*i.e.,*

$$y = x \tan(\ln Cx).$$

73) Solve the equation

$$x\frac{d\left\{y\right\}}{dx}=2y$$

We take both sides of

$$x\frac{d\left\{y\right\}}{dx}=2y$$

divided by $x$ and obtain

$$\frac{d\left\{y\right\}}{dx}=2\frac{y}{x}.$$

Let

$$z=\frac{y}{x},$$

$$\frac{d\left\{y\right\}}{dx}=2\frac{y}{x}$$

can be written as

$$\frac{d\left\{y\right\}}{dx}=2z\equiv f(z).$$

When we apply $f(z)=2z$ to Equation (87), we get

$$\begin{aligned}\ln x+c&=\int\frac{dz}{2z-z}\\&=\int\frac{dz}{z}\\&=\ln z\\\therefore\ln Cx&=\ln z\\\therefore Cx&=z\end{aligned}$$

We now put

$$z=\frac{y}{x}$$

back into

$$Cx=z,$$

we get

$$Cx=\frac{y}{x}$$

which can be re-written as

$$y=Cx^2.$$

74) Solve the following equation

$$\frac{d\left\{y\right\}}{dx}=\frac{1-y^2}{2xy-\sin y}$$

We can re-write the equation as

$$\begin{aligned}&(y^2-1)dx+(2xy-\sin y)dy\\&\equiv f(x,y)dx+g(x,y)dy=0.\end{aligned}$$

Since

$$\frac{d\left\{f(x,y)\right\}}{dy}=\frac{d\left\{y^2-1\right\}}{dy}=2y$$

and

$$\frac{d\{g(x,y)\}}{dx} = \frac{d\{2xy - \sin y\}}{dx} = 2y$$

the answer can be obtained from Equation (89) or Equation (93).

**First approach**

In order to find $U(x,y)$, first we apply

$$\begin{aligned} U(x,y) &= \int f(x,y)dx + h(y) \\ &= \int (y^2-1)dx + h(y) \\ &= (y^2-1)x + h(y) \end{aligned}$$

then we find $h(y)$ from

$$\begin{aligned} \frac{d\{U(x,y)\}}{dy} &= \frac{d\{(y^2-1)x + h(y)\}}{dy} \\ &= 2xy + \frac{\partial h(y)}{\partial y} \\ \frac{d\{U(x,y)\}}{dy} &= g(x,y) = 2xy - \sin y \\ \therefore 2xy + \frac{d\{h(y)\}}{dy} &= 2xy - \sin y \\ \therefore \frac{d\{h(y)\}}{dy} &= -\sin y \\ \therefore \int dh(y) &= \int (-\sin y)dy \\ \therefore h(y) &= \cos y \end{aligned}$$

Thus, the answer is

$$(y^2-1)x + \cos y = c$$

**Second approach**

When we use the alternative approach in Equation (93)

$$\begin{aligned} U(x,y) &= \int_{x_0}^{x} f(x,y)dx + \int_{y_0}^{y} g(x_0,y)dy \\ &= \int_{x_0}^{x} (y^2-1)dx + \int_{y_0}^{y} (2x_0y - \sin y)dy \\ &= (y^2-1)\,[x]_{x_0}^{x} + \left[x_0y^2 + \cos y\right]_{y_0}^{y} \\ &= (y^2-1)x - (y^2-1)x_0 + \left[x_0y^2 + \cos y\right] - \left[x_0y_0^2 + \cos y_0\right] \\ &= (y^2-1)x + \cos y - x_0 - x_0y_0^2 - \cos y_0 \\ &\therefore (y^2-1)x + \cos y + C = c \\ &\therefore (y^2-1)x + \cos y = c \end{aligned}$$

75) Solve the following equation

$$-\frac{y}{x^2}dx + \frac{1}{x}dy = 0$$

Since

$$\frac{d\left\{f(x,y)\right\}}{dy}$$
$$=\frac{d\left\{-\frac{y}{x^2}\right\}}{dy}$$
$$=-\frac{1}{x^2}$$

and

$$\frac{d\left\{g(x,y)\right\}}{dx}$$
$$=\frac{d\left\{\frac{1}{x}\right\}}{dx}$$
$$=-\frac{1}{x^2}$$
$$=\frac{d\left\{f(x,y)\right\}}{dy},$$

the answer can be obtained from Equation (89) or Equation (93).

**First approach**

In order to find $U(x,y)$, first we apply

$$\begin{aligned}U(x,y) &= \int f(x,y)dx + h(y)\\ &= \int\left(-\frac{y}{x^2}\right)dx + h(y)\\ &= -y\int\frac{dx}{x^2} + h(y)\\ &= -\frac{1}{-2+1}yx^{-1} + h(y)\\ &= \frac{y}{x} + h(y)\end{aligned}$$

then we find $h(y)$ from

$$\frac{d\left\{U(x,y)\right\}}{dy}$$
$$=\frac{d\left\{\frac{y}{x}+h(y)\right\}}{dy}$$
$$=\frac{1}{x}+\frac{\partial h(y)}{\partial y}=\frac{1}{x}+\frac{d\left\{h(y)\right\}}{dy}$$

At the same time

$$\frac{d\left\{U(x,y)\right\}}{dy}$$
$$=g(x,y)$$
$$=\frac{1}{x}$$

Therefore

$$\frac{1}{x} + \frac{d\{h(y)\}}{dy} = \frac{1}{x}$$
$$\therefore \frac{d\{h(y)\}}{dy} = 0$$
$$\therefore \int dh(y) = \int 0dy$$
$$\therefore h(y) = C$$

Thus, from Equation (90), the answer is

$$U(x,y) = \frac{y}{x} + C = c,$$

*i.e.,*

$$\frac{y}{x} = c$$
$$\therefore y = cx$$

**Second approach**
When we use the alternative approach in Equation (93)

$$\begin{aligned} U(x,y) &= \int_{x_0}^{x} f(x,y)dx + \int_{y_0}^{y} g(x_0,y)dy \\ &= \int_{x_0}^{x} -\frac{y}{x^2}dx + \int_{y_0}^{y} \frac{1}{x_0}dy \\ &= y\left[\frac{1}{x}\right]_{x_0}^{x} + \frac{1}{x_0}[y]_{y_0}^{y} \\ &= y\left[\frac{1}{x} - \frac{1}{x_0}\right] + \frac{1}{x_0}[y - y_0] \\ &= \frac{y}{x} - \frac{y}{x_0} + \frac{y}{x_0} - \frac{y_0}{x_0} \\ &= \frac{y}{x} - \frac{y_0}{x_0} = c \\ \therefore \frac{y}{x} &= \frac{y_0}{x_0} + c = c \end{aligned}$$

76) Solve the following equation

$$\frac{d\{y\}}{dx} = -\frac{x^2 + y}{y^2 + x}$$

Hint

$$\frac{d\{y\}}{dx} = \frac{P(x,y)}{Q(x,y)}$$

is the same as

$$Q(x,y)dy = P(x,y)dx.$$

We can re-write the equation as

$$\begin{aligned} &(x^2 + y)dx + (y^2 + x)dy \\ &\equiv f(x,y)dx + g(x,y)dy = 0. \end{aligned}$$

Since

$$\frac{d\left\{f(x,y)\right\}}{dy} = \frac{d\left\{x^2+y\right\}}{dy} = 1$$

and

$$\frac{d\left\{g(x,y)\right\}}{dx} = \frac{d\left\{y^2+x\right\}}{dx} = 1 = \frac{d\left\{f(x,y)\right\}}{dy},$$

the answer can be obtained from Equation (89) or Equation (93).

**First approach**

In order to find $U(x, y)$, first we apply

$$\begin{aligned} U(x,y) &= \int f(x,y)dx + h(y) \\ &= \int (x^2+y)dx + h(y) \\ &= \frac{1}{3}x^3 + xy + h(y) \end{aligned}$$

then we find $h(y)$ from

$$\begin{aligned} \frac{d\left\{U(x,y)\right\}}{dy} &= \frac{d\left\{\frac{1}{3}x^3+xy+h(y)\right\}}{dy} \\ &= x + \frac{\partial h(y)}{\partial y} \\ \frac{d\left\{U(x,y)\right\}}{dy} &= g(x,y) = y^2 + x \\ \therefore x + \frac{d\left\{h(y)\right\}}{dy} &= y^2 + x \\ \therefore \frac{d\left\{h(y)\right\}}{dy} &= y^2 \\ \therefore \int dh(y) &= \int y^2 dy \\ \therefore h(y) &= \frac{1}{3}y^3 \end{aligned}$$

Thus, the answer is

$$\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 = c$$

**Second approach**

When we use the alternative approach in Equation (93)

$$
\begin{aligned}
U(x,y) &= \int_{x_0}^{x} f(x,y)dx + \int_{y_0}^{y} g(x_0,y)dy \\
&= \int_{x_0}^{x} (x^2+y)dx + \int_{y_0}^{y} (y^2+x_0)dy \\
&= \left[\frac{x^3}{3} + yx\right]_{x_0}^{x} + [\frac{y^3}{3} + x_0 y]_{y_0}^{y} \\
&= \frac{x^3}{3} + yx - \left(\frac{x_0^3}{3} + yx_0\right) + \frac{y^3}{3} + x_0 y - \left(\frac{y_0^3}{3} + x_0 y_0\right) \\
&= \frac{x^3}{3} + yx - \frac{x_0^3}{3} - yx_0 + \frac{y^3}{3} + x_0 y - \frac{y_0^3}{3} - x_0 y_0 \\
&= \frac{x^3}{3} + yx - \frac{x_0^3}{3} + \frac{y^3}{3} - \frac{y_0^3}{3} - x_0 y_0 \\
&\therefore \frac{x^3}{3} + yx + \frac{y^3}{3} + C = c \\
&\therefore \frac{x^3}{3} + yx + \frac{y^3}{3} = c
\end{aligned}
$$

77) Solve the following equation

$$\frac{d\{y\}}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

We can re-write the equation as

$$
\begin{aligned}
&(x^2 - 2y)dx + (y^2 - 2x)dy \\
&\equiv f(x,y)dx + g(x,y)dy \\
&= 0.
\end{aligned}
$$

Since

$$\frac{d\{f(x,y)\}}{dy} = \frac{d\{x^2 - 2y\}}{dy} = -2$$

and

$$\frac{d\{g(x,y)\}}{dx} = \frac{d\{y^2 - 2x\}}{dx} = -2$$

the answer can be obtained from Equation (89) or Equation (93).

**First approach**

In order to find $U(x,y)$, first we apply

$$
\begin{aligned}
U(x,y) &= \int f(x,y)dx + h(y) \\
&= \int (x^2 - 2y)dx + h(y) \\
&= \frac{1}{3}x^3 - 2yx + h(y)
\end{aligned}
$$

then we find $h(y)$ from

$$\begin{aligned}
\frac{d\{U(x,y)\}}{dy} &= \frac{d\left\{\frac{1}{3}x^3 - 2xy + h(y)\right\}}{dy} \\
&= -2x + \frac{\partial h(y)}{\partial y} \\
\frac{d\{U(x,y)\}}{dy} &= g(x,y) = y^2 - 2x \\
\therefore -2x + \frac{d\{h(y)\}}{dy} &= y^2 - 2x \\
\therefore \frac{d\{h(y)\}}{dy} &= y^2 \\
\therefore \int dh(y) &= \int y^2 dy \\
\therefore h(y) &= \frac{1}{3}y^3
\end{aligned}$$

Thus, the answer is

$$\frac{1}{3}x^3 - 2xy + \frac{1}{3}y^3 = c$$

**Second approach**
When we use the alternative approach in Equation (93)

$$\begin{aligned}
U(x,y) &= \int_{x_0}^{x} f(x,y)dx + \int_{y_0}^{y} g(x_0,y)dy \\
&= \int_{x_0}^{x} (x^2 - 2y)dx + \int_{y_0}^{y} (y^2 - 2x_0)dy \\
&= \left[\frac{x^3}{3} - 2yx\right]_{x_0}^{x} + \left[\frac{y^3}{3} - 2x_0 y\right]_{y_0}^{y} \\
= \left[\frac{x^3}{3} - 2yx\right] - \left[\frac{x_0^3}{3} - 2yx_0\right] &+ \left[\frac{y^3}{3} - 2x_0 y\right] - \left[\frac{y_0^3}{3} - 2x_0 y_0\right] \\
&= \frac{x^3}{3} - 2yx + \frac{y^3}{3} - \frac{x_0^3}{3} - \frac{y_0^3}{3} + 2x_0 y_0 \\
\therefore \frac{x^3}{3} - 2yx + \frac{y^3}{3} + C &= c \\
\therefore \frac{x^3}{3} - 2yx + \frac{y^3}{3} &= c
\end{aligned}$$

78) Solve the following equation

$$\frac{d\{y\}}{dx} = \frac{2x+y}{2y-x}$$

We can re-write the equation as

$$\begin{aligned}
&(2x+y)dx + (x-2y)dy \\
&\equiv f(x,y)dx + g(x,y)dy \\
&= 0.
\end{aligned}$$

Since

$$\frac{d\left\{f(x,y)\right\}}{dy}=\frac{d\left\{2x+y\right\}}{dy}=1$$

and $\frac{d\left\{g(x,y)\right\}}{dx}=\frac{d\left\{x-2y\right\}}{dx}=1=\frac{d\left\{f(x,y)\right\}}{dy}$, the answer can be obtained from Equation (89) or Equation (93).

**First approach**

In order to find $U(x,y)$, first we apply

$$\begin{aligned}U(x,y)&=\int f(x,y)dx+h(y)\\&=\int(2x+y)dx+h(y)\\&=x^2+yx+h(y)\end{aligned}$$

then we find $h(y)$ from

$$\begin{aligned}\frac{d\left\{U(x,y)\right\}}{dy}&=\frac{d\left\{x^2+yx+h(y)\right\}}{dy}\\&=x+\frac{\partial h(y)}{\partial y}\\\frac{d\left\{U(x,y)\right\}}{dy}&=g(x,y)=x-2y\\\therefore x+\frac{d\left\{h(y)\right\}}{dy}&=x-2y\\\therefore\frac{d\left\{h(y)\right\}}{dy}&=-2y\\\therefore\int dh(y)&=\int(-2y)dy\\\therefore h(y)&=-y^2\end{aligned}$$

Thus, the answer is

$$x^2+yx-y^2=c$$

**Second approach**

When we use the alternative approach in Equation (93)

$$\begin{aligned}U(x,y)&=\int_{x_0}^{x}f(x,y)dx+\int_{y_0}^{y}g(x_0,y)dy\\&=\int_{x_0}^{x}(2x+y)dx+\int_{y_0}^{y}(x_0-2y)dy\\&=\left[x^2+yx\right]_{x_0}^{x}+\left[x_0y-y^2\right]_{y_0}^{y}\\=\left[x^2+yx\right]-\left[x_0^2+yx_0\right]&+\left[x_0y-y^2\right]-\left[x_0y_0-y_0^2\right]\\x^2+yx-y^2&-x_0^2-x_0y_0+y_0^2\\\therefore x^2+yx-y^2+C&=c\\\therefore x^2+yx-y^2&=c\end{aligned}$$

79) Solve the following equation

$$\frac{x}{y^2}\frac{d\left\{y\right\}}{dx}+\frac{1}{y}-x^3y=0$$

The given equation can be manipulated as

$$x\frac{d\{y\}}{dx} + y = x^3y^3$$
$$\therefore \frac{d\{y\}}{dx} + \frac{y}{x} = x^2y^3$$

Compared with Equation (94), we obtain

$$p(x) = \frac{1}{x}$$
$$q(x) = x^2$$
$$\alpha = 3$$

Thus using Equation (85) and Equation (97),Equation (98) which are

$$P(x) = (1-3)\frac{1}{x} = \frac{-2}{x}$$
$$Q(x) = -2x^2$$

we first solve for

$$Y = y^{1-3} = y^{-2}$$

First, we find $\Phi(x)$ in Equation (86):

$$\Phi(x) = \mathfrak{e}^{\int \frac{-2}{x}dx}$$
$$= \mathfrak{e}^{-2\int \frac{1}{x}dx}$$
$$= \mathfrak{e}^{-2\ln x}$$
$$= \mathfrak{e}^{\ln x^{-2}}$$
$$= x^{-2}$$

Thus

$$Y = \frac{1}{x^{-2}}\left[\int x^{-2}\cdot(-2x^2)dx + c\right]$$
$$= \frac{1}{x^{-2}}\left[\int -2dx + c\right]$$
$$= x^2\left[-2x + c\right]$$

Since $Y = y^{-2}$, the solution is

$$y^{-2} = x^2\left[-2x + c\right]$$

80) Solve the following equation

$$4x\frac{d\{y\}}{dx} + 2y - x\cdot y^5 = 0$$

The given equation can be manipulated as

$$4x\frac{d\{y\}}{dx} + 2y - x\cdot y^5 = 0$$
$$\therefore \frac{d\{y\}}{dx} + \frac{2y}{4x} - \frac{x\cdot y^5}{4x} = 0$$
$$\therefore \frac{d\{y\}}{dx} + \frac{y}{2x} = \frac{y^5}{4}$$

Compared with Equation (94), we obtain

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{1}{4}$$
$$\alpha = 5$$

Thus using Equation (85) and Equation (97),Equation (98) which are

$$P(x) = (1-5)\frac{1}{2x} = \frac{-4}{2x} = \frac{-2}{x}$$
$$Q(x) = (1-5)\cdot\frac{1}{4} = -1$$

we first solve for

$$Y = y^{1-5} = y^{-4}$$

First, we find $\Phi(x)$ in Equation (86):

$$\begin{aligned}\Phi(x) &= \mathfrak{e}^{\int \frac{-2}{x}dx} \\ &= \mathfrak{e}^{-2\int \frac{1}{x}dx} \\ &= \mathfrak{e}^{-2\ln x} \\ &= \mathfrak{e}^{\ln x^{-2}} \\ &= x^{-2}\end{aligned}$$

Thus

$$\begin{aligned}Y &= \frac{1}{x^{-2}}\left[\int x^{-2}\cdot(-1)dx + c\right] \\ &= \frac{1}{x^{-2}}\left[-\int x^{-2}dx + c\right] \\ &= \frac{1}{x^{-2}}\left[-\frac{1}{-1}x^{-2+1} + c\right] \\ &= \frac{1}{x^{-2}}\left[x^{-1} + c\right] \\ &= x^{2}\left[x^{-1} + c\right] \\ &= x + cx^2\end{aligned}$$

Since $Y = y^{-4}$, the solution is

$$y^{-4} = x + cx^2$$

81) Solve the following equation

$$dy = 2xy(x^2y^2 - 1)dx$$

The given equation is re-written as

$$\frac{d\{y\}}{dx} + 2xy = 2x^3y^3$$

which becomes Equation (94) when

$$p(x) = 2x$$
$$q(x) = 2x^3$$
$$\alpha = 3$$

$Y = y^{1-3} = y^{-2}$ can be obtained from Equation (85) with Equation (97),Equation (98) which are

$$P(x) = -2 \cdot 2x = -4x$$
$$Q(x) = -2 \cdot 2x^3 = -4x^3$$

Using Equation (86),

$$\begin{aligned}\Phi(x) &= \mathfrak{e}^{\int -4x dx} \\ &= \mathfrak{e}^{-4 \cdot \frac{x^2}{2}} \\ &= \mathfrak{e}^{-2x^2}\end{aligned}$$

Then, using Equation (85),

$$Y = \frac{1}{\mathfrak{e}^{-2x^2}} \left[ \int \mathfrak{e}^{-2x^2} \cdot (-4x^3) dx + c \right]$$

Since $\int \mathfrak{e}^{-2x^2} dx$ is not obtained with ease, we now set

$$\begin{aligned} z &= -2x^2 \\ \therefore \frac{z}{-2} &= x^2 \\ \therefore \frac{d\{z\}}{dx} &= -2 \cdot 2x = -4x \\ \therefore dz &= -4x dx \\ \therefore \frac{dz}{-4x} &= dx \end{aligned}$$

Substituting $dx = \frac{dz}{-4x}$ and $x^2 = \frac{z}{-2}$ into

$$Y = \frac{1}{\mathfrak{e}^{-2x^2}} \left[ \int \mathfrak{e}^{-2x^2} \cdot (-4x^3) dx + c \right]$$

, we get

$$\begin{aligned} Y &= \frac{1}{\mathfrak{e}^{-2x^2}} \left[ \int \mathfrak{e}^{z} \cdot (-4x^3) \cdot \frac{dz}{-4x} + c \right] \\ &= \frac{1}{\mathfrak{e}^{-2x^2}} \left[ \int \mathfrak{e}^{z} \cdot x^2 \cdot dz + c \right] \\ &= \frac{1}{\mathfrak{e}^{-2x^2}} \left[ \int \mathfrak{e}^{z} \cdot \frac{z}{-2} \cdot dz + c \right] \\ &= \frac{1}{\mathfrak{e}^{-2x^2}} \left[ \frac{1}{-2} \int z\mathfrak{e}^{z} dz + c \right] \\ &= \frac{1}{\mathfrak{e}^{-2x^2}} \left[ \frac{1}{-2} \left\{ z \int \mathfrak{e}^{z} dz - \int \left( \frac{d\{z\}}{dz} \int \mathfrak{e}^{z} dz \right) dz \right\} + c \right] \\ &= \frac{1}{\mathfrak{e}^{z}} \left[ \frac{1}{-2} \left\{ z\mathfrak{e}^{z} - \int \mathfrak{e}^{z} dz \right\} + c \right] \\ &= \mathfrak{e}^{-z} \left[ \frac{1}{-2} \left\{ (z-1)\mathfrak{e}^{z} \right\} + c \right] \end{aligned}$$

$$= \frac{1}{-2}\left\{(z-1)\right\} + c\mathfrak{e}^{-z}$$
$$= \frac{1-z}{2} + c\mathfrak{e}^{-z}$$
$$= \frac{1-(-2x^2)}{2} + c\mathfrak{e}^{-(-2x^2)}$$
$$= \frac{1+2x^2}{2} + c\mathfrak{e}^{2x^2}$$
$$\therefore y^{-2} = \frac{1+2x^2}{2} + c\mathfrak{e}^{2x^2}$$

82) Solve the following equation

$$y = x\frac{d\left\{y\right\}}{dx} - \ln\left(\frac{d\left\{y\right\}}{dx}\right)$$

The given equation is the same type as Equation (99). Thus we set

$$f\left(\frac{d\left\{y\right\}}{dx}\right) = -\ln\left(\frac{d\left\{y\right\}}{dx}\right)$$

Equation (100) gives the general solution:

$$y = ax - \ln\left(a\right)$$

We now differentiate the general solution with respect to $a$:

$$\frac{\partial\left\{y\right\}}{\partial a} = \frac{\partial\left\{ax - \ln\left(a\right)\right\}}{\partial a}$$
$$\therefore 0 = \frac{\partial\left\{ax\right\}}{\partial a} - \frac{\partial\left\{\ln\left(a\right)\right\}}{\partial a}$$
$$\therefore 0 = x - \frac{1}{a}$$
$$\therefore \frac{1}{a} = x$$
$$\therefore a = \frac{1}{x}$$

By substituting $a = \frac{1}{x}$ into the general solution as in Equation (101) we obtain

$$y = x\cdot\frac{1}{x} - \ln\left(\frac{1}{x}\right)$$
$$\therefore y = 1 - \left\{\ln(1) - \ln(x)\right\}$$
$$\therefore y = 1 - \left\{0 - \ln(x)\right\}$$
$$\therefore y = 1 + \ln(x)$$

Therefore the solutions are

$$y = ax - \ln\left(a\right)$$

and

$$y = 1 + \ln(x)$$

where $a$ is an arbitrary constant.

83) Solve the following equation

$$3\left(\frac{d\left\{y\right\}}{dx}\right)^3 + x\frac{d\left\{y\right\}}{dx} - y = 0$$

When the given equation is re-written as

$$y = x\frac{d\left\{y\right\}}{dx} + 3\left(\frac{d\left\{y\right\}}{dx}\right)^3$$

this equation is the same type as Equation (99) with

$$f(\frac{d\left\{y\right\}}{dx}) = 3\left(\frac{d\left\{y\right\}}{dx}\right)^3$$

The general solution is

$$y = ax + 3a^3$$

By differentiating this general solution with respect to $a$

$$\begin{aligned}\frac{\partial\left\{y\right\}}{\partial a} &= \frac{\partial\left\{ax + 3a^3\right\}}{\partial a}\\ \therefore 0 &= x + 9a^2\\ \therefore x &= -9a^2\end{aligned}$$

Here, we can not solve for $a$ because $\pm\sqrt{-x/9}$ may be imaginary numbers. Substituting $x = -9a^2$ into the general solution ,

$$\begin{aligned}y &= a(-9a^2) + 3a^3\\ &= -9a^3 + 3a^3\\ &= -6a^3\end{aligned}$$

Thus, the solutions are

$$y = ax + 3a^3$$

and

$$(x, y) = (-9a^2, -6a^3)$$

which is a point, not a function.

84) Solve the following equation

$$y = x\frac{d\left\{y\right\}}{dx} - \mathfrak{e}^{\frac{d\left\{y\right\}}{dx}}$$

The given equation is the same type as Equation (99). Thus we set

$$f\left(\frac{d\left\{y\right\}}{dx}\right) = -\mathfrak{e}^{\frac{d\left\{y\right\}}{dx}}$$

Equation (100) gives the general solution:

$$y = ax - \mathfrak{e}^a$$

We now differentiate the general solution with respect to $a$:

$$\begin{aligned}\frac{\partial\{y\}}{\partial a} &= \frac{\partial\{ax-\mathfrak{e}^a\}}{\partial a}\\ \therefore 0 &= \frac{\partial\{ax\}}{\partial a} - \frac{\partial\{\mathfrak{e}^a\}}{\partial a}\\ \therefore 0 &= x-\mathfrak{e}^a\\ \therefore \mathfrak{e}^a &= x\\ \therefore \ln(\mathfrak{e}^a) &= \ln(x)\\ \therefore a &= \ln(x)\end{aligned}$$

By substituting $a=\ln(x)$ into the general solution as in Equation (101) we obtain

$$\begin{aligned}y &= x\ln(x)-\mathfrak{e}^{\ln(x)}\\ \therefore y &= x\ln(x)-x\end{aligned}$$

Therefore the solutions are

$$y = ax-\mathfrak{e}^a$$

and

$$y = x\ln(x)-x$$

where $a$ is an arbitrary constant.

85) Solve the following equation

$$y = x\frac{d\{y\}}{dx} + \left(\frac{d\{y\}}{dx}\right)^2$$

The given equation is the same type as Equation (99). Thus we set

$$f\left(\frac{d\{y\}}{dx}\right) = \left(\frac{d\{y\}}{dx}\right)^2$$

Equation (100) gives the general solution:

$$y = ax+(a)^2$$

We now differentiate the general solution with respect to $a$:

$$\begin{aligned}\frac{\partial\{y\}}{\partial a} &= \frac{\partial\{ax+a^2\}}{\partial a}\\ \therefore 0 &= \frac{\partial\{ax\}}{\partial a} + \frac{\partial\{a^2\}}{\partial a}\\ \therefore 0 &= x+2a\\ \therefore -2a &= x\\ \therefore a &= \frac{x}{-2}\end{aligned}$$

By substituting $a = \frac{x}{-2}$ into the general solution as in Equation (101) we obtain

$$y = ax + (a)^2$$
$$\therefore y = x \cdot \frac{x}{-2} + \left(\frac{x}{-2}\right)^2$$
$$\therefore y = \frac{x^2}{-2} + \frac{x^2}{4}$$
$$\therefore y = \frac{-2x^2}{4} + \frac{x^2}{4}$$
$$\therefore y = \frac{-2x^2 + x^2}{4}$$
$$\therefore y = \frac{-x^2}{4}$$

Therefore the solutions are

$$y = ax + (a)^2$$

and

$$y = \frac{-x^2}{4}$$

where $a$ is an arbitrary constant.

86) Solve the following equation

$$y = x\frac{d\{y\}}{dx} + \cos\left(\frac{d\{y\}}{dx}\right)$$

The given equation is the same type as Equation (99). Thus we set

$$f\left(\frac{d\{y\}}{dx}\right) = \cos\left(\frac{d\{y\}}{dx}\right)$$

Equation (100) gives the general solution:

$$y = ax + \cos(a)$$

We now differentiate the general solution with respect to $a$:

$$\frac{\partial\{y\}}{\partial a} = \frac{\partial\{ax + \cos(a)\}}{\partial a}$$
$$\therefore 0 = \frac{\partial\{ax\}}{\partial a} + \frac{\partial\{\cos(a)\}}{\partial a}$$
$$\therefore 0 = x - \sin(a)$$
$$\therefore \sin(a) = x$$

Since we can not solve $\sin(a) = x$ for $a$ with ease, the particular solution is expressed using two equations by substituting $x = \sin(a)$ into $y = ax + \cos(a)$

$$y = ax + \cos(a)$$
$$\therefore y = a \cdot \sin(a) + \cos(a)$$

Therefore the solutions are

$$y = ax + \cos(a)$$

and

$$y = a\sin(a) + \cos(a), x = \sin(a)$$

where $a$ is an arbitrary constant.