Name: ID Number:

1) Find $-7-(+3)$

$$-7-(+3)=-7-3=-10$$

[5 mark]

2) Find $10-3\cdot(-7)$

$$10-3\cdot(-7)=10-(-21)=10+21=31$$

[5 mark]

3) Find $\sqrt{3^2+(-4)^2}$

$$\sqrt{3^2+(-4)^2}=\sqrt{9+16}=\sqrt{25}=5$$

[5 mark]

4) Find $9\cdot 6+3\cdot(-7)+(-4)\cdot(-2)$

$$9\cdot 6+3\cdot(-7)+(-4)\cdot(-2)=54-21+8=41$$

[5 mark]

5) Find θ when θ satisfies $1=2\cos\theta$

$$\begin{aligned} 1 &= 2\cos\theta \\ \therefore \cos\theta &= \frac{1}{2} \\ \therefore \theta &= \cos^{-1}\frac{1}{2} \\ \therefore \theta &= \frac{\pi}{3} \end{aligned}$$

[5 mark]

6) Find t,s and a when they satisfy

$$4+2s=2t; \quad -3s=a+t; \quad -2-s=-4+3t$$

Using the first and third equations, we can obtain s and t. $s=2-3t$ is substituted into the first equation:

$$\begin{aligned} 4+2s &= 2t \\ \therefore 4+2(2-3t) &= 2t \\ \therefore 4+4-6t &= 2t \\ \therefore 8 &= 2t+6t \\ \therefore 8 &= 8t \\ \therefore 1 &= t \end{aligned}$$

Substituting $t=1$ into $s=2-3t$, we obtain

$$s=2-3\cdot 1=2-3=-1$$

By substituting $(s,t)=(-1,1)$ into $-3s=a+t$

$$\begin{aligned} -3s &= a+t \\ \therefore -3\cdot(-1) &= a+1 \\ \therefore 3 &= a+1 \\ \therefore 3-1 &= a \\ \therefore a &= 2 \end{aligned}$$

[5 mark]

7) If $\boldsymbol{p}=9\boldsymbol{i}-7\boldsymbol{j}+5\boldsymbol{k}$ and $\boldsymbol{q}=-8\boldsymbol{i}+3\boldsymbol{j}-2\boldsymbol{k}$ express

a) $\boldsymbol{p}+\boldsymbol{q}$ in terms of $\boldsymbol{i}$,$\boldsymbol{j}$, and $\boldsymbol{k}$

$$\begin{aligned} \boldsymbol{p}+\boldsymbol{q} &= \begin{pmatrix} 9 \\ -7 \\ 5 \end{pmatrix} + \begin{pmatrix} -8 \\ 3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 9-8 \\ -7+3 \\ 5-2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} \end{aligned}$$

Keep your concentration until the third calculation , *i.e.*, the calculation of z component, finishes. You tend to

make a mistake in handling $\pm$ in the calculation of z component. Therefore $\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{i}-4\boldsymbol{j}+3\boldsymbol{k}$

[5 mark]

b) $\boldsymbol{p}-\boldsymbol{q}$ in terms of $\boldsymbol{i}$,$\boldsymbol{j}$, and $\boldsymbol{k}$

$$\boldsymbol{p}-\boldsymbol{q}=\begin{pmatrix}9\\-7\\5\end{pmatrix}-\begin{pmatrix}-8\\3\\-2\end{pmatrix}$$
$$=\begin{pmatrix}9-(-8)\\-7-(+3)\\5-(-2)\end{pmatrix}=\begin{pmatrix}9+8\\-7-3\\5+2\end{pmatrix}=\begin{pmatrix}17\\-10\\7\end{pmatrix}$$

Therefore $\boldsymbol{p}-\boldsymbol{q}=17\boldsymbol{i}-10\boldsymbol{j}+7\boldsymbol{k}$

[5 mark]

8) Consider the vectors $\boldsymbol{c}=4\boldsymbol{i}-5\boldsymbol{j}+10\boldsymbol{k}$ and $\boldsymbol{d}=-6\boldsymbol{i}+\boldsymbol{j}-7\boldsymbol{k}$ together with the scalar $\lambda=3$. Find

a) $\boldsymbol{c}-\lambda\boldsymbol{d}$ expressed in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$

$$\boldsymbol{c}-\lambda\boldsymbol{d}=\begin{pmatrix}4\\-5\\10\end{pmatrix}-3\cdot\begin{pmatrix}-6\\1\\-7\end{pmatrix}$$
$$=\begin{pmatrix}4\\-5\\10\end{pmatrix}-\begin{pmatrix}3\cdot(-6)\\3\cdot 1\\3\cdot(-7)\end{pmatrix}$$
$$=\begin{pmatrix}4\\-5\\10\end{pmatrix}-\begin{pmatrix}-18\\3\\-21\end{pmatrix}$$
$$=\begin{pmatrix}4-(-18)\\-5-3\\10-(-21)\end{pmatrix}$$
$$=\begin{pmatrix}22\\-8\\31\end{pmatrix}$$
$$=22\boldsymbol{i}-8\boldsymbol{j}+31\boldsymbol{k}$$

[5 mark]

b) the magnitude of $\boldsymbol{c}$

$$|\boldsymbol{c}|=\sqrt{4^2+(-5)^2+10^2}$$
$$=\sqrt{16+25+100}$$
$$=\sqrt{141}$$

Do not forget squaring the values.

$$\sqrt{4^2+(-5)^2+10^2}\neq\sqrt{4+(-5)+10}$$

[5 mark]

c) a unit vector parallel to $\boldsymbol{c}$ **The exam question is talking about c, not $c-\lambda d$. Read your exam question carefully.**

$$\hat{\boldsymbol{n}}=\frac{\boldsymbol{c}}{|\boldsymbol{c}|}=\frac{1}{\sqrt{141}}(4\boldsymbol{i}-5\boldsymbol{j}+10\boldsymbol{k})$$

[5 mark]

9) Points R, S, and T have coordinates $(-4,0,-1)$, $(5,3,-5)$ and $(2,-7,-3)$ respectively. Find

a) the scalar product $\overrightarrow{RS}\cdot\overrightarrow{RT}$.

$$\overrightarrow{RS}\neq\overrightarrow{OR}+\overrightarrow{OS}$$

but

$$\overrightarrow{RS}=\overrightarrow{RO}+\overrightarrow{OS}=\underline{-}\overrightarrow{OR}+\overrightarrow{OS}$$

$$
\begin{aligned}
\overrightarrow{RS} &= \overrightarrow{RO} + \overrightarrow{OS} \\
&= -\overrightarrow{OR} + \overrightarrow{OS} \\
&= -\boldsymbol{r} + \boldsymbol{s} \\
&= -\begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \\ -5 \end{pmatrix} \\
&= \begin{pmatrix} -(-4)+5 \\ 3 \\ -(-1)-5 \end{pmatrix} \\
&= \begin{pmatrix} 9 \\ 3 \\ -4 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{RT} &= \overrightarrow{RO} + \overrightarrow{OT} \\
&= -\overrightarrow{OR} + \overrightarrow{OT} \\
&= -\boldsymbol{r} + \boldsymbol{t} \\
&= -\begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \\ -3 \end{pmatrix} \\
&= \begin{pmatrix} -(-4)+2 \\ -7 \\ -(-1)-3 \end{pmatrix} \\
&= \begin{pmatrix} 6 \\ -7 \\ -2 \end{pmatrix}
\end{aligned}
$$

[5 mark]

$$
\begin{aligned}
\therefore \overrightarrow{RS} \cdot \overrightarrow{RT} &= \begin{pmatrix} 9 \\ 3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -7 \\ -2 \end{pmatrix} \\
&= 9 \cdot 6 + 3 \cdot (-7) + (-4) \cdot (-2) \\
&= 54 - 21 + 8 = 41
\end{aligned}
$$

The outcome of the scalar product is a scalar. Please check whether or not your answer is a scalar. Do not put $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, in your answer such as $54\boldsymbol{i} - 21\boldsymbol{j} + 8\boldsymbol{k} \leftarrow$ wrong

[5 mark]

b) the vector product $\overrightarrow{RS} \times \overrightarrow{RT}$.

$$
\begin{pmatrix} \overrightarrow{RS} & \overrightarrow{RT} \end{pmatrix} = \begin{pmatrix} 9 & 6 \\ 3 & -7 \\ -4 & -2 \end{pmatrix}
$$

$$
\begin{aligned}
\overrightarrow{RS} \times \overrightarrow{RT} &= \begin{vmatrix} 3 & -7 \\ -4 & -2 \end{vmatrix} \boldsymbol{i} \\
&+ \begin{vmatrix} -4 & -2 \\ 9 & 6 \end{vmatrix} \boldsymbol{j} \\
&+ \begin{vmatrix} 9 & 6 \\ 3 & -7 \end{vmatrix} \boldsymbol{k} \\
&= \{3 \cdot (-2) - (-4) \cdot (-7)\}\boldsymbol{i} \\
&+ \{(-4) \cdot 6 - 9 \cdot (-2)\}\boldsymbol{j} \\
&+ \{9 \cdot (-7) - 3 \cdot 6\}\boldsymbol{k} \\
&= \{-6 - 28\}\boldsymbol{i} \\
&+ \{-24 - (-18)\}\boldsymbol{j} \\
&+ \{-63 - 18)\}\boldsymbol{k} \\
&= -34\boldsymbol{i} - 6\boldsymbol{j} - 81\boldsymbol{k}
\end{aligned}
$$

The outcome of the vector product is a vector. Please check whether or not your answer is a vector. Do not add three components together in your answer such as $-34 - 6 - 81 = -121 \leftarrow$ wrong

[5 mark]

c) the angle between the vectors $\overrightarrow{RS}$ and $\overrightarrow{RT}$.

Do not mix up the scalar product and the vector product with respect to $\sin$ and $\cos$

$$\overrightarrow{RS} \cdot \overrightarrow{RT} = \left|\overrightarrow{RS}\right| \cdot \left|\overrightarrow{RT}\right| \cdot \cos\theta$$

$$|\overrightarrow{RS} \times \overrightarrow{RT}| = \left|\overrightarrow{RS}\right| \cdot \left|\overrightarrow{RT}\right| \cdot |\sin\theta|$$

Do not swap around $\sin$ and $\cos$ between these two formula.

$$\begin{aligned}\left|\overrightarrow{RS}\right| &= \sqrt{9^2+3^2+(-4)^2} \\ &= \sqrt{81+9+16} \\ &= \sqrt{106} \\ \left|\overrightarrow{RT}\right| &= \sqrt{6^2+(-7)^2+(-2)^2} \\ &= \sqrt{36+49+4} \\ &= \sqrt{89}\end{aligned}$$

$$\begin{aligned}\therefore \overrightarrow{RS} \cdot \overrightarrow{RT} &= \left|\overrightarrow{RS}\right| \cdot \left|\overrightarrow{RT}\right| \cdot \cos\theta \\ \therefore 41 &= \sqrt{106} \cdot \sqrt{89} \cdot \cos\theta \\ \therefore \frac{41}{\sqrt{106} \cdot \sqrt{89}} &= \cos\theta \\ \therefore \theta = \cos^{-1}\frac{41}{\sqrt{106} \cdot \sqrt{89}} &= 1.12363 \text{radians}\end{aligned}$$

[5 mark]

10) A line l is parallel to $\boldsymbol{p} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$, passing through $A(4, 0, -2)$. A line m is parallel to $\boldsymbol{q} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, passing through $B(0, a, -4)$. Find the Cartesian coordinate of the crossing point and a when l and m are crossing each other.

Develop a plan to solve the problem.

a) **produce a Cartesian coordiate of a point L on a line l with a parameter s**

b) **produce a Cartesian coordiate of a point M on a line m with a parameter t**

c) **make these two Cartesian coordinate equal to find a, s, and t**

d) **put the value of s into the Cartesian coordinate of a point L**

Using an arbitrary variable s, the line l can be expressed as

$$\overrightarrow{OA} + s\boldsymbol{p}$$
$$= \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} + s\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \cdot s \\ -3 \cdot s \\ -1 \cdot s \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 2s \\ -3s \\ -s \end{pmatrix}$$
$$= \begin{pmatrix} 4+2s \\ 0-3s \\ -2-s \end{pmatrix}$$
$$= \begin{pmatrix} 4+2s \\ -3s \\ -2-s \end{pmatrix}$$

[5 mark]

Using an arbitrary variable t, the line m can be expressed as

$$\overrightarrow{OB} + t\boldsymbol{q}$$
$$= \begin{pmatrix} 0 \\ a \\ -4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ a \\ -4 \end{pmatrix} + \begin{pmatrix} 2 \cdot t \\ 1 \cdot t \\ 3 \cdot t \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ a \\ -4 \end{pmatrix} + \begin{pmatrix} 2t \\ t \\ 3t \end{pmatrix}$$
$$= \begin{pmatrix} 0+2t \\ a+t \\ -4+3t \end{pmatrix}$$
$$= \begin{pmatrix} 2t \\ a+t \\ -4+3t \end{pmatrix}$$

[5 mark]

When l and m are crossing, there exists s and t which satisfy

$$\begin{pmatrix} 4+2s \\ -3s \\ -2-s \end{pmatrix} = \begin{pmatrix} 2t \\ a+t \\ -4+3t \end{pmatrix}$$

From this, we obtain the following 3 equations:

$$4 + 2s = 2t$$
$$-3s = a + t$$
$$-2 - s = -4 + 3t$$
$$\therefore -s = -2 + 3t$$
$$\therefore s = 2 - 3t$$

[5 mark]

Using the first and third equations, we can obtain s and t. $s = 2 - 3t$ is substituted into the first equation:

$$4 + 2s = 2t$$
$$\therefore 4 + 2(2 - 3t) = 2t$$
$$\therefore 4 + 4 - 6t = 2t$$
$$\therefore 8 = 2t + 6t$$
$$\therefore 8 = 8t$$
$$\therefore 1 = t$$

Substituting $t = 1$ into $s = 2 - 3t$, we obtain

$$s = 2 - 3 \cdot 1 = 2 - 3 = -1$$

By substituting $(s, t) = (-1, 1)$ into $-3s = a + t$

$$-3s = a + t$$
$$\therefore -3 \cdot (-1) = a + 1$$
$$\therefore 3 = a + 1$$
$$\therefore 3 - 1 = a$$
$$\therefore a = 2$$

[5 mark]

Thus the Cartesian coordinate of the crossing point is

$$\begin{aligned} & \begin{pmatrix} 4+2s \\ -3s \\ -2-s \end{pmatrix} \\ = & \begin{pmatrix} 4+2\cdot(-1) \\ -3\cdot(-1) \\ -2-(-1) \end{pmatrix} \\ = & \begin{pmatrix} 4-2 \\ 3 \\ -2+1 \end{pmatrix} \\ = & \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \end{aligned}$$

[5 mark]

Name: ID Number:

1) Find $\sqrt{x^2+y^2}$ when $x = 2$, $y = -8$.

$$\begin{aligned}\sqrt{x^2+y^2} &= \sqrt{2^2+(-8)^2}\\ &= \sqrt{4+64}\\ &= \sqrt{68}\\ &= \sqrt{4}\cdot\sqrt{17}\\ &= 2\sqrt{17}\end{aligned}$$

[1 mark]

2) Find θ when $\tan\theta = -4$.

$$\begin{aligned}\tan\theta &= -4\\ \theta &= \tan^{-1}(-4)\\ \theta &= -1.33\end{aligned}$$

No need to add π to θ as the Cartesian coordinate is not given.

[1 mark]

3) Find $\sqrt{x^2+y^2+z^2}$ when $x = -1$, $y = -\sqrt{3}$, $z = 2$.

$$\sqrt{x^2+y^2+z^2} = \sqrt{1^2+(-\sqrt{3})^2+2^2} = \sqrt{1+3+4} = \sqrt{8} = 2\sqrt{2}$$

[1 mark]

4) Complete the square for x and y in $x^2+y^2-10y = 0$

$$\begin{aligned}x^2+y^2-10y &= 0\\ x^2+(y-5)^2-5^2 &= 0\\ x^2+(y-5)^2 &= 5^2\end{aligned}$$

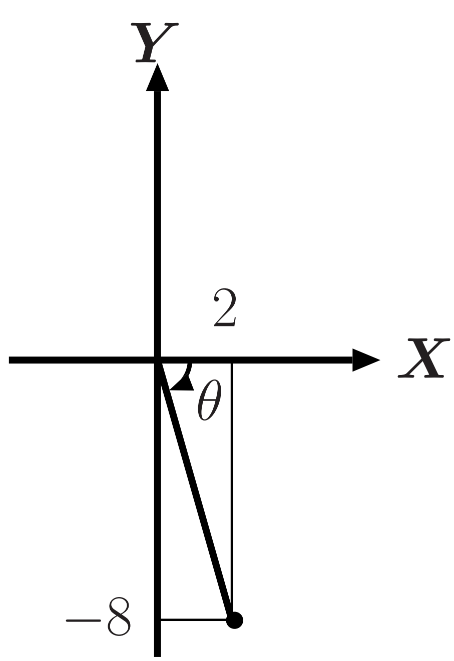
[1 mark]

5) Find θ when $\cos\theta = \frac{1}{\sqrt{2}}$.

$$\begin{aligned}\cos\theta &= \frac{1}{\sqrt{2}}\\ \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) &= \frac{\pi}{4}\end{aligned}$$

[1 mark]

6) Calculate the equivalent polar coordinates of the following Cartesian coordinate of $(2, -8)$



$$
\begin{aligned}
r &= \sqrt{x^2+y^2} \\
&= \sqrt{2^2+(-8)^2} \\
&= \sqrt{4+64} \\
&= \sqrt{68} \\
&= \sqrt{4}\cdot\sqrt{17} \\
&= 2\sqrt{17}
\end{aligned}
$$

[1 mark]

$$
\begin{aligned}
\tan\theta &= \frac{y}{x} \\
\tan\theta &= \frac{-8}{2} \\
\tan\theta &= -4 \\
\theta &= \tan^{-1}(-4) \\
\theta &= -1.33
\end{aligned}
$$

Since this is in the fourth quadrant the answer is valid.

[1 mark]

7) In Cartesian coordinates a point P is given by $x=-1$, $y=-\sqrt{3}$, $z=2$. Give the position of P in spherical coordinates. We need to find out r, θ and ϕ which

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{1^2+(-\sqrt{3})^2+2^2} = \sqrt{1+3+4} = \sqrt{8} = 2\sqrt{2}$$

[1 mark]

$$
\begin{aligned}
\cos\theta &= \frac{z}{r} = \frac{2}{\sqrt{8}} \\
\therefore \theta &= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}
\end{aligned}
$$

[1 mark]

$$
\begin{aligned}
\tan\phi &= \frac{y}{x} = \frac{-\sqrt{3}}{-1} = \sqrt{3} \\
\therefore \phi &= \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}, \frac{4\pi}{3}
\end{aligned}
$$

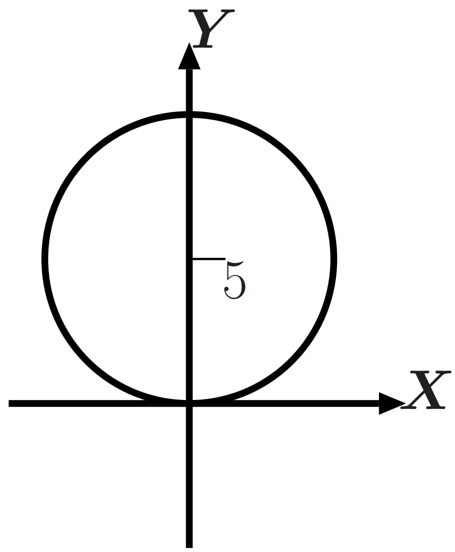
However ϕ is in the 3rd quadrant. Thus $\phi = \frac{4\pi}{3}$. **Check the location of the point by drawing the Cartesian coordiante!! From your drawing, obtain the rough idea on ϕ. The procedure to this type of problems is**

a) **Draw the Cartesian coordinate figure and place the point in question**
b) **Estimate the rough idea on ϕ, you do not need to worry about θ as there is no ambiguity involved as long as you follow the formula to calculate θ**
c) **Calculate ϕ and check whether or not ϕ is similar to your estimate. If not, add π to obtain the correct ϕ**

[1 mark]

8) Find the Cartesian form of

$$r = 10\cdot\sin\theta$$



$$\therefore r = 10 \cdot \frac{y}{r} \because \sin\theta = \frac{y}{r}$$
$$\therefore r^2 = 10y$$
$$\therefore x^2 + y^2 = 10y \because r^2 = x^2 + y^2$$
$$\therefore x^2 + y^2 - 10y = 0$$

Procedure is as follows

a) **Get rid of θ using $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{x}{r}$**

b) **Get rid of r using $r^2 = x^2 + y^2$**

[1 mark]

By completing the square,

$$x^2 + (y-5)^2 - 5^2 = 0$$
$$\therefore x^2 + (y-5)^2 = 5^2$$

[1 mark]

9) Calculate the equivalent exponential form of $4 - \jmath 2$ and draw the argand diagram.

Please remember the meaning of Argand diagram which consists of r and θ

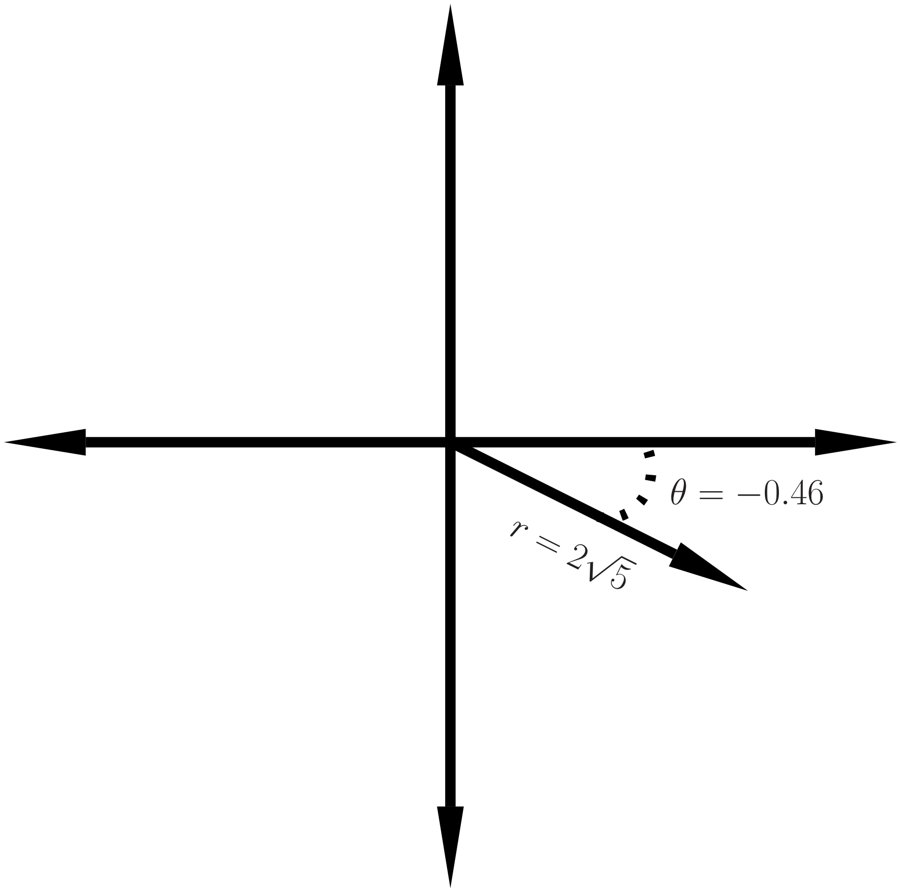
$$r = \sqrt{a^2 + b^2}$$
$$= \sqrt{4^2 + (-2)^2}$$
$$= \sqrt{16 + 4}$$
$$= \sqrt{20}$$
$$= \sqrt{4} \cdot \sqrt{5}$$
$$= 2\sqrt{5}$$

[1 mark]

$$\tan\theta = \frac{b}{a}$$
$$\therefore \theta = \tan^{-1}\frac{b}{a}$$
$$= \tan^{-1}\frac{-2}{4}$$
$$= \tan^{-1}\frac{-1}{2}$$
$$= -0.46$$

Since this is in the fourth quadrant the answer is valid.

[1 mark]



[1 mark]

10) The point p is at $z = r\mathfrak{e}^{\jmath t}$. We rotate p about origin and the angle of the rotation is $\pi/3$. The new position of p is $\hat{p}$. Calculate the Cartesian coordinates of $\hat{p}$.

The point p's original angle is t, not 0. Thus the new angle after the rotation is $t + \frac{\pi}{3}$. The Cartesian coordiate expression is either $a + \jmath b$ or (a, b), not $(a, \jmath b)$. $z = r\mathfrak{e}^{\jmath t}$ is rotated by $\pi/3$.

This can be expressed as

$$\begin{aligned} z &= r\mathfrak{e}^{\jmath(t+\pi/3)} \\ &= r(\cos(t+\pi/3) + \jmath\sin(t+\pi/3)) \end{aligned}$$

[1 mark]

Thus the coordinate of $\hat{p}$ is

$$(r\cos(t+\pi/3), r\sin(t+\pi/3))$$

[1 mark]

11) Using De Moivre's theorem, write $\left(\sqrt{3} + \jmath\sqrt{3}\right)^4$ in the form $\alpha \pm \jmath\beta$.

The strategy to tackle this problem is

a) change $\sqrt{3} + \jmath\sqrt{3}$ to the exponential form
b) use Equation (28) to get the form of $r(\cos\theta + \jmath\sin\theta)$
c) Apply Equation (34)

First let's find r.

$$r = \sqrt{(\sqrt{3})^2 + (\sqrt{3})^2} = \sqrt{3+3} = \sqrt{6}$$

[1 mark]

Now to work out θ. We know that $\tan\theta = \dfrac{\sqrt{3}}{\sqrt{3}} = 1$.

$$\theta = \tan^{-1} 1 = \frac{1}{4}\pi$$

[1 mark]

Now using De Moivre's theorem.

$$
\begin{aligned}
\left(\sqrt{3}-\jmath\sqrt{3}\right)^4 &= \left[\sqrt{6}(\cos(\frac{1}{4}\pi)+\jmath\sin(\frac{1}{4}\pi))\right]^4 \\
&= (\sqrt{6})^4(\cos(\frac{1}{4}\pi)+\jmath\sin(\frac{1}{4}\pi))^4 \\
&= (\sqrt{6})^4(\cos(4\cdot\frac{1}{4}\pi)+\jmath\sin(4\cdot\frac{1}{4}\pi)) \\
&= (\sqrt{6})^2\cdot(\sqrt{6})^2(\cos(4\cdot\frac{1}{4}\pi)+\jmath\sin(4\cdot\frac{1}{4}\pi)) \\
&= 6\cdot 6(\cos(4\cdot\frac{1}{4}\pi)+\jmath\sin(4\cdot\frac{1}{4}\pi)) \\
&= 36(\cos(\pi)+\jmath\sin(\pi)) \\
&= 36(-1+\jmath 0) \\
&= -36+\jmath 0
\end{aligned}
$$

Therefore $\alpha = -36$ and $\beta = 0$.

[1 mark]

Name: ID Number:

1) Find $\frac{\partial \{y\}}{\partial x}$ of $y = 5x^7$.

$$\frac{\partial \{y\}}{\partial x} = \frac{\partial \left\{5x^7\right\}}{\partial x} = 35x^6$$

[1 mark]

2) Express z using x and y when x, y and z satisfy $2x + y + xz + 2yz = 0$

$$\begin{aligned} 2x + y + xz + 2yz &= 0 \\ \therefore 2x + y + (x + 2y)z &= 0 \\ \therefore (x + 2y)z &= -2x - y \\ \therefore z = \frac{-2x - y}{x + 2y} &= -\frac{2x + y}{x + 2y} \end{aligned}$$

[1 mark]

$\frac{-2x + y}{x + 2y} \neq -\frac{2x + y}{x + 2y} = \frac{-2x - y}{x + 2y} = \frac{-(2x + y)}{x + 2y}$ **Pay extra attention to the location of "$-$" around and in the fraction.**

3) Find x when $x^2 + xy + y^2 = 3$ and $y = -2x$

$$\begin{aligned} x^2 + x(-2x) + (-2x)^2 &= 3 \\ \therefore x^2 - 2x^2 + 4x^2 &= 3 \\ \therefore 3x^2 &= 3 \\ \therefore x &= \pm 1 \end{aligned}$$

Do not forget $\pm$ when you obtain x from $x^2 = \alpha$

[1 mark]

4) Find $\frac{\partial \{y\}}{\partial x}$ of $y = 1 - x^{\frac{1}{2}}$.

$$\begin{aligned} \frac{\partial \left\{1 - x^{\frac{1}{2}}\right\}}{\partial x} &= \frac{\partial \{1\}}{\partial x} - \frac{\partial \left\{x^{\frac{1}{2}}\right\}}{\partial x} = 0 - \frac{1}{2}x^{\frac{1}{2}-1} \\ &= -\frac{1}{2}x^{\frac{1}{2}-\frac{2}{2}} = -\frac{1}{2}x^{-\frac{1}{2}} \end{aligned}$$

[1 mark]

5) Find y when $x^2 + xy + y^2 = 3$ and $x = -1$

$$\begin{aligned} (-1)^2 + (-1) \cdot y + y^2 &= 3 \\ \therefore 1 - y + y^2 &= 3 \\ \therefore y^2 - y - 2 &= 0 \\ \therefore y = \frac{1 \pm \sqrt{1 - 4(-2)}}{2} = \frac{1 \pm \sqrt{1 + 8}}{2} = \frac{1 \pm 3}{2} &= 2, -1 \end{aligned}$$

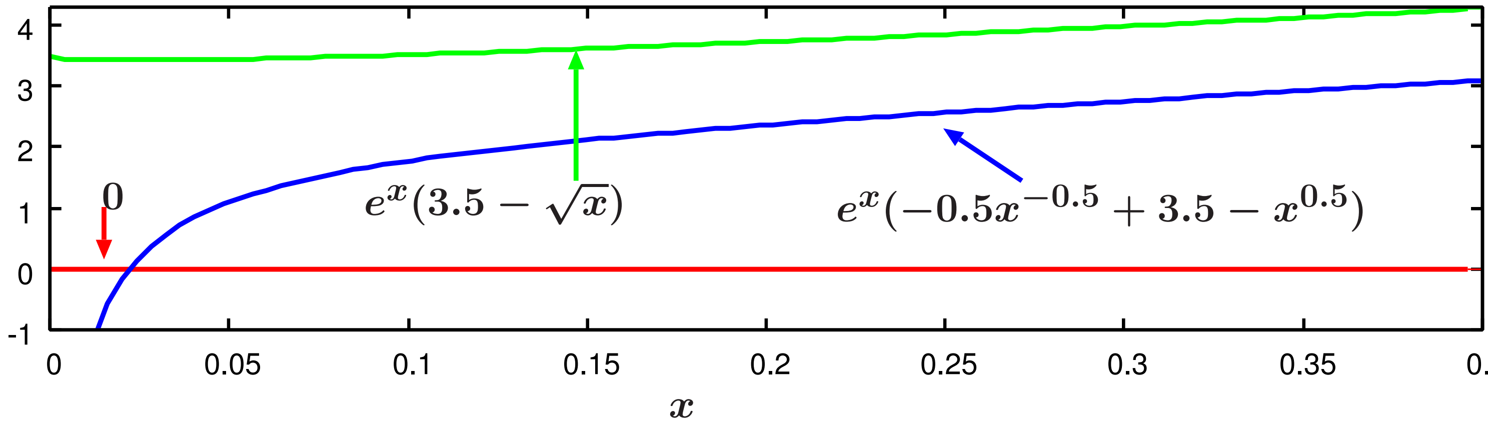
It is better for you to remember the answer of $Ax^2 + Bx + C = 0$ as $x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$.

The determinant $B^2 - 4AC \geq 0 \rightarrow Ax^2 + Bx + C$ can be factorized using real values.

The determinant $B^2 - 4AC < 0 \rightarrow Ax^2 + Bx + C$ can not be factorized with ease.

[1 mark]

6) Differentiate $f(x) = \mathfrak{e}^x (3.5 - \sqrt{x})$ with regard to x.



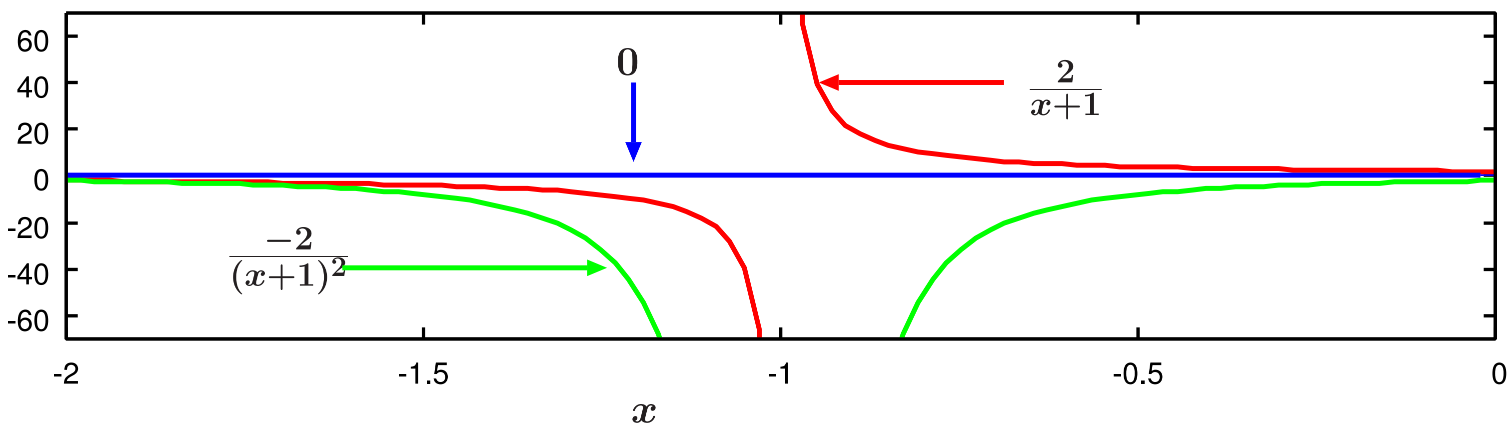
$$\frac{\partial\{f(x)\}}{\partial x} = \mathfrak{e}^x \frac{\partial\{3.5 - \sqrt{x}\}}{\partial x} + \frac{\partial\{\mathfrak{e}^x\}}{\partial x}(3.5 - \sqrt{x})$$

[1 mark]

$$= \mathfrak{e}^x\left(-\frac{1}{2}x^{-\frac{1}{2}}\right) + \mathfrak{e}^x(3.5 - \sqrt{x})$$
$$= \mathfrak{e}^x\left(-\frac{1}{2}x^{-\frac{1}{2}} + 3.5 - \sqrt{x}\right)$$

[1 mark]

7) Differentiate $f(x) = \frac{2}{x+1}$ with regard to x



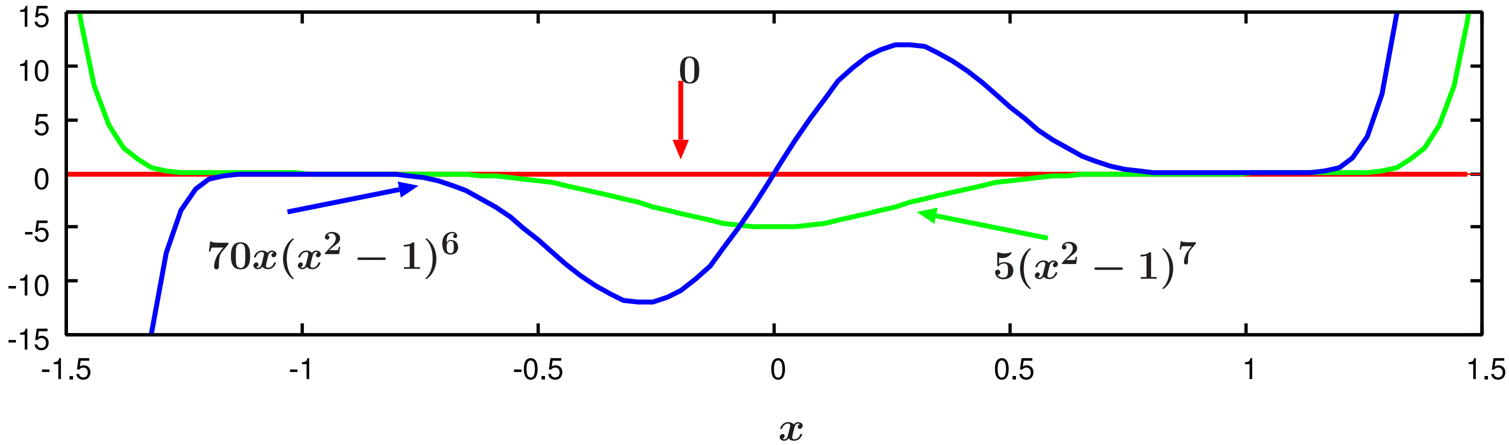
$$\frac{\partial\{f(x)\}}{\partial x} = \frac{\partial\left\{\frac{2}{x+1}\right\}}{\partial x}$$
$$= \frac{\frac{\partial\{2\}}{\partial x}\cdot(x+1) - 2\cdot\frac{\partial\{x+1\}}{\partial x}}{(x+1)^2}$$
$$= \frac{0\cdot(x+1) - 2\cdot 1}{(x+1)^2}$$
$$= \frac{-2}{(x+1)^2}$$

Do not mix up the differentiation and the integration.

$$\frac{\partial\left\{\frac{1}{x}\right\}}{\partial x} = -\frac{1}{x^2}. \quad \int \frac{1}{x}dx = \ln x.$$

[1 mark]

8) Find $\frac{\partial\{y\}}{\partial x}$ of $y = 5(x^2 - 1)^7$.



First let $u = x^2 - 1$. Therefore the function becomes $y = 5(u)^7$. Differentiate both of these equations.

$$u = x^2 - 1$$
$$\frac{\partial \{u\}}{\partial x} = 2x$$
$$y = 5(u)^7$$
$$\frac{\partial \{y\}}{\partial u} = 35(u)^6$$

[1 mark]

Now using the chain rule formula below we can find $\frac{\partial \{y\}}{\partial x}$

$$\frac{\partial \{y\}}{\partial x} = \frac{\partial \{y\}}{\partial u} \cdot \frac{\partial \{u\}}{\partial x}$$
$$= 2x \cdot 35(u)^6$$
$$= 70x(x^2-1)^6 \because u = x^2 - 1$$

[1 mark]

9) Evaluate the limit $\lim_{x \to 0} \frac{\sin x}{x}$. You may use $\sin(0) = 0$ and $\cos(0) = 1$.

When you see $\lim_{x \to 0} \frac{f(x)}{g(x)}$, first of all, try putting $x = 0$ into $\frac{f(x)}{g(x)}$.

Then you shall face $\frac{0}{0} \Leftarrow$ mathematically undefined.

Finally you declare that you use $\lim_{x \to 0} \frac{f'(x)}{g'(x)}$ instead.

When $x = 0$, we find

$$\sin x|_{x=0} = 0$$

and

$$x|_{x=0} = 0.$$

Therefore we use L'Hôpitals rule as follows.

$$\lim_{x \to 0} \frac{\sin x}{x} = \left. \frac{\frac{\partial \{\sin x\}}{\partial x}}{\frac{\partial \{x\}}{\partial x}} \right|_{x=0} = \left. \frac{\cos(x)}{1} \right|_{x=0}$$

[1 mark]

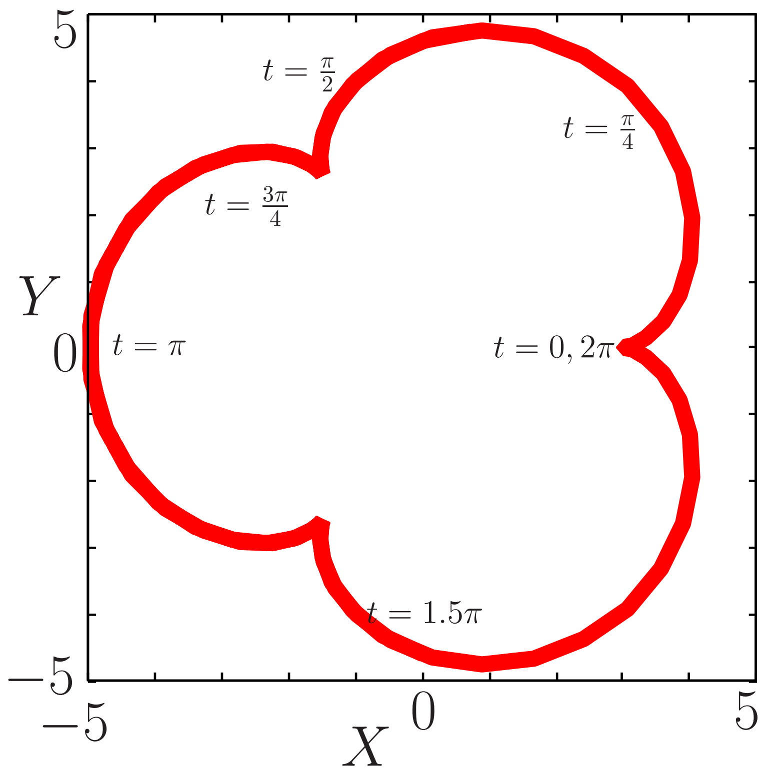
$$= \frac{1}{1} = 1$$

[1 mark]

10) y is the function of x. Using a parameter t, x and y are expressed as

$$x = 4\cos(t) - \cos(4t)$$
$$y = 4\sin(t) - \sin(4t)$$

Express $\frac{\partial \{y\}}{\partial x}$ using t.



$$\frac{\partial \{y\}}{\partial x} \neq \frac{y}{x}$$

Since x and y are expressed using t, we can find $\frac{\partial \{x\}}{\partial t}$ and $\frac{\partial \{y\}}{\partial t}$ with ease. So we are going to use $\frac{\partial \{x\}}{\partial t}$ and $\frac{\partial \{y\}}{\partial t}$ to produce $\frac{\partial \{y\}}{\partial x}$ as follows:

$$\frac{\partial \{y\}}{\partial x} = \frac{\partial \{y\}}{\partial t} \cdot \frac{\partial \{t\}}{\partial x} = \frac{\partial \{y\}}{\partial t} \cdot (\frac{\partial \{x\}}{\partial t})^{-1}$$

[1 mark]

$\frac{\partial \{x\}}{\partial t}$ and $\frac{\partial \{y\}}{\partial t}$ are:

$$\frac{\partial \{x\}}{\partial t} = -4\sin(t) + 4\sin(4t)$$

[1 mark]

$$\frac{\partial \{y\}}{\partial t} = 4\cos(t) - 4\cos(4t)$$

[1 mark]

$\frac{\partial \{y\}}{\partial x}$ are

$$\begin{aligned} &\frac{\partial \{y\}}{\partial x} \\ &= \frac{\partial \{y\}}{\partial t} \cdot \frac{\partial \{t\}}{\partial x} \\ &= \frac{\partial \{y\}}{\partial t} \cdot (\frac{\partial \{x\}}{\partial t})^{-1} \\ &= (4\cos(t) - 4\cos(4t)) \cdot (-4\sin(t) + 4\sin(4t))^{-1} \\ &= (\cos(t) - \cos(4t)) \cdot (-\sin(t) + \sin(4t))^{-1} \\ &= \frac{\cos(t) - \cos(4t)}{-\sin(t) + \sin(4t)} \end{aligned}$$

[1 mark]

11) For $x^2 + xy + y^2 = 3$

a) find $\frac{\partial \{y\}}{\partial x}$

Implicit differentiation sees all the variables such as x,y,z,s,t as a function.

$$\frac{\partial \{f \cdot g\}}{\partial x} = \frac{\partial \{f\}}{\partial x} g + f \frac{\partial \{g\}}{\partial x}$$
$$\frac{\partial \{x \cdot y\}}{\partial x} = \underline{\frac{\partial \{x\}}{\partial x}} y + x \frac{\partial \{y\}}{\partial x} = \underline{y} + x \frac{\partial \{y\}}{\partial x}$$

Do not forget to differentiate both x and y

First of all we find $\frac{\partial \{y\}}{\partial x}$ as follows:

$$\begin{aligned}
\frac{\partial \{x^2 + xy + y^2\}}{\partial x} &= \frac{\partial \{3\}}{\partial x} \\
\therefore \frac{\partial \{x^2\}}{\partial x} + \frac{\partial \{xy\}}{\partial x} + \frac{\partial \{y^2\}}{\partial x} &= 0 \\
\therefore 2x + \frac{\partial \{x\}}{\partial x} y + x \frac{\partial \{y\}}{\partial x} + 2y \frac{\partial \{y\}}{\partial x} &= 0 \\
\therefore 2x + y + x \frac{\partial \{y\}}{\partial x} + 2y \frac{\partial \{y\}}{\partial x} &= 0
\end{aligned}$$

[1 mark]

$$\therefore (x + 2y) \frac{\partial \{y\}}{\partial x} = -(2x + y)$$

$$\therefore \frac{\partial \{y\}}{\partial x} = -\frac{2x + y}{x + 2y}$$

[1 mark]

b) evaluate y at $x = 0$
First of all, we find the value of y When $x = 0$ as follows:

$$\begin{aligned}
x^2 + xy + y^2 &= 3 \\
\therefore (0)^2 - 0 + y^2 &= 3 \\
\therefore y^2 &= 3 \\
\therefore y &= \pm\sqrt{3}
\end{aligned}$$

[1 mark]

c) evaluate $\frac{\partial \{y\}}{\partial x}$ at $x = 0$

$$\left.\frac{\partial \{y\}}{\partial x}\right|_{(x,y)=(0,\pm\sqrt{3})} = -\left.\frac{2x + y}{x + 2y}\right|_{(x,y)=(0,\pm\sqrt{3})} = -\frac{0 + y}{0 + 2y} = -\frac{y}{2y} = -\frac{1}{2}$$

[1 mark]

I. Prerequisites

In order to successfully complete this Engineering Mathematics course you must be competent with the following material. If you are unfamiliar with the any of the following material it is recommended that you attempt some practice questions before undertaking the main course material.

1) **Logarithms**

$$\begin{aligned}
\log_a(x) = m &\equiv a^m = x \\
\log(x) &\equiv \log_{10}(x) \\
\ln(x) &\equiv \log_e(x) \\
\log_a(a) &= 1 \\
\log_a(m \cdot n) &= \log_a(m) + \log_a(n) \\
\log_a\left(\frac{m}{n}\right) &= \log_a(m) - \log_a(n) \\
\log_a(m^n) &= n \cdot \log_a(m) \\
\log_a b &= \frac{\log_c b}{\log_c a}
\end{aligned}$$

2) **Indices**

$$\begin{aligned}
a^m \cdot a^n &= a^{(m+n)} \\
\frac{a^m}{a^n} &= a^{(m-n)} \\
(a^m)^n &= a^{(m \cdot n)} \\
a^{-m} &= \frac{1}{a^m} \\
a^{(m/n)} &= \sqrt[n]{a^m} \\
a^0 &= 1 \\
a^1 &= a
\end{aligned}$$

3) **Trigonometric Identities**

$$\begin{aligned}
y = \sin^{-1} x = \arcsin x &\iff x = \sin y \\
y = \cos^{-1} x = \arccos x &\iff x = \cos y \\
y = \tan^{-1} x = \arctan x &\iff x = \tan y \\
\text{cosec } x &= \frac{1}{\sin x} \\
\sec x &= \frac{1}{\cos x} \\
\cot x &= \frac{1}{\tan x} \\
y = \text{cosec }^{-1} x \iff x = \text{ cosec } y &= \frac{1}{\sin y} \\
y = \sec^{-1} x \iff x = \sec y &= \frac{1}{\cos y} \\
y = \cot^{-1} x \iff x = \cot y &= \frac{1}{\tan y} \\
\tan(x) &= \frac{\sin(x)}{\cos(x)} \\
\sin^2(x) + \cos^2(x) &= 1 \\
\sec^2(x) &= 1 + \tan^2(x) \\
\sin(A \pm B) &= \sin(A)\cos(B) \pm \cos(A)\sin(B) \\
\cos(A \pm B) &= \cos(A)\cos(B) \mp \sin(A)\sin(B) \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \\
\sin(2A) &= 2\sin(A)\cos(A) \\
\cos(2A) &= \cos^2(A) - \sin^2(A) \\
&= 2\cos^2(A) - 1 \\
&= 1 - 2\sin^2(A) \\
\tan(2A) &= \frac{2\tan(A)}{1 - \tan^2(A)} \\
2\sin(A)\cos(B) &= \sin(A+B) + \sin(A-B) \\
2\cos(A)\sin(B) &= \sin(A+B) - \sin(A-B) \\
2\cos(A)\cos(B) &= \cos(A+B) + \cos(A-B) \\
-2\sin(A)\sin(B) &= \cos(A+B) - \cos(A-B) \\
\cos(x) &= \frac{\mathfrak{e}^{\jmath x} + \mathfrak{e}^{-\jmath x}}{2} \\
\sin(x) &= \frac{\mathfrak{e}^{\jmath x} - \mathfrak{e}^{-\jmath x}}{2\jmath}
\end{aligned}$$

4) **Hyperbolic Identities**

$$\cosh(x) = (\mathfrak{e}^{x} + \mathfrak{e}^{-x})/2$$
$$x = \cosh^{-1}\left(\frac{\mathfrak{e}^{x} + \mathfrak{e}^{-x}}{2}\right)$$
$$\sinh(x) = (\mathfrak{e}^{x} - \mathfrak{e}^{-x})/2$$
$$x = \sinh^{-1}\left(\frac{\mathfrak{e}^{x} - \mathfrak{e}^{-x}}{2}\right) \tanh(x) = (\mathfrak{e}^{x} - \mathfrak{e}^{-x})/(\mathfrak{e}^{x} + \mathfrak{e}^{-x})$$
$$\cosh^2(A) - \sinh^2(A) = 1$$

When you need $\cosh^{-1}(\alpha)$ where α is a real number,

$$\frac{\mathfrak{e}^{x} + \mathfrak{e}^{-x}}{2} = \alpha$$
$$\therefore \mathfrak{e}^{x} + \mathfrak{e}^{-x} = 2\alpha$$
$$\therefore \mathfrak{e}^{2x} + 1 = 2\alpha\mathfrak{e}^{x}$$
$$\therefore \mathfrak{e}^{2x} - 2\alpha\mathfrak{e}^{x} + 1 = 0$$
$$\therefore \mathfrak{e}^{x} = \alpha \pm \sqrt{\alpha^2 - 1}$$
$$\therefore x = \ln(\alpha \pm \sqrt{\alpha^2 - 1})$$

When you need $\sinh^{-1}(\alpha)$ where α is a real number,

$$\frac{\mathfrak{e}^{x} - \mathfrak{e}^{-x}}{2} = \alpha$$
$$\therefore \mathfrak{e}^{x} - \mathfrak{e}^{-x} = 2\alpha$$
$$\therefore \mathfrak{e}^{2x} - 1 = 2\alpha\mathfrak{e}^{x}$$
$$\therefore \mathfrak{e}^{2x} - 2\alpha\mathfrak{e}^{x} - 1 = 0$$
$$\therefore \mathfrak{e}^{x} = \alpha \pm \sqrt{\alpha^2 + 1}$$
$$\therefore x = \ln(\alpha \pm \sqrt{\alpha^2 + 1})$$
$$\therefore x = \ln(\alpha + \sqrt{\alpha^2 + 1})(\because A > 0 \text{ for } \ln A)$$

5) **Completing the Square**

$$4x^2 - 2x - 5 = 0$$

We can solve the above equation by completing the square as follows

$$4x^2 - 2x - 5 = 0$$
$$4x^2 - 2x = 5$$
$$x^2 - \frac{1}{2}x = \frac{5}{4}$$
$$\left(x - \frac{1}{4}\right)^2 - \frac{1}{16} = \frac{5}{4}$$
$$\left(x - \frac{1}{4}\right)^2 = \frac{5}{4} + \frac{1}{16}$$
$$\left(x - \frac{1}{4}\right)^2 = \frac{21}{16}$$
$$\therefore x = \frac{1}{4} \pm \sqrt{\frac{21}{16}}$$

6) **Quadratic Equation**
We can use completing the square to derive the quadratic equation.

$$ax^2 + bx + c = 0$$
$$ax^2 + bx = -c$$
$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$
$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = -\frac{c}{a}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$
$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$
\begin{aligned}
x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

7) **Polynomial Long Division**
If we know one factor of a polynomial equation, in order to find out the other factor we perform a division. In this example we know that $x^2 - 9x - 10$ has a factor of $x + 1$. Therefore

$$
\begin{array}{rrr}
 & & x - 10 \\
x + 1 \,) & x^2 & -9x - 10 \\
-) & x^2 & +x \\
\hline
 & & -10x - 10 \\
-) & & -10x - 10 \\
\hline
 & & 0 \quad 0
\end{array}
$$

Thus, we find the other factor to be

$$x - 10$$

In order to confirm this is correct we can multiply this factor by the known factor to find the original polynomial.

$$
\begin{aligned}
(x - 10)(x + 1) &= x^2 + x - 10x - 10 \\
&= x^2 - 9x - 10
\end{aligned}
$$

8) **Area of a Triangle in Vector Form**
When a triangle is defined with two sides $|\boldsymbol{p}|$ and $|\boldsymbol{q}|$ and the angle between these two sides is θ, the area of triangle is

$$\frac{1}{2}|\boldsymbol{p}| \cdot |\boldsymbol{q}| \cdot \sin\theta$$

9) **Inequalities**

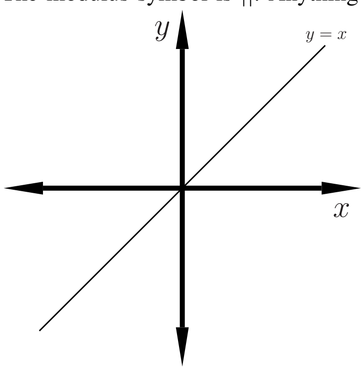
Symbol	Meaning
$<$	is less than
$>$	is greater than
$\leq$	is less than or equal to
$\geq$	is greater than or equal to

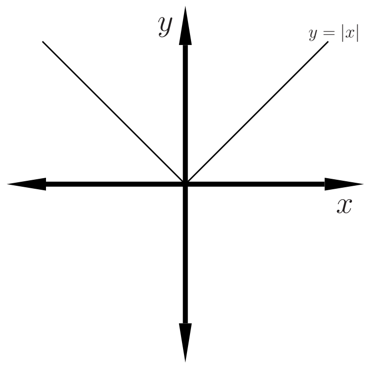
The one rule for inequalities is if you multiply or divide by a negative number the inequality sign is reversed as follows

$$
\begin{aligned}
-ax + c &\leq d \\
-ax &\leq d - c \\
x &\geq -\frac{(d - c)}{a}
\end{aligned}
$$

$$
\begin{aligned}
\frac{x}{-e} - f &> g \\
\frac{x}{-e} &> g + f \\
x &< -e(g + f)
\end{aligned}
$$

10) **Modulus**
The modulus symbol is $||$. Anything that is enclosed within this can not evaluate to a negative number. For example $|-4 + 2| = 2$.



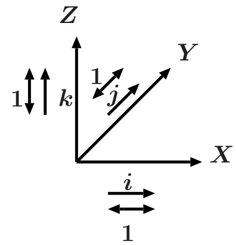


II. Key points on vectors

Key Points

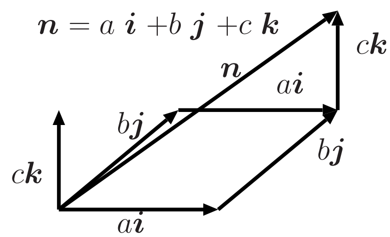
$\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are unit vector in x, y, and z directions respectively. $\jmath$ is $\sqrt{-1}$.

1) A vector has a x component, y component, and z component
   - A vector is expressed as $\boldsymbol{i}$ when it has only a x component and its modulus is 1.
   - A vector is expressed as $\boldsymbol{j}$ when it has only a y component and its modulus is 1.
   - A vector is expressed as $\boldsymbol{k}$ when it has only a z component and its modulus is 1.



2) When a vector has an amount of a in x component, an amount of b in y component, and an amount of c in z component, the vector can be expressed as

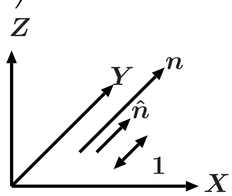
$$\begin{aligned} \boldsymbol{n} &= a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} \\ &\equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix} \end{aligned} \tag{1}$$



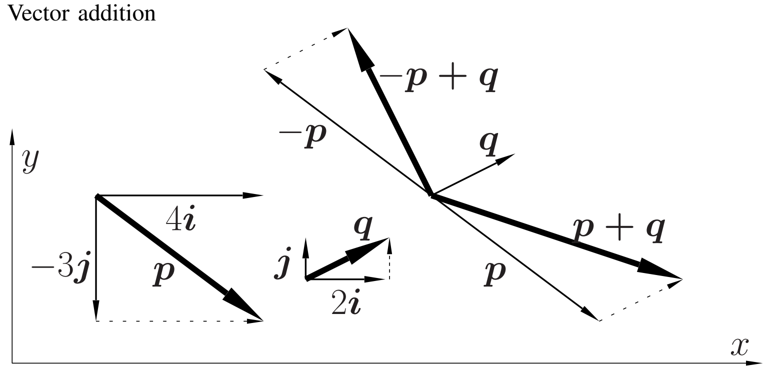
3) A unit vector can be found by dividing a vector by its modulus.

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{|\boldsymbol{n}|} \tag{2}$$

where $|\boldsymbol{n}|$ is $\sqrt{a^2+b^2+c^2}$ when $\boldsymbol{n} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} \equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.



4) Vector addition



When there are two vectors

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and

$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

the addition of the vectors is

$$\boldsymbol{a}+\boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix} \tag{3}$$

5) The position vector of P with coordinates (a, b, c) is

$$\overrightarrow{OP} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} \tag{4}$$

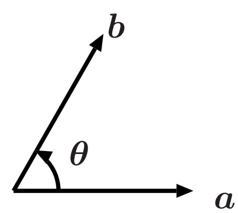
6) When there are two vectors

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and

$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

and these two vectors subtend an angle θ,



the scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1\cdot b_1 + a_2\cdot b_2 + a_3\cdot b_3 = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta \tag{5}$$

7) When there are two vectors

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and

$$\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

and these two vectors subtend an angle θ, the vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{a}\times\boldsymbol{b} &= \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\boldsymbol{i} \\ &+ \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}\boldsymbol{j} \\ &+ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\boldsymbol{k} \\ &= (a_2b_3 - a_3b_2)\boldsymbol{i} \\ &+(a_3b_1 - a_1b_3)\boldsymbol{j} \\ &+(a_1b_2 - a_2b_1)\boldsymbol{k} \\ &= |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\hat{\boldsymbol{n}} \end{aligned} \tag{6}$$

where $\hat{\boldsymbol{n}}$ is a unit vector and the direction of $\hat{\boldsymbol{n}}$ is the same as $\boldsymbol{a}\times\boldsymbol{b}$ in Fig. 1.

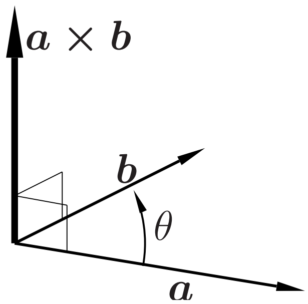


Fig. 1. $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$

8) The vector equation of the line which goes through a point A and is parallel to a vector c is

$$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{c} \tag{7}$$

where t is the real number. Please note that 'x','y','z' are not involved in the vector equation. The cartesian form of Equation (7) is obtained as follows:

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} &= t \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
\therefore \begin{pmatrix} x - a_1 \\ y - a_2 \\ z - a_3 \end{pmatrix} &= t \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}
\end{aligned}$$

This can be expressed in the scalar manner as

$$\begin{aligned}
x - a_1 &= tc_1 \\
\therefore \frac{x - a_1}{c_1} &= t \\
y - a_2 &= tc_2 \\
\therefore \frac{y - a_2}{c_2} &= t \\
z - a_3 &= tc_3 \\
\therefore \frac{z - a_3}{c_3} &= t
\end{aligned}$$

By getting rid of t in these three equations, we get the cartesian equation:

$$\frac{x - a_1}{c_1} = \frac{y - a_2}{c_2} = \frac{z - a_3}{c_3} \tag{8}$$

9) The vector equation of the line through points A and B with position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ is

$$\boldsymbol{r} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}) \tag{9}$$

where t is the real number. Please note that 'x','y','z' are not involved in the vector equation. When $0 \leq t \leq 1$, then $\boldsymbol{r}$ is in-between A and B. The cartesian form of Equation (9) is obtained as follows:

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \left(\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right) \\
\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix} \\
\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} &= t \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix} \\
\therefore \begin{pmatrix} x - a_1 \\ y - a_2 \\ z - a_3 \end{pmatrix} &= t \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}
\end{aligned}$$

This can be expressed in the scalar manner as

$$\begin{aligned}
x - a_1 &= t(b_1 - a_1) \\
\therefore \frac{x - a_1}{b_1 - a_1} &= t \\
y - a_2 &= t(b_2 - a_2) \\
\therefore \frac{y - a_2}{b_2 - a_2} &= t \\
z - a_3 &= t(b_3 - a_3)
\end{aligned}$$

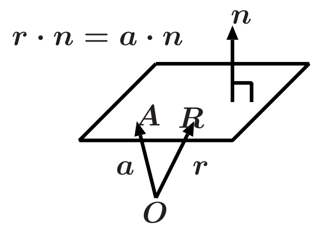
$$\therefore \frac{z-a_3}{b_3-a_3} = t$$

By getting rid of t in these three equations, we get the cartesian equation:

$$\frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3} \tag{10}$$

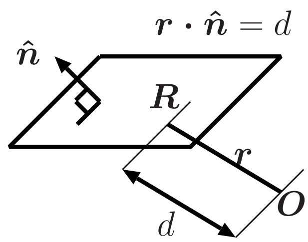
10) A plane perpendicular to the vector $\boldsymbol{n}$ and passing through the point with position vector $\boldsymbol{a}$, has equation

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n} \tag{11}$$



11) A plane with unit normal $\hat{\boldsymbol{n}}$, which has a perpendicular distance d from the origin is given by

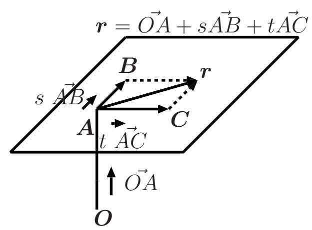
$$\boldsymbol{r} \cdot \hat{\boldsymbol{n}} = d \tag{12}$$



12) A plane which goes through $A(\boldsymbol{a})$, $B(\boldsymbol{b})$ and $C(\boldsymbol{c})$ is given by

$$\boldsymbol{r} = \overrightarrow{OA} + s\overrightarrow{AB} + t\overrightarrow{AC} \tag{13}$$

If the point $R(\boldsymbol{r})$ is inside of the triangle ABC then $0 \leq s$, $0 \leq t$, and $s + t \leq 1$.

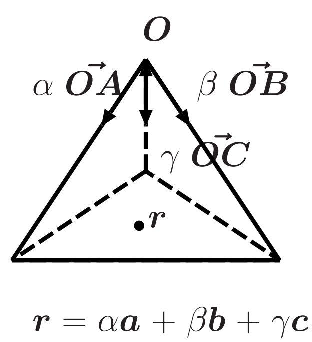


13) A point $R(\boldsymbol{r})$ which is inside the tetrahedron O, $A(\boldsymbol{a})$, $B(\boldsymbol{b})$ and $C(\boldsymbol{c})$ is given by

$$\boldsymbol{r} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c} \tag{14}$$

where α, β, γ are real numbers and satisfy

$$\alpha + \beta + \gamma < 1,\ 0 < \alpha,\ 0 < \beta,\ 0 < \gamma \tag{15}$$



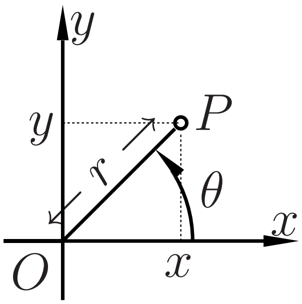


Fig. 2. The relashionship between polar and Cartesian coordinates

III. Key points on coordinates

Key Points

1) If the Cartesian coordinates of a point P are (x, y) then P can be located on a Cartesian plane as indicated in Fig. 2. r is the distance of P from the origin and θ is the angle, measured anti-clockwise, which the line OP makes when measured from the positive $x-$direction. If (x, y) are the Cartesian coordinates and $[r, \theta]$ the polar coordinates of a point P, then

$$x = r\cos\theta, \quad y = r\sin\theta \tag{16}$$

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = y/x \tag{17}$$

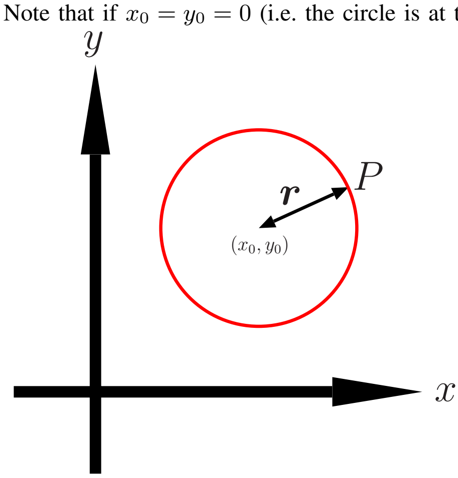
2) If the Cartesian coordinates (x, y) are any point P on a circle of radius r whose centre is at the origin. Then since $\sqrt{x^2 + y^2}$ is the distance of P from the origin, the equation of the circle is,

$$r = \sqrt{x^2 + y^2}, \quad x^2 + y^2 = r^2 \tag{18}$$

3) If the Cartesian coordinates (x, y) are any point P on a circle of radius r whose centre is (x_0, y_0). Then since $\sqrt{(x - x_o)^2 + (y - y_0)^2}$ is the distance of P from the origin, the equation of the circle is,

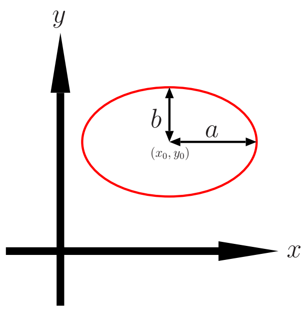
$$r = \sqrt{(x - x_o)^2 + (y - y_0)^2}, \quad (x - x_o)^2 + (y - y_0)^2 = r^2 \tag{19}$$

Note that if $x_0 = y_0 = 0$ (i.e. the circle is at the origin) then Equation (19) reduces to Equation (18).



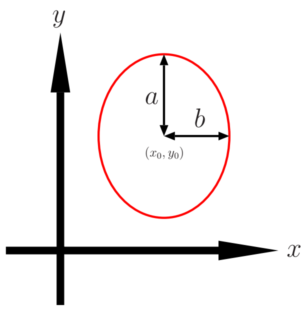
4) An ellipse with centre (x_0, y_0) satifies the equation

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \tag{20}$$



or

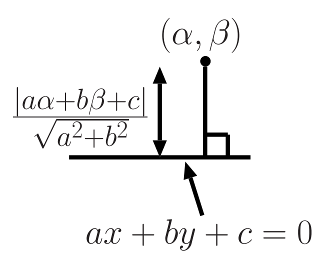
$$\frac{(x-x_0)^2}{b^2} + \frac{(y-y_0)^2}{a^2} = 1 \tag{21}$$



The parameter b is called the semiminor axis by analogy with the parameter a, which is called the semimajor axis (assuming $a > b$). When the major axis is horizontal use Equation (20). If on the other hand the major axis is vertical use Equation (21).

5) The minimum distance between a point $Q(\alpha, \beta)$ and a line $ax + by + c = 0$ is expressed as

$$\frac{|a\alpha + b\beta + c|}{\sqrt{a^2 + b^2}} \tag{22}$$



Proof: The line $ax + by + c = 0$ goes through the point $R(\boldsymbol{r})$ where

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -\frac{c}{b} \end{pmatrix}$$

and it is parallel to

$$\boldsymbol{l} = \begin{pmatrix} b \\ -a \end{pmatrix}$$

A point $P(\boldsymbol{p})$ on the line can be written as

$$\boldsymbol{p} = \boldsymbol{r} + t\boldsymbol{l}$$

where t is a real value. Since

$$\overrightarrow{QP} \perp \boldsymbol{l}$$

we can express this as the following equation:

$$\begin{aligned} \overrightarrow{QP} \cdot \boldsymbol{l} \\ = (\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{l} \\ = (\boldsymbol{r} + t\boldsymbol{l} - qv) \cdot \boldsymbol{l} \\ = (\boldsymbol{r} - \boldsymbol{q}) \cdot \boldsymbol{l} + t|\boldsymbol{l}|^2 = 0 \\ \therefore t = \frac{(\boldsymbol{q} - \boldsymbol{r}) \cdot \boldsymbol{l}}{|\boldsymbol{l}|^2} \end{aligned}$$

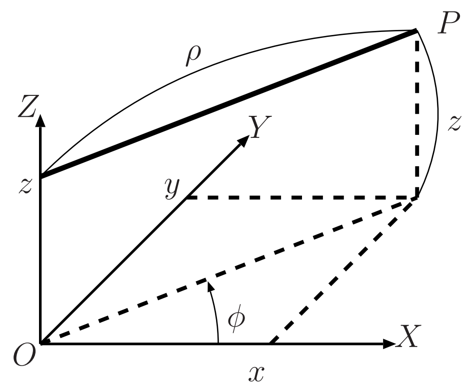


Fig. 3. The relashionship between Cylindrical and Cartesian coordinates

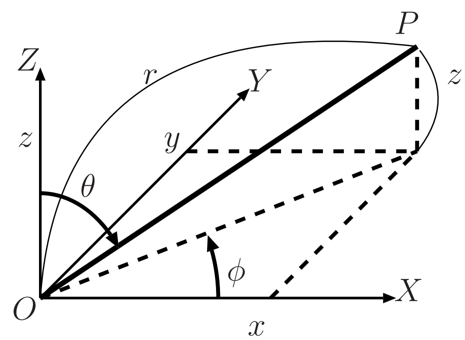


Fig. 4. The relashionship between Spherical and Cartesian coordinates

Now we need to get $\overrightarrow{QP}$ as follows:

$$
\begin{aligned}
|\overrightarrow{QP}|^2 &= |\boldsymbol{p}-\boldsymbol{q}|^2 \\
&= |\boldsymbol{r}+t\boldsymbol{l}-\boldsymbol{q}|^2 \\
&= |\boldsymbol{r}|^2+|\boldsymbol{q}|^2+t^2|\boldsymbol{l}|^2+2t\boldsymbol{r}\boldsymbol{l}-2t\boldsymbol{l}\boldsymbol{q}-2\boldsymbol{r}\boldsymbol{q} \\
&= |\boldsymbol{r}|^2+|\boldsymbol{q}|^2+\frac{((\boldsymbol{q}-\boldsymbol{r})\cdot\boldsymbol{l})^2}{|\boldsymbol{l}|^4}\cdot|\boldsymbol{l}|^2+2\frac{(\boldsymbol{q}-\boldsymbol{r})\cdot\boldsymbol{l}}{|\boldsymbol{l}|^2}(\boldsymbol{r}\boldsymbol{l}-\boldsymbol{l}\boldsymbol{q})-2\boldsymbol{r}\boldsymbol{q} \\
&= |\boldsymbol{r}|^2+|\boldsymbol{q}|^2+\frac{((\boldsymbol{q}-\boldsymbol{r})\cdot\boldsymbol{l})^2}{|\boldsymbol{l}|^2}-2\frac{(\boldsymbol{q}-\boldsymbol{r})\cdot\boldsymbol{l}}{|\boldsymbol{l}|^2}(\boldsymbol{q}-\boldsymbol{r})\boldsymbol{l}-2\boldsymbol{r}\boldsymbol{q} \\
&= |\boldsymbol{r}|^2+|\boldsymbol{q}|^2+\frac{((\boldsymbol{q}-\boldsymbol{r})\cdot\boldsymbol{l})^2}{|\boldsymbol{l}|^2}-2\frac{((\boldsymbol{q}-\boldsymbol{r})\boldsymbol{l})^2}{|\boldsymbol{l}|^2}-2\boldsymbol{r}\boldsymbol{q} \\
&= |\boldsymbol{r}|^2+|\boldsymbol{q}|^2-\frac{((\boldsymbol{q}-\boldsymbol{r})\cdot\boldsymbol{l})^2}{|\boldsymbol{l}|^2}-2\boldsymbol{r}\boldsymbol{q} \\
&= \frac{|a\alpha+b\beta+c|^2}{a^2+b^2} \\
\therefore |\overrightarrow{QP}| &= \frac{|a\alpha+b\beta+c|}{\sqrt{a^2+b^2}}
\end{aligned}
$$

6) 3D Cylindrical polar coordinate (ρ,ϕ,z) in Fig. 3 can be obtained from

$$\rho=\sqrt{x^2+y^2};\phi=\tan^{-1}\left(\frac{y}{x}\right) \tag{23}$$

$$\because x=\rho\cos\phi, y=\rho\sin\phi \tag{24}$$

You need to draw a diagram to determine the correct ϕ

7) 3D Spherical polar coordinate (r,θ,ϕ) in Fig. 4 can be obtained from

$$r=\sqrt{x^2+y^2+z^2};\theta=\cos^{-1}\left(\frac{z}{r}\right);\phi=\tan^{-1}\left(\frac{y}{x}\right) \tag{25}$$

$$\because x=r\sin\theta\cos\phi; y=r\sin\theta\sin\phi; z=r\cos\theta \tag{26}$$

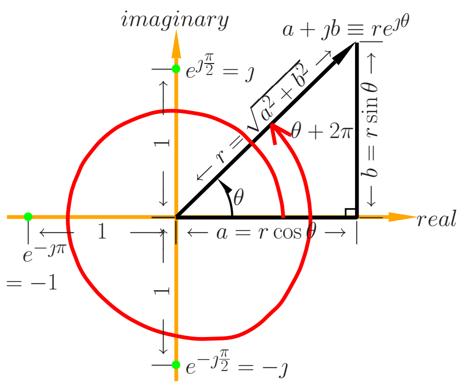
You need to draw a diagram to determine the correct ϕ. θ should satisfy $0\le\theta\le\pi$ without a diagram

IV. Key points on complex numbers

Key Points

1) The symbol $\jmath$ is such that

$$\jmath^2 = -1 \qquad \jmath = \sqrt{-1} \tag{27}$$



2) In Argand diagram, the complex number $a + \jmath b$ can be expressed as

$$a + \jmath b = re^{\jmath\theta} = r(\cos\theta + \jmath\sin\theta) \tag{28}$$

where

$$r = |a + \jmath b| = \sqrt{a^2 + b^2} \qquad \tan\theta = \frac{b}{a} \tag{29}$$

$$a = r\cos\theta \qquad b = r\sin\theta \tag{30}$$

Be careful: $a^2 - b^2 + 2ab\jmath = (a + \jmath b)^2 \neq |a + \jmath b|^2 = a^2 + b^2$

3) From the figure, $\pm\jmath$ can be expressed as

$$\jmath = \mathfrak{e}^{\frac{\pi}{2}\jmath}, -\jmath = \mathfrak{e}^{-\frac{\pi}{2}\jmath} \tag{31}$$

4) If $a + \jmath b$ is any complex number then its complex conjugate is

$$a - \jmath b \tag{32}$$

5) In the Argand diagram, the argument can be $2\pi n$ rotated to have an identical value:

$$\mathfrak{e}^{\jmath\theta} = \mathfrak{e}^{\jmath(\theta + 2\pi n)} \tag{33}$$

where n is an integer.

6) De Moivre's theorem

$$(r\mathfrak{e}^{\jmath\theta})^n = [r(\cos\theta + \jmath\sin\theta)]^n = r^n(\cos n\theta + \jmath\sin n\theta) = r^n\mathfrak{e}^{\jmath n\theta} \tag{34}$$

7) nth roots of complex numbers

If

$$z^n = r\mathfrak{e}^{\jmath\theta} = r(\cos\theta + \jmath\sin\theta)$$

then

$$z = \sqrt[n]{r}\mathfrak{e}^{\jmath(\theta + 2k\pi)/n} \qquad k = 0, \pm1, \pm2, \ldots \tag{35}$$

In other words, if

$$a\mathfrak{e}^{\jmath b} = c\mathfrak{e}^{\jmath d}$$

then

$$a = c$$
$$b = d + 2n\pi$$

8) If $a + \jmath b = c + \jmath d$, where a, b, c, and d, are real, then we can say

$$a = c, b = d \tag{36}$$

If $a + \jmath b = 0$, then $a = b = 0$

9) $\cosh x$ and $\sinh x$ are defined as

$$\cosh x = \frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}, \sinh x = \frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2} \tag{37}$$

$$x = \cosh^{-1}\left(\frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}\right), x = \sinh^{-1}\left(\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2}\right)$$

$$\tanh(x) = (\mathfrak{e}^x - \mathfrak{e}^{-x})/(\mathfrak{e}^x + \mathfrak{e}^{-x})$$
$$\cosh^2(A) - \sinh^2(A) = 1$$

When you need $\cosh^{-1}(\alpha)$ where α is a real number, using $x = \cosh^{-1}\left(\frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2}\right)$ we get

$$\begin{aligned}\frac{\mathfrak{e}^x + \mathfrak{e}^{-x}}{2} &= \alpha \\ \therefore \mathfrak{e}^x + \mathfrak{e}^{-x} &= 2\alpha \\ \therefore \mathfrak{e}^{2x} + 1 &= 2\alpha\mathfrak{e}^x \\ \therefore \mathfrak{e}^{2x} - 2\alpha\mathfrak{e}^x + 1 &= 0 \\ \therefore \mathfrak{e}^x &= \alpha \pm \sqrt{\alpha^2 - 1} \\ \therefore x = \cosh^{-1}(\alpha) &= \ln(\alpha \pm \sqrt{\alpha^2 - 1})\end{aligned}$$

When you need $\sinh^{-1}(\alpha)$ where α is a real number, using $x = \sinh^{-1}\left(\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2}\right)$ we get

$$\begin{aligned}\frac{\mathfrak{e}^x - \mathfrak{e}^{-x}}{2} &= \alpha \\ \therefore \mathfrak{e}^x - \mathfrak{e}^{-x} &= 2\alpha \\ \therefore \mathfrak{e}^{2x} - 1 &= 2\alpha\mathfrak{e}^x \\ \therefore \mathfrak{e}^{2x} - 2\alpha\mathfrak{e}^x - 1 &= 0 \\ \therefore \mathfrak{e}^x &= \alpha \pm \sqrt{\alpha^2 + 1} \\ \therefore x &= \ln(\alpha \pm \sqrt{\alpha^2 + 1}) \\ \therefore x = \sinh^{-1}(\alpha) &= \ln(\alpha + \sqrt{\alpha^2 + 1})(\because A > 0 \text{ for } \ln A)\end{aligned}$$

10) $\cos\theta$ and $\sin\theta$ are defined as

$$\cos\theta = \frac{\mathfrak{e}^{\jmath\theta} + \mathfrak{e}^{-\jmath\theta}}{2}, \sin\theta = \frac{\mathfrak{e}^{\jmath\theta} - \mathfrak{e}^{-\jmath\theta}}{2\jmath} \tag{38}$$

Proof: We know that

$$\mathfrak{e}^{\jmath\theta} = \cos\theta + \jmath\sin\theta \qquad ①$$

By replacing $\jmath$ in ① with $-\jmath$ we get

$$\mathfrak{e}^{-\jmath\theta} = \cos\theta - \jmath\sin\theta \qquad ②$$

① + ② gives us

$$\begin{aligned}\mathfrak{e}^{\jmath\theta} + \mathfrak{e}^{-\jmath\theta} &= 2\cos\theta \\ \therefore \frac{\mathfrak{e}^{\jmath\theta} + \mathfrak{e}^{-\jmath\theta}}{2} &= \cos\theta\end{aligned}$$

① - ② gives us

$$\begin{aligned}\mathfrak{e}^{\jmath\theta} - \mathfrak{e}^{-\jmath\theta} &= 2\jmath\sin\theta \\ \therefore \frac{\mathfrak{e}^{\jmath\theta} - \mathfrak{e}^{-\jmath\theta}}{2\jmath} &= \sin\theta\end{aligned}$$

V. Key points on differentiation

Key points

1) Product rule

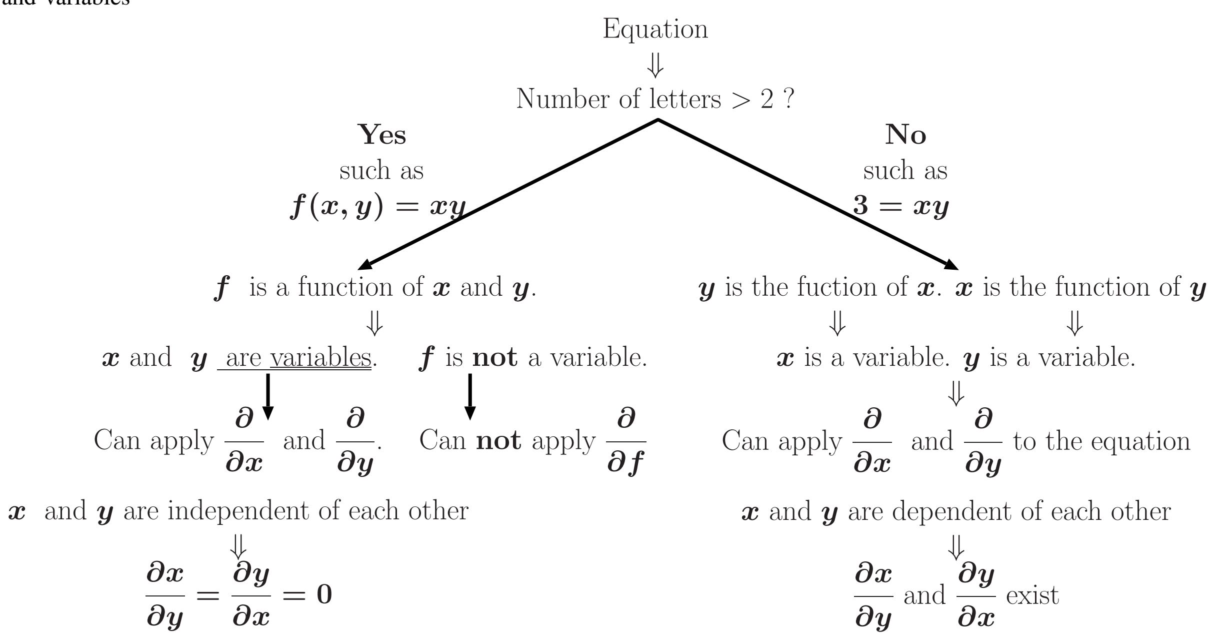
$$\frac{\partial\left\{f(x)g(x)\right\}}{\partial x}=f(x)\frac{\partial\left\{g(x)\right\}}{\partial x}+\frac{\partial\left\{f(x)\right\}}{\partial x}g(x) \tag{39}$$

2) Chain rule
   a) When $y=f(u)$ and $u=g(x)$,

$$\frac{\partial\left\{y\right\}}{\partial x}=\frac{\partial\left\{u\right\}}{\partial x}\cdot\frac{\partial\left\{y\right\}}{\partial u} \tag{40}$$

   It is important that you know the fundamental differentiable functions of Equation (46) $\sim$ Equation (54) so that a complicated function can be simplified to one of the fundamental functions of Equation (46) $\sim$ Equation (54). For example, if you know that 5^x can be differentiable, you can change $\dfrac{\partial\left\{5^{x^4-2}\right\}}{\partial x}$ to $\dfrac{\partial\left\{5^X\right\}}{\partial x}$ where $X=x^4-2$.

   b) Function and variables



   c) When W is a function of x, y and z and x, y, z are the function of s and t, $\dfrac{\partial\left\{W\right\}}{\partial t}$ and $\dfrac{\partial\left\{W\right\}}{\partial s}$ can not be directly calculated but can be calculated as follows:

$$\frac{\partial\left\{W\right\}}{\partial t}=\frac{\partial\left\{W\right\}}{\partial x}\cdot\frac{\partial\left\{x\right\}}{\partial t}+\frac{\partial\left\{W\right\}}{\partial y}\cdot\frac{\partial\left\{y\right\}}{\partial t}+\frac{\partial\left\{W\right\}}{\partial z}\cdot\frac{\partial\left\{z\right\}}{\partial t}$$

$$\frac{\partial\left\{W\right\}}{\partial s}=\frac{\partial\left\{W\right\}}{\partial x}\cdot\frac{\partial\left\{x\right\}}{\partial s}+\frac{\partial\left\{W\right\}}{\partial y}\cdot\frac{\partial\left\{y\right\}}{\partial s}+\frac{\partial\left\{W\right\}}{\partial z}\cdot\frac{\partial\left\{z\right\}}{\partial s}$$

   d) When W is a function of x, y and z, the total differential dW can be obtained by

$$dW=\frac{\partial\left\{W\right\}}{\partial x}dx+\frac{\partial\left\{W\right\}}{\partial y}dy+\frac{\partial\left\{W\right\}}{\partial z}dz$$

   e) When W is a function of x, y and z, the gradient ∇W is defined as

$$\nabla W=\frac{\partial\left\{W\right\}}{\partial x}\boldsymbol{i}+\frac{\partial\left\{W\right\}}{\partial y}\boldsymbol{j}+\frac{\partial\left\{W\right\}}{\partial y}\boldsymbol{k}=\begin{pmatrix}\dfrac{\partial\left\{W\right\}}{\partial x}\\[2ex]\dfrac{\partial\left\{W\right\}}{\partial y}\\[2ex]\dfrac{\partial\left\{W\right\}}{\partial z}\end{pmatrix}$$

3) Quotient rule

$$\frac{\partial\left\{\dfrac{f(x)}{g(x)}\right\}}{\partial x}=\frac{\dfrac{\partial\left\{f(x)\right\}}{\partial x}g(x)-f(x)\dfrac{\partial\left\{g(x)\right\}}{\partial x}}{(g(x))^2} \tag{41}$$

   Check if $g(x)$ is really a function. If $g(x)$ is a constant, you do not have to use the quatient rule. If $f(x)$ and $g(x)$ are polynomial, check the order of $f(x)$ and $g(x)$. If the order of $f(x)$ is higher than that of $g(x)$ then modify $\dfrac{f(x)}{g(x)}$ so that the order of the numerator of the resultant function is always lower than the order of denominator.

4) When x and y are the function of t,

$$\frac{\partial\left\{y\right\}}{\partial x}=\frac{\partial\left\{y\right\}}{\partial t}\cdot\frac{\partial\left\{t\right\}}{\partial x}=\frac{\partial\left\{y\right\}}{\partial t}\cdot\left(\frac{\partial\left\{x\right\}}{\partial t}\right)^{-1}$$

and

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial \left\{ \dfrac{\partial \{y\}}{\partial x} \right\}}{\partial x} = \frac{\partial \{t\}}{\partial x} \frac{\partial \left\{ \dfrac{\partial \{y\}}{\partial x} \right\}}{\partial t} = \left(\frac{\partial \{x\}}{\partial t} \right)^{-1} \frac{\partial \left\{ \dfrac{\partial \{y\}}{\partial x} \right\}}{\partial t}$$

5) Let $F(x)$ and $G(y)$ the function of x and y, respectively.
   a) $\dfrac{\partial \{y\}}{\partial x}$ for $F(x) + G(y) = 0$ is obtained as

$$\begin{aligned} F(x) + G(y) &= 0 \\ \therefore \frac{\partial \{F(x) + G(y)\}}{\partial x} &= \frac{\partial \{0\}}{\partial x} \\ \therefore \frac{\partial \{F(x)\}}{\partial x} + \frac{\partial \{G(y)\}}{\partial x} &= 0 \\ \therefore \frac{\partial \{F(x)\}}{\partial x} + \frac{\partial \{y\}}{\partial x} \frac{\partial \{G(y)\}}{\partial y} &= 0 \\ \therefore \frac{\partial \{y\}}{\partial x} &= -\frac{\dfrac{\partial \{F(x)\}}{\partial x}}{\dfrac{\partial \{G(y)\}}{\partial y}} \end{aligned}$$

   b) $\dfrac{\partial \{y\}}{\partial x}$ for $F(x) \cdot G(y) = 0$ is obtained as

$$\begin{aligned} F(x) \cdot G(y) &= 0 \\ \therefore \frac{\partial \{F(x) \cdot G(y)\}}{\partial x} &= \frac{\partial \{0\}}{\partial x} \\ \therefore \frac{\partial \{F(x)\}}{\partial x} G(y) + F(x) \frac{\partial \{G(y)\}}{\partial x} &= 0 \\ \therefore \frac{\partial \{F(x)\}}{\partial x} G(y) + F(x) \cdot \frac{\partial \{y\}}{\partial x} \frac{\partial \{G(y)\}}{\partial y} &= 0 \\ \therefore \frac{\partial \{y\}}{\partial x} &= -\frac{\dfrac{\partial \{F(x)\}}{\partial x} G(y)}{F(x) \dfrac{\partial \{G(y)\}}{\partial y}} \end{aligned}$$

6) When a graph has a local minimum and local maximum at (x_m, y_m), $\dfrac{\partial \{y\}}{\partial x}|_{(x,y)=(x_m,y_m)} = 0$. Furthermore, if $\dfrac{\partial^2 y}{\partial x^2}|_{(x,y)=(x_m,y_m)} > 0$, then (x_m, y_m) is the local minimum point. If $\dfrac{\partial^2 y}{\partial x^2}|_{(x,y)=(x_m,y_m)} < 0$, then (x_m, y_m) is the local maximum point.
7) L'Hôpital's Rule
   Let's assume we have a function of

$$y = f(x) = \frac{P(x)}{Q(x)}.$$

   If we want $\lim\limits_{x \to a} f(x)$ but we find out $P(a) = Q(a) = 0$ then we can still find $f(a)$ by

$$\lim_{x \to a} f(x) = \frac{P'(a)}{Q'(a)}.$$

   Proof: When we use Equation (83) we can write

$$P(a+h) = P(a) + h \left. \frac{\partial \{P\}}{\partial x} \right|_{x=a} + \cdots$$

   and

$$Q(a+h) = Q(a) + h \left. \frac{\partial \{Q\}}{\partial x} \right|_{x=a} + \cdots$$

   Then we can get the limit as

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \lim_{h \to 0} \frac{P(a+h)}{Q(a+h)} = \lim_{h \to 0} \frac{P(a) + h \left. \dfrac{\partial \{P\}}{\partial x} \right|_{x=a}}{Q(a) + h \left. \dfrac{\partial \{Q\}}{\partial x} \right|_{x=a}} = \lim_{h \to 0} \frac{0 + h \left. \dfrac{\partial \{P\}}{\partial x} \right|_{x=a}}{0 + h \left. \dfrac{\partial \{Q\}}{\partial x} \right|_{x=a}} = \lim_{h \to 0} \frac{h \left. \dfrac{\partial \{P\}}{\partial x} \right|_{x=a}}{h \left. \dfrac{\partial \{Q\}}{\partial x} \right|_{x=a}} = \frac{\left. \dfrac{\partial \{P\}}{\partial x} \right|_{x=a}}{\left. \dfrac{\partial \{Q\}}{\partial x} \right|_{x=a}}$$

8) Newton-Raphson method The crossing point between $y = f(x)$ and X axis can be estimated in an iterative manner as is shown in Fig. 5. The $(n+1)$th guess of the crossing point is obtained using nth guess as in Equation (42).

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{42}$$

9) Multivariable higher order differentiation

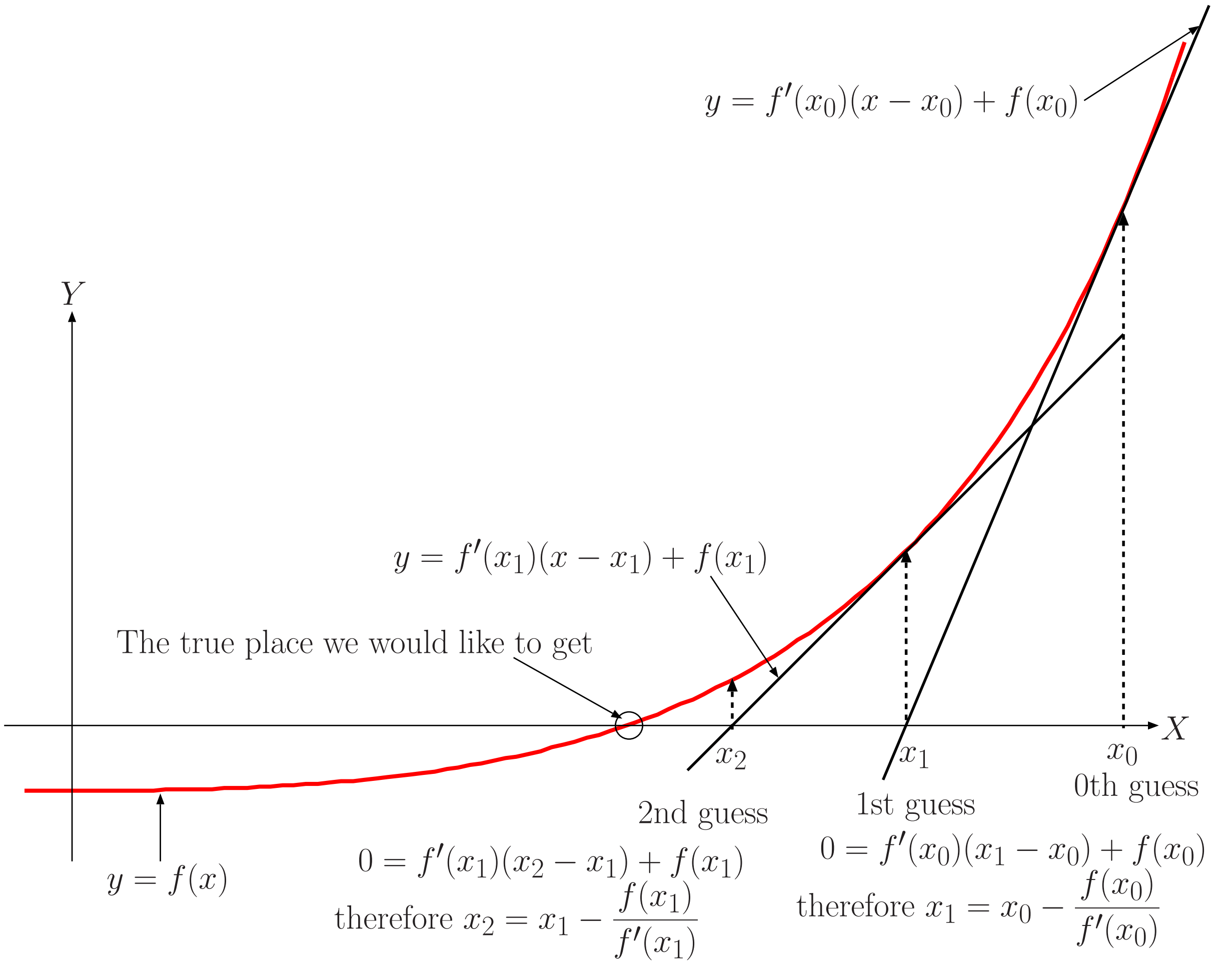


Fig. 5. Estimation of the crossing point between $y = f(x)$ and X axis.

$$\frac{\partial^2 f(x,y)}{\partial x^2} = \frac{\partial \left\{ \frac{\partial \{f(x,y)\}}{\partial x} \right\}}{\partial x} \tag{43}$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial \left\{ \frac{\partial \{f(x,y)\}}{\partial x} \right\}}{\partial y} \tag{44}$$

Please pay attention

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} \neq \frac{\partial \{f(x,y\}}{\partial y} \cdot \frac{\partial \{f(x,y)\}}{\partial x}.$$

Please also be aware the following difference: Let

$$f(x,y) = axy + bx + cy.$$

When we need $\frac{\partial \{f(x,y\}}{\partial x}$, then you assume x and y are independent and we obtain

$$\frac{\partial \{f(x,y)\}}{\partial x} = ay + b$$

but if we need $\frac{\partial \{y\}}{\partial x}$ for $f(x,y) = 0$, then $f(x,y) = 0$ tells you that x and y are dependent of each other and xy can be regarded as the multiplication of two function x and y and then we obtain

$$\begin{aligned}
\frac{\partial \{f(x,y)\}}{\partial x} &= \frac{\partial \{0\}}{\partial x} \\
\therefore \frac{\partial \{axy + bx + cy\}}{\partial x} &= 0 \\
\therefore a\frac{\partial \{x\}}{\partial x} y + ax\frac{\partial \{y\}}{\partial x} + b\frac{\partial \{x\}}{\partial x} + c\frac{\partial \{y\}}{\partial x} &= 0 \\
\therefore ay + ax\frac{\partial \{y\}}{\partial x} + b + c\frac{\partial \{y\}}{\partial x} &= 0 \\
\therefore (ax + c)\frac{\partial \{y\}}{\partial x} &= -ay - b
\end{aligned}$$

$$\therefore \frac{\partial \{y\}}{\partial x} = \frac{-ay-b}{ax+c}$$

10) Local minimum and local maximum
When $f(x,y)$ has a local minimum or a local maximum at $x=a$ and $y=b$, then $f(x,y)$ satisfies:

$$\left.\frac{\partial \{f(x,y)\}}{\partial x}\right|_{x=a,y=b} = 0, \qquad \left.\frac{\partial \{f(x,y)\}}{\partial y}\right|_{x=a,y=b} = 0 \tag{45}$$

This does NOT mean that if $\frac{\partial \{f(a,b)\}}{\partial x} = 0$, $\frac{\partial \{f(a,b)\}}{\partial y} = 0$, then $f(a,b)$ is a local minimum or a local maximum.
When $\frac{\partial \{f(a,b)\}}{\partial x} = 0$, $\frac{\partial \{f(a,b)\}}{\partial y} = 0$ is satisfied;

a) $f(a,b)$ is the local maximum when

$$\frac{\partial^2 f(a,b)}{\partial x^2}\frac{\partial^2 f(a,b)}{\partial y^2} - \left(\frac{\partial^2 f(a,b)}{\partial y \partial x}\right)^2 > 0$$

and $\frac{\partial^2 f(a,b)}{\partial x^2} < 0$

b) $f(a,b)$ is the local minimum when

$$\frac{\partial^2 f(a,b)}{\partial x^2}\frac{\partial^2 f(a,b)}{\partial y^2} - \left(\frac{\partial^2 f(a,b)}{\partial y \partial x}\right)^2 > 0$$

and $\frac{\partial^2 f(a,b)}{\partial x^2} > 0$

c) $f(a,b)$ is a saddle point when

$$\frac{\partial^2 f(a,b)}{\partial x^2}\frac{\partial^2 f(a,b)}{\partial y^2} - \left(\frac{\partial^2 f(a,b)}{\partial y \partial x}\right)^2 < 0$$

d) We do not know whether or not $f(a,b)$ is a local maximum or minimum when

$$\frac{\partial^2 f(a,b)}{\partial x^2}\frac{\partial^2 f(a,b)}{\partial y^2} - \left(\frac{\partial^2 f(a,b)}{\partial y \partial x}\right)^2 = 0$$

Attention: $\frac{\partial^2 f}{\partial y \partial x}$ is different from $\frac{\partial \{f\}}{\partial x} \cdot \frac{\partial \{f\}}{\partial y}$.

Basic derivative:

$$\frac{\partial \{x^\alpha\}}{\partial x} = \alpha x^{\alpha-1} \tag{46}$$

Attention: When you see a fraction, get rid of a fraction such as $\frac{1}{x^a}$ immediately by changing it to x^{-a}.

$$\frac{\partial \{x^a\}}{\partial x} = a \cdot x^{a-1} \tag{47}$$

$$\frac{\partial \{\mathfrak{e}^{kx}\}}{\partial x} = k\mathfrak{e}^{kx} \tag{48}$$

$$\frac{\partial \{\ln(kx)\}}{\partial x} = \frac{1}{x} \tag{49}$$

$$\frac{\partial \{\log_a(kx)\}}{\partial x} = \frac{1}{x \ln a} \tag{50}$$

$$\frac{\partial \{a^x\}}{\partial x} = a^x \ln a \tag{51}$$

$$\frac{\partial \{\sin kx\}}{\partial x} = k \cos kx \tag{52}$$

$$\frac{\partial \{\cos kx\}}{\partial x} = -k \sin kx \tag{53}$$

$$\frac{\partial \{\tan kx\}}{\partial x} = \frac{k}{\cos^2 kx} \tag{54}$$

VI. Key points on integration

Key points

1) Integral by Parts

$$\int_a^b f(x)\cdot g(x)dx = \left[f(x)\cdot\int g(x)dx\right]_a^b - \int_a^b\left(\frac{\partial\{f(x)\}}{\partial x}\cdot\int g(x)dx\right)dx \tag{55}$$

Hint: Let $f(x)$ equate the polynominal part or logarithmic part of the intergral.

$\int \sin^n x dx$ and $\int \cos^n x dx$ can be obtained using "Integral by Parts" in order to reduce the power as follows

$$\int \cos^n x dx = \int \cos^{n-1} x\cdot\cos x dx = f(x)\cdot\int g(x)dx - \int\left(\frac{\partial\{f(x)\}}{\partial x}\cdot\int g(x)dx\right)dx$$
$$= \cos^{n-1}x\cdot\int\cos x dx - \int\left(\frac{\partial\{\cos^{n-1}x\}}{\partial x}\cdot\int\cos x dx\right)dx = \cos^{n-1}x\cdot\sin x - \int\left((n-1)\cos^{n-2}x(-\sin x)\cdot\sin x\right)dx$$
$$= \cos^{n-1}x\cdot\sin x + (n-1)\int\left(\cos^{n-2}x\cdot\sin^2 x\right)dx = \cos^{n-1}x\cdot\sin x + (n-1)\int\left(\cos^{n-2}x\cdot(1-\cos^2 x)\right)dx$$
$$= \cos^{n-1}x\cdot\sin x + (n-1)\int\left(\cos^{n-2}x - \cos^n x\right)dx = \cos^{n-1}x\cdot\sin x + (n-1)\int\cos^{n-2}x dx - (n-1)\int\cos^n x dx$$
$$\therefore \int\cos^n x dx + (n-1)\int\cos^n x dx = \cos^{n-1}x\cdot\sin x + (n-1)\int\cos^{n-2}x dx$$
$$\therefore n\int\cos^n x dx = \cos^{n-1}x\cdot\sin x + (n-1)\int\cos^{n-2}x dx$$

$$\int \sin^n x dx = \int \sin^{n-1} x\cdot\sin x dx = f(x)\cdot\int g(x)dx - \int\left(\frac{\partial\{f(x)\}}{\partial x}\cdot\int g(x)dx\right)dx$$
$$= \sin^{n-1}x\cdot\int\sin x dx - \int\left(\frac{\partial\{\sin^{n-1}x\}}{\partial x}\cdot\int\sin x dx\right)dx = \sin^{n-1}x\cdot(-\cos x) - \int\left((n-1)\sin^{n-2}x(\cos x)\cdot(-\cos x)\right)dx$$
$$= -\sin^{n-1}x\cdot\cos x + \int\left((n-1)\sin^{n-2}x(\cos^2 x)\right)dx = -\sin^{n-1}x\cdot\cos x + (n-1)\int\left(\sin^{n-2}x(1-\sin^2 x)\right)dx$$
$$= -\sin^{n-1}x\cdot\cos x + (n-1)\int\left(\sin^{n-2}x - \sin^n x\right)dx = -\sin^{n-1}x\cdot\cos x + (n-1)\int\left(\sin^{n-2}x\right)dx - (n-1)\int\left(\sin^n x\right)dx$$
$$\therefore \int\sin^n x dx + (n-1)\int\left(\sin^n x\right)dx = -\sin^{n-1}x\cdot\cos x + (n-1)\int\left(\sin^{n-2}x\right)dx$$
$$\therefore n\int\left(\sin^n x\right)dx = -\sin^{n-1}x\cdot\cos x + (n-1)\int\left(\sin^{n-2}x\right)dx$$

2) Integral by substitution

When a function $f(x)$ can be written as $h(g(x))\frac{\partial\{g(x)\}}{\partial x}$, you can let $t = g(x)$ therefore, $\frac{\partial\{t\}}{\partial x} = \frac{\partial\{g(x)\}}{\partial x}$.

$$\int f(x)dx = \int h(g(x))\frac{\partial\{g(x)\}}{\partial x}dx = \int h(t)\frac{\partial\{t\}}{\partial x}dx = \int h(t)dt \tag{56}$$

- For $\int \sin^{2m+1} x dx$, set $t = \cos x$.
- For $\int \sin^{2m} x dx$, set $t = \sin x$.
- For $\int \cos^{2m+1} x dx$, set $t = \sin x$.
- For $\int \cos^{2m} x dx$, set $t = \cos x$.

where m is an integer. But in case of even power such as $2m$, it is better to decrease the power such as

$$\sin^4 x = \left(\sin^2 x\right)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1-2\cos 2x+\cos^2 2x}{4} = \frac{1-2\cos 2x}{4} + \frac{1}{4}\cos^2 2x$$
$$= \frac{1-2\cos 2x}{4} + \frac{1}{4}\frac{1+\cos 4x}{2} = \frac{2-4\cos 2x}{8} + \frac{1+\cos 4x}{8} = \frac{3-4\cos 2x+\cos 4x}{8}$$

If the power is higher than 4, then use "Integral by Parts" as shown above.

When we carry out $\int_{x_L}^{x_H} f(x)dx$, the procedure of 'integral by substitution' is as follows

a) set the new variable θ for substitution such as $x = \frac{\mathfrak{e}^{\theta} - \mathfrak{e}^{-\theta}}{2}$

b) find the relationship between dx and $d\theta$ such as $dx = \frac{\mathfrak{e}^{\theta} + \mathfrak{e}^{-\theta}}{2}d\theta$

c) find the range for the new variable θ

$$x_{\mathrm{L}} = \frac{e^{\theta} - e^{-\theta}}{2} \rightarrow \theta_{\mathrm{L}} = \ln(x_{\mathrm{L}} + \sqrt{x_{\mathrm{L}}^2 + 1})$$
$$x_{\mathrm{H}} = \frac{e^{\theta} - e^{-\theta}}{2} \rightarrow \theta_{\mathrm{H}} = \ln(x_{\mathrm{H}} + \sqrt{x_{\mathrm{H}}^2 + 1})$$

d) manipulate the original function $f(x)$ to remove x . $f(x) \Longrightarrow g(\theta)$

e) calculate the final modified integral such as $\int_{\theta_{\mathrm{L}}}^{\theta_{\mathrm{H}}} g(\theta)\frac{e^{\theta}+e^{-\theta}}{2}d\theta$

3) Integral of $f(x)^k \frac{\partial\{f(x)\}}{\partial x}$ for $k=-1$, *i.e.*, $\int \frac{f'(x)}{f(x)}dx$

$$\int \frac{1}{f(x)}\frac{\partial\{f(x)\}}{\partial x}dx = \ln|f(x)| + c \tag{57}$$

Proof:

$$\frac{\partial\{\ln|f(x)|\}}{\partial x} = \frac{\partial\{\ln|A|\}}{\partial x}(\because A \triangleq f(x)) = \frac{\partial\{A\}}{\partial x}\frac{\partial\{\ln|A|\}}{\partial A} = \frac{\partial\{f(x)\}}{\partial x}\frac{1}{A} = \frac{f'(x)}{f(x)}$$
$$\therefore \frac{f'(x)}{f(x)} = \frac{\partial\{\ln|f(x)|\}}{\partial x}$$
$$\therefore \int \frac{f'(x)}{f(x)}dx = \int \frac{\partial\{\ln|f(x)|\}}{\partial x}dx = \int \partial(\ln|f(x)|) = \ln|f(x)|$$

4) Integral of $f(x)^k \frac{\partial\{f(x)\}}{\partial x}$ for $k \neq -1$

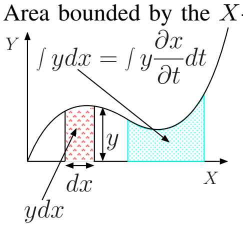
$$\int f(x)^k \cdot \frac{\partial\{f(x)\}}{\partial x}dx = \frac{1}{k+1}f(x)^{k+1} + c \tag{58}$$

5) $P(x)$ and $Q(x)$ are the mth and nth order polynomials, respectively.

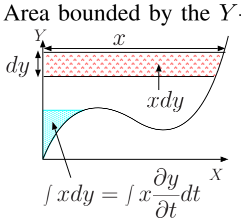
- When $m > n$, $\int \frac{P(x)}{Q(x)}dx$ can be obtained as follows:
  a) Find the answer of $A(x)$ and the remainer $R(x)$ of $\frac{P(x)}{Q(x)}$ which satisfy $P(x) = Q(x)A(x) + R(x)$
  b) Find the answer of C and the remainder of E of $\frac{R(x)}{Q'(x)}$ which satisfy $R(x) = C \cdot Q'(x) + E$
  c) $\int \frac{P(x)}{Q(x)}dx = \int \left(A(x) + C\frac{Q'(x)}{Q(x)} + \frac{E}{Q(x)}\right)dx = \int A(x)dx + C\ln|Q(x)| + \int \frac{E}{Q(x)}dx$
- When $m < n$, $\int \frac{P(x)}{Q(x)}dx$ can be obtained as follows:
  a) Find the answer of C and the remainder of E of $\frac{P(x)}{Q'(x)}$ which satisfy $P(x) = C \cdot Q'(x) + E$
  b) $\int \frac{P(x)}{Q(x)}dx = \int \left(C\frac{Q'(x)}{Q(x)} + \frac{E}{Q(x)}\right)dx = C\ln|Q(x)| + \int \frac{E}{Q(x)}dx$

6) Calculation of Area

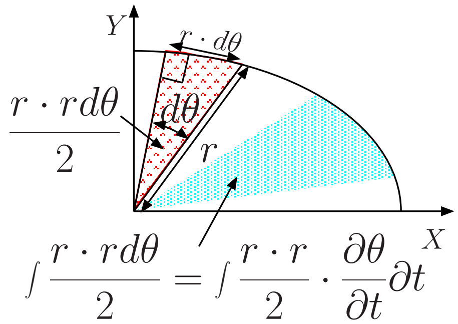
a) Area bounded by the X-axis



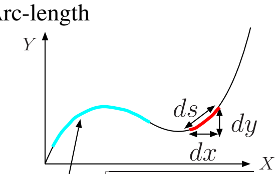
b) Area bounded by the Y-axis



c) Area in polar coordinates

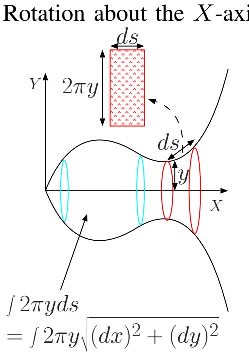


7) Arc-length

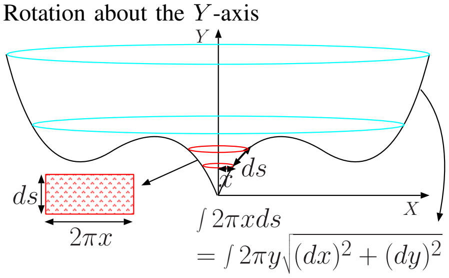


$$\int ds = \int \sqrt{(dx)^2 + (dy)^2}$$
$$= \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy$$
$$= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

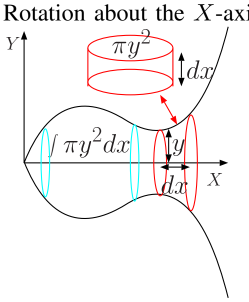
8) Surface area of solid of revolution
   a) Rotation about the X-axis



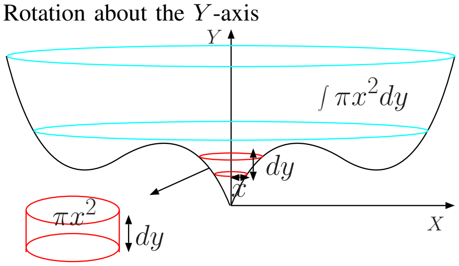
   b) Rotation about the Y-axis



9) Volume of solid of revolution
   a) Rotation about the X-axis



   b) Rotation about the Y-axis



10) Line integrals of a function which has dx,dy, and dz.

Consider a curve C. The position vector of a point on the curve C is written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$
$$a \le t \le b$$

Denote

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and its derivative with respect to t as

$$\frac{\partial \{\boldsymbol{r}\}}{\partial t} = \begin{pmatrix} \frac{\partial \{x\}}{\partial t} \\ \frac{\partial \{y\}}{\partial t} \\ \frac{\partial \{z\}}{\partial t} \end{pmatrix}.$$

When a vector function is expressed as

$$\mathbf{F}(\boldsymbol{r}) = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

a line integral of $\mathbf{F}(\boldsymbol{r})$ over a curve C is defined by

$$\int_C \mathbf{F} \cdot d\boldsymbol{r} = \int_{t=a}^{t=b} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \cdot \frac{\partial \{\boldsymbol{r}\}}{\partial t} dt$$
$$= \int_{t=a}^{t=b} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \{x\}}{\partial t} \\ \frac{\partial \{y\}}{\partial t} \\ \frac{\partial \{z\}}{\partial t} \end{pmatrix} dt$$
$$= \int_{t=a}^{t=b} (F_x \frac{\partial \{x\}}{\partial t} + F_y \frac{\partial \{y\}}{\partial t} + F_z \frac{\partial \{z\}}{\partial t}) dt \quad (59)$$
$$= \int (F_x dx + F_y dy + F_z dz) \quad (60)$$
$$= \int_{x=\hat{a}}^{x=\hat{b}} (F_x + F_y \frac{dy}{dx} + F_z \frac{dz}{dx}) dx \quad (61)$$

Thus the procedure to solve the line integral is

a) Express x, y, z using t
b) Express $\mathbf{F}$ as the function of t
c) Express $\frac{\partial \{\boldsymbol{r}\}}{\partial t} = \begin{pmatrix} \frac{\partial \{x\}}{\partial t} \\ \frac{\partial \{y\}}{\partial t} \\ \frac{\partial \{z\}}{\partial t} \end{pmatrix}$ using t
d) Put all of them into $\int \mathbf{F} \cdot \frac{\partial \{\boldsymbol{r}\}}{\partial t} dt$

11) Line integrals of a function (which does not have dx, dy or dz explicitly) with respect to arc length.
Consider a curve C. The position vector of a point on the curve C is written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$
$$a \le t \le b$$

Denoting

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and its derivative with respect to t as

$$\frac{\partial \{\boldsymbol{r}\}}{\partial t} = \begin{pmatrix} \frac{\partial \{x\}}{\partial t} \\ \frac{\partial \{y\}}{\partial t} \\ \frac{\partial \{z\}}{\partial t} \end{pmatrix}$$

the line integral of a function with respect to arc length is defined by

$$\int_C f(x,y,z)ds$$
$$= \int_{t=a}^{t=b} f(x,y,z)\sqrt{\left(\frac{\partial\{x\}}{\partial t}\right)^2+\left(\frac{\partial\{y\}}{\partial t}\right)^2+\left(\frac{\partial\{z\}}{\partial t}\right)^2}dt \tag{62}$$

where

$$ds = \sqrt{\left(\frac{\partial\{x\}}{\partial t}\right)^2+\left(\frac{\partial\{y\}}{\partial t}\right)^2+\left(\frac{\partial\{z\}}{\partial t}\right)^2}dt$$

The procedure to solve this type of the line integral is

a) Express x, y, z using t
b) Express $f(x,y,z)$ as the function of t
c) Express $\frac{\partial\{\boldsymbol{r}\}}{\partial t} = \begin{pmatrix} \frac{\partial\{x\}}{\partial t} \\ \frac{\partial\{y\}}{\partial t} \\ \frac{\partial\{z\}}{\partial t} \end{pmatrix}$ using t
d) Put all of them into

$$\int_{t=a}^{t=b} f(x,y,z)\sqrt{\left(\frac{\partial\{x\}}{\partial t}\right)^2+\left(\frac{\partial\{y\}}{\partial t}\right)^2+\left(\frac{\partial\{z\}}{\partial t}\right)^2}dt$$

12) Multiple integration

$$I = \int_a^b \int_c^d \int_e^f f(x,y,z)dxdydz$$

has the following range:

$$e \le x \le f$$
$$c \le y \le d$$
$$a \le z \le b$$

The procedure for the calculation is

a)

$$A = \int_e^f f(x,y,z)dx$$

b)

$$B = \int_a^b \int_c^d Ady$$

c)

$$I = \int_a^b Bdz$$

Please be aware that

$$\int_a^b \int_c^d \int_e^f fdxdydz \ne \int_a^b fdx \times \int_c^d fdy \times \int_e^f fdz$$

Integrals of common functions.

Some are very similar to the fundamental functions for differentiation. So please do not mix up!, especially signs such as + or −.

$$n \ne -1 \quad \text{and} \quad \int kx^n dx = \frac{1}{n+1}\cdot kx^{n+1}+c \tag{63}$$
$$n = -1 \quad \text{and} \quad \int kx^n dx = \int \frac{k}{x}dx = k\ln|x|+c \tag{64}$$
$$\int \cos kx dx = \frac{1}{k}\sin kx + c \tag{65}$$
$$\int \sin kx dx = -\frac{1}{k}\cos kx + c \tag{66}$$
$$\int \tan kx dx = -\frac{1}{k}\ln|\cos kx| + c \tag{67}$$
$$\int \mathfrak{e}^{kx}dx = \frac{1}{k}\mathfrak{e}^{kx}+c \tag{68}$$
$$\int a^{kx}dx = \frac{a^{kx}}{k\ln a}+c(a>0) \tag{69}$$
$$\int \cos^2(kx)dx = \frac{1}{2k}(kx+\sin(kx)\cos(kx)) \tag{70}$$
$$\int \frac{1}{\cos^2(kx)}dx = \frac{\tan kx}{k} \tag{71}$$

$$\int \frac{1}{\sin^2(kx)} dx = -\frac{1}{k \tan kx} \tag{72}$$

$$\int \sin^2(kx) dx = \frac{1}{2k}(kx - \sin(kx)\cos(kx)) \tag{73}$$

$$\int \ln kx dx = x \ln kx - x \tag{74}$$

$$\int \frac{dx}{\sqrt{x^2 - k^2}} = \cosh^{-1}\left(\frac{x}{k}\right) \tag{75}$$

$$\int \frac{dx}{\sqrt{x^2 + k^2}} = \sinh^{-1}\left(\frac{x}{k}\right) \tag{76}$$

$$\int \frac{dx}{\sqrt{k^2 - x^2}} = \sin^{-1}\left(\frac{x}{k}\right) \tag{77}$$

$$\int \frac{dx}{x^2 + k^2} = \frac{1}{k} \tan^{-1}\left(\frac{x}{k}\right) \tag{78}$$

VII. Key points on sequences and series

Key points

1) Sequences and Series

   a) Arithmetic progressions. Consider a sequence that starts at r and we add d each time. This forms the Arithemtic series as follows.

$$\begin{aligned} a_1 &= r \\ a_2 &= r + d \\ a_3 &= r + 2d \\ a_4 &= r + 3d \\ &\cdots \\ a_n &= r + (n-1)d \end{aligned}$$

   Here d is the difference or common difference between successive terms. The sum of an arthimetic progression is as follows.

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + a_4 + a_5 + a_n \\ S_n &= r + (r+d) + (r+2d) + \cdots + r + (n-1)d \end{aligned}$$

$$S_n = rn + \frac{n(n-1)d}{2} \tag{79}$$

   b) Geometric progressions. Suppose we let the first term equal a and times each successive term by r then we get.

$$\begin{aligned} a_1 &= a \\ a_2 &= ar \\ a_3 &= ar^2 \\ a_4 &= ar^3 \\ a_5 &= ar^4 \\ &\cdots \\ a_n &= ar^{n-1} \end{aligned}$$

   To find the sum of this progression to n terms, we sum all the terms up until n.

$$S_n = a + ar + ar^2 + ar^3 + ar^4, + \cdots + ar^{n-1}$$

   Since $r \cdot S_n$ is written as

$$rS_n = ar + ar^2 + ar^3 + ar^4, + \cdots + ar^{n-1} + ar^n$$

   Using these two equations, we calculate $S_n - rS_n$ as follows:

$$S_n - rS_n = a - ar^n$$

   This leads to :

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r} \tag{80}$$

   If $-1 < r < 1$ therefore the sum to infinity of an geomteric series is given by the following

$$S_\infty = \frac{a}{1 - r} \tag{81}$$

2) Taylor series with one variable.
   A Taylor series is a series expansion of a function about a point. A one-dimensional Taylor series is an expansion of a real function $f(x)$ about the point $x = a$ upto terms of degree n in h ($|h| \ll 1$) which is given by

$$\begin{aligned} f(x) = f(a) + (x-a)\left.\frac{\partial f}{\partial x}\right|_{x=a} + \frac{(x-a)^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=a} \\ + \frac{(x-a)^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=a} + \cdots + \frac{(x-a)^n}{n!}\left.\frac{\partial^n f}{\partial x^n}\right|_{x=a} \end{aligned} \tag{82}$$

   or by substituing $x = a + h$ into Equation (82) we get the following taylor polynomial of degree n:

$$\begin{aligned} f(a+h) = f(a) + h\left.\frac{\partial f}{\partial x}\right|_{x=a} + \frac{h^2}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=a} \\ + \frac{h^3}{3!}\left.\frac{\partial^3 f}{\partial x^3}\right|_{x=a} + \cdots + \frac{h^n}{n!}\left.\frac{\partial^n f}{\partial x^n}\right|_{x=a} \end{aligned} \tag{83}$$

   If $a = 0$, the expansion is known as a Maclaurin series.
   In the end, in order to obtain the taylor series

   a) Obtain $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, $\ldots$ $\frac{\partial^n f}{\partial x^n}$
   b) Substitute $x = a$ into $f(x)$, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x^2}$, $\ldots$ $\frac{\partial^n f}{\partial x^n}$
   c) Put all of them into Equation (83).

3) Taylor series with two variables.

The taylor series for two variables is very similar to that of one variable the same method is used to find the series. The Taylor series expansion about the point (a, b), where a and b are known constants, up to and including terms of degree three in h and k ($|h| \ll 1$ and $|k| \ll 1$) where, in the usual notation, $x = a + h$ and $y = b + k$ is expressed as

$$f(a+h, b+k) = \tag{84}$$

$$f(a,b) + h\frac{\partial\{f(x,y)\}}{\partial x}\bigg|_{\substack{x=a\\y=b}} + k\frac{\partial\{f(x,y)\}}{\partial y}\bigg|_{\substack{x=a\\y=b}}$$

$$+\frac{1}{2!}\left[h^2\frac{\partial^2 f(x,y)}{\partial x^2}\bigg|_{\substack{x=a\\y=b}} + 2hk\frac{\partial^2 f(x,y)}{\partial y\partial x}\bigg|_{\substack{x=a\\y=b}} + k^2\frac{\partial^2 f(x,y)}{\partial y^2}\bigg|_{\substack{x=a\\y=b}}\right]$$

$$+\frac{1}{3!}\left[h^3\frac{\partial^3 f(x,y)}{\partial x^3}\bigg|_{\substack{x=a\\y=b}} + 3h^2k\frac{\partial^3 f(x,y)}{\partial y\partial x^2}\bigg|_{\substack{x=a\\y=b}} + 3hk^2\frac{\partial^3 f(x,y)}{\partial y^2\partial x}\bigg|_{\substack{x=a\\y=b}} + k^3\frac{\partial^3 f(x,y)}{\partial y^3}\bigg|_{\substack{x=a\\y=b}}\right]$$

In the end, in order to obtain the taylor series

a) Obtain $\frac{\partial\{f(x,y)\}}{\partial x}, \frac{\partial\{f(x,y)\}}{\partial y}$ and if you need the second degree, then obtain $\frac{\partial^2 f(x,y)}{\partial x^2}, \frac{\partial^2 f(x,y)}{\partial y\partial x}, \frac{\partial^2 f(x,y)}{\partial y^2}$ as well, and if you need the third degree, then obtain $\frac{\partial^3 f(x,y)}{\partial x^3}, \frac{\partial^3 f(x,y)}{\partial y\partial x^2}, \frac{\partial^3 f(x,y)}{\partial y^2\partial x}, \frac{\partial^3 f(x,y)}{\partial y^3}$ as well.

b) Substitute $x = a$ and $y = b$ into $\frac{\partial\{f(x,y)\}}{\partial x}, \frac{\partial\{f(x,y)\}}{\partial y}, \frac{\partial^2 f(x,y)}{\partial x^2}, \frac{\partial^2 f(x,y)}{\partial y\partial x}, \frac{\partial^2 f(x,y)}{\partial y^2}, \frac{\partial^3 f(x,y)}{\partial x^3}, \frac{\partial^3 f(x,y)}{\partial y\partial x^2}, \frac{\partial^3 f(x,y)}{\partial y^2\partial x}, \frac{\partial^3 f(x,y)}{\partial y^3}$

c) Put all of them into Equation (84).

VIII. Key points on ordinary differential equations

Key points

1) The solution of the equation $\frac{\partial\{y\}}{\partial x} = f(x)g(y)$ may be found from separating the variables and integrating

$$\int \frac{1}{g(y)} dy = \int f(x) dx \tag{85}$$

Procedure:

a) Allocate $f(x)$ and $g(x)$

b) Calculate

$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

2) When f can be written as a function of $y/x \triangleq z$, the solution of the equation $\frac{\partial\{y\}}{\partial x} = f(y/x)$ may be found as

$$\int \frac{dz}{f(z) - z} = \int \frac{1}{x} dx = \ln x + c \tag{86}$$

Procedure:

a) Find $f(\frac{y}{x})$

b) Calculate

$$\int \frac{dz}{f(z) - z} \triangleq g(z)$$

c) Set $\ln(x) + c = g(z)$

d) Replace z with $\frac{y}{x}$ so that $\ln(x) + c = g(\frac{y}{x})$ is the answer

Proof: $y/x \triangleq z$ can be written as $y = zx$. Thus $\frac{\partial\{y\}}{\partial x} = \frac{\partial\{z\}}{\partial x}x + z\frac{\partial\{x\}}{\partial x} = x\frac{\partial\{z\}}{\partial x} + z$. Thus $\frac{\partial\{y\}}{\partial x} = f(y/x) = f(z)$ can be written as

$$\begin{aligned} x\frac{\partial\{z\}}{\partial x} + z &= f(z) \\ \therefore x\frac{\partial\{z\}}{\partial x} &= f(z) - z \\ \therefore \frac{1}{x}dx &= \frac{1}{f(z) - z}dz \\ \therefore \int \frac{1}{f(z) - z}dz &= \int \frac{1}{x}dx = \ln x + c \end{aligned}$$

3) When the differential equation can be written as $f(x, y)dx + g(x, y)dy = 0$ and if

$$\frac{\partial\{f(x, y)\}}{\partial y} = \frac{\partial\{g(x, y)\}}{\partial x}, \tag{87}$$

then there is a function $U(x, y)$ which satisfies

$$\begin{aligned} dU(x, y) &= \frac{\partial\{U(x, y)\}}{\partial x}dx + \frac{\partial\{U(x, y)\}}{\partial y}dy \\ &\equiv f(x, y)dx + g(x, y)dy = 0 \end{aligned} \tag{88}$$

$dU(x, y) = 0$ gives

$$U(x, y) = c \tag{89}$$

which is the answer. In order to find $U(x, y)$, we first perform

$$U(x, y) = \int f(x, y)dx + h(y) \tag{90}$$

then we find $h(y)$ from

$$\begin{aligned} \frac{\partial\{U(x, y)\}}{\partial y} &= \frac{\partial\left\{\int f(x, y)dx + h(y)\right\}}{\partial y} \\ &= g(x, y) \end{aligned} \tag{91}$$

The alternative approach to obtain $U(x, y)$ is

$$U(x, y) = \int_{x_0}^{x} f(x, y)dx + \int_{y_0}^{y} g(x_0, y)dy \tag{92}$$

where x_0 and y_0 are arbitrary constants. Please be aware of $g(\underline{x_0}, y)$ which is not $g(\underline{x}, y)$ x_0 and y_0 can be added into c in Equation (89) as they are arbitrary constants.

Procedure:

a) Allocate $f(x, y)$ and $g(x, y)$

b) Confirm

$$\frac{\partial\{f(x, y)\}}{\partial y} = \frac{\partial\{g(x, y)\}}{\partial x}$$

c) Apply $\int_{x_0}^{x} f(x, y)dx + \int_{y_0}^{y} g(x_0, y)dy = c$

d) Merge all the terms which have x_0 and y_0

Proof: Let's assume there is a function

$$U(x,y)=\int_{x_0}^{x} f(x,y)dx+\int_{y_0}^{y} g(x_0,y)dy=c \qquad ①$$

When you calculate $\int_{x_0}^{x} f(x,y)dx$, you assume y is a constant and let it be y_0. Thus we can write

$$\int f(x,y)dx \equiv \int f(x,y_0)dx \triangleq F(x,y_0) \qquad ②$$

In the similar way we can write

$$\int g(x_0,y)dy \triangleq G(x_0,y) \qquad ③$$

By putting ② and ③ into ①, we get

$$U(x,y) = F(x,y_0)-F(x_0,y_0)+G(x_0,y)-G(x_0,y_0)=c \qquad ④$$

Since $U(x,y)=c$ from ①, we can write

$$\partial U(x,y)=\frac{\partial\{U(x,y)\}}{\partial x}dx+\frac{\partial\{U(x,y)\}}{\partial y}dy=0 \qquad ⑤$$

Using ④, we obtain $\frac{\partial\{U(x,y)\}}{\partial x}$ and $\frac{\partial\{U(x,y)\}}{\partial y}$ as follows:

$$\frac{\partial\{U(x,y)\}}{\partial x}=f(x,y_0) \qquad ⑥$$
$$\frac{\partial\{U(x,y)\}}{\partial y}=g(x_0,y) \qquad ⑦$$

By putting ⑥ and ⑦ into ⑤, we get

$$\frac{\partial\{U(x,y)\}}{\partial x}dx+\frac{\partial\{U(x,y)\}}{\partial y}dy = f(x,y_0)dx+g(x_0,y)dy=0 \qquad ⑥$$

Now since

$$\frac{\partial\{f(x,y_0)\}}{\partial y}=\frac{\partial\{g(x_0,y)\}}{\partial x}(=0) \qquad ⑦$$

we can conclude that ① satisfies ⑥ and ⑦. In other words, when ⑥ and ⑦ are given, we can say ① is valid.

4) When the differential equation can be written as $\frac{\partial\{y\}}{\partial x}+P(x)y=Q(x)$ then the answer is

$$y=\frac{1}{\Phi(x)}\left[\int \Phi(x)Q(x)dx+c\right] \tag{93}$$

where

$$\Phi(x)=\mathfrak{e}^{\int P(x)dx} \tag{94}$$

Procedure:

a) Allocate $P(x)$ and $Q(x)$
b) Calculate $A=\int P(x)dx$
c) Obtain $\Phi(x)=\mathfrak{e}^{A}$
d) Calculate $B=\int \Phi(x)Q(x)dx$
e) Obtain the general solution $y=\frac{1}{\Phi(x)}[B+c]$
f) Apply the condition to $y=\frac{1}{\Phi(x)}[B+c]$ in order to find out c and thus the particular solution

Proof:

When we multiply $\frac{\partial\{y\}}{\partial x}+P(x)y=Q(x)$ with $\Phi(x)$, we get:
$\Phi(x)\frac{\partial\{y\}}{\partial x}+\Phi(x)P(x)y=\Phi(x)Q(x)$. Since,

$$\frac{\partial\{\Phi(x)\}}{\partial x}=\frac{\partial\left\{\mathfrak{e}^{\int P(x)dx}\right\}}{\partial x} = \mathfrak{e}^{\int P(x)dx}\frac{\partial\left\{\int P(x)dx\right\}}{\partial x} = \mathfrak{e}^{\int P(x)dx}P(x) = \Phi(x)P(x),$$

$$\begin{aligned}\Phi(x)Q(x) &= \Phi(x)\frac{\partial\{y\}}{\partial x} + \Phi(x)P(x)y \\ &= \Phi(x)\frac{\partial\{y\}}{\partial x} + \frac{\partial\{\Phi(x)\}}{\partial x}y \\ &= \frac{\partial\{y\Phi(x)\}}{\partial x}\end{aligned}$$

because $\frac{\partial\{y\}}{\partial x} + P(x)y = Q(x)$ and $\frac{\partial\{\Phi(x)\}}{\partial x} = \Phi(x)P(x)$.
When we integrate $\frac{\partial\{y\Phi(x)\}}{\partial x} = \Phi(x)Q(x)$ with respect to x,

$$\begin{aligned}\int \frac{\partial\{y\Phi(x)\}}{\partial x}dx &= \int \Phi(x)Q(x)dx \\ \therefore y\Phi(x) &= \int \Phi(x)Q(x)dx + c \\ \therefore y &= \frac{1}{\Phi(x)}\left[\int \Phi(x)Q(x)dx + c\right]\end{aligned}$$

5) The solution of Jean Bernoulli equation

$$\frac{\partial\{y\}}{\partial x} + p(x)y = q(x)y^{\alpha} \qquad (\alpha \neq 0, 1) \tag{95}$$

is obtained by solving

$$\frac{\partial\{Y\}}{\partial x} + (1-\alpha)p(x)Y = (1-\alpha)q(x) \tag{96}$$

where

$$Y = y^{1-\alpha}. \tag{97}$$

In other words, Y ($= y^{1-\alpha}$, be aware that this is not y but Y!!) is obtained from $Y = \frac{1}{\Phi(x)}\left[\int \Phi(x)Q(x)dx + c\right]$ where $\Phi(x) = \mathfrak{e}^{\int P(x)dx}$ and $P(x) = (1-\alpha)p(x)$ and $Q(x) = (1-\alpha)q(x)$. The steps to the solution are:

a) allocate $p(x)$ and $q(x)$
b) identify the value of α
c) allocate $P(x) = (1-\alpha)p(x)$ and $Q(x) = (1-\alpha)q(x)$
d) calculate $\int P(x)dx$
e) calculate $\Phi(x) = \mathfrak{e}^{\int P(x)dx}$
f) calculate $y^{1-\alpha} = \frac{1}{\Phi(x)}[\int \Phi(x)Q(x)dx + c]$

Proof:

$$\begin{aligned}\frac{\partial\{y\}}{\partial x} + p(x)y &= q(x)y^{\alpha} \\ \therefore y^{-\alpha}\frac{\partial\{y\}}{\partial x} + p(x)y\cdot y^{-\alpha} &= q(x) \\ \therefore y^{-\alpha}\frac{\partial\{y\}}{\partial x} + p(x)y^{1-\alpha} &= q(x)\end{aligned}$$

Since

$$\begin{aligned}\frac{\partial\{y^{1-\alpha}\}}{\partial x} &= \frac{\partial\{y^{1-\alpha}\}}{\partial y}\frac{\partial\{y\}}{\partial x} \\ &= (1-\alpha)y^{1-\alpha-1}\frac{\partial\{y\}}{\partial x} \\ &= (1-\alpha)y^{-\alpha}\frac{\partial\{y\}}{\partial x} \\ \therefore \frac{1}{1-\alpha}\frac{\partial\{y^{1-\alpha}\}}{\partial x} &= y^{-\alpha}\frac{\partial\{y\}}{\partial x}\end{aligned}$$

we can manipulate the equation as follows:

$$\begin{aligned}y^{-\alpha}\frac{\partial\{y\}}{\partial x} + p(x)y^{1-\alpha} &= q(x) \\ \therefore \frac{1}{1-\alpha}\frac{\partial\{y^{1-\alpha}\}}{\partial x} + p(x)y^{1-\alpha} &= q(x) \\ \therefore \frac{\partial\{y^{1-\alpha}\}}{\partial x} + (1-\alpha)p(x)y^{1-\alpha} & \\ &= (1-\alpha)q(x) \\ \therefore \frac{\partial\{Y\}}{\partial x} + (1-\alpha)p(x)Y &= (1-\alpha)q(x)\end{aligned}$$

The answer can be obtained from Equation (93) where

$$P(x) = (1-\alpha)p(x) \tag{98}$$
$$Q(x) = (1-\alpha)q(x) \tag{99}$$

6) Clairaut type

$$y = x\frac{\partial\{y\}}{\partial x} + f\left(\frac{\partial\{y\}}{\partial x}\right) \tag{100}$$

can be solved as follows:

a) Allocate $f(\frac{\partial\{y\}}{\partial x})$
b) Write down the general solution of

$$y = ax + f(a)$$

which is the answer!. State a is a constant value.
c) Differentiate

$$y = ax + f(a)$$

with respect to a
d) Express a as a function of x, let's say $a = g(x)$
e) Insert $a = g(x)$ into the general solution to get a particular solution of

$$y = x \cdot g(x) + f(g(x))$$

Proof:

$$\begin{aligned}
\frac{\partial\{y\}}{\partial x} &= \frac{\partial\left\{x\frac{\partial\{y\}}{\partial x} + f\left(\frac{\partial\{y\}}{\partial x}\right)\right\}}{\partial x} \\
&= \frac{\partial\{x\}}{\partial x}\frac{\partial\{y\}}{\partial x} + x\frac{\partial^2 y}{\partial x^2} + \frac{\partial\left\{f\left(\frac{\partial\{y\}}{\partial x}\right)\right\}}{\partial x} \\
&= \frac{\partial\{y\}}{\partial x} + x\frac{\partial^2 y}{\partial x^2} + \frac{\partial\left\{f\left(\frac{\partial\{y\}}{\partial x}\right)\right\}}{\partial\left\{\frac{\partial\{y\}}{\partial x}\right\}}\frac{\partial\left\{\frac{\partial\{y\}}{\partial x}\right\}}{\partial x} \\
&= \frac{\partial\{y\}}{\partial x} + x\frac{\partial^2 y}{\partial x^2} + \frac{\partial\left\{f\left(\frac{\partial\{y\}}{\partial x}\right)\right\}}{\partial\left\{\frac{\partial\{y\}}{\partial x}\right\}}\frac{\partial^2 y}{\partial x^2} \\
\therefore 0 &= x\frac{\partial^2 y}{\partial x^2} + \frac{\partial\left\{f\left(\frac{\partial\{y\}}{\partial x}\right)\right\}}{\partial\left\{\frac{\partial\{y\}}{\partial x}\right\}}\frac{\partial^2 y}{\partial x^2} \\
\therefore 0 &= \left(x + \frac{\partial\left\{f\left(\frac{\partial\{y\}}{\partial x}\right)\right\}}{\partial\left\{\frac{\partial\{y\}}{\partial x}\right\}}\right)\frac{\partial^2 y}{\partial x^2}
\end{aligned}$$

Thus we obtain

$$\frac{\partial^2 y}{\partial x^2} = 0$$

or

$$x + \frac{\partial\left\{f\left(\frac{\partial\{y\}}{\partial x}\right)\right\}}{\partial\left\{\frac{\partial\{y\}}{\partial x}\right\}} = 0$$

From $\frac{\partial^2 y}{\partial x^2} = 0$ we obtain

$$\begin{aligned}
\frac{\partial^2 y}{\partial x^2} &= 0 \\
\therefore \frac{\partial\left\{\frac{\partial\{y\}}{\partial x}\right\}}{\partial x} &= 0 \\
\therefore \partial(\frac{\partial\{y\}}{\partial x}) &= 0 \cdot \partial x \\
\therefore \int d(\frac{\partial\{y\}}{\partial x}) &= \int 0 \cdot dx \\
\therefore \frac{\partial\{y\}}{\partial x} &= a \\
\therefore dy &= a \cdot dx \\
\therefore \int dy &= \int a \cdot dx \\
\therefore y &= ax + b \\
\therefore \frac{\partial\{y\}}{\partial x} = \frac{\partial\{ax+b\}}{\partial x} &= a
\end{aligned}$$

where a and b are the arbitrary constants. Substituting $y = ax + b$ and $\frac{\partial\{y\}}{\partial x} = a$ into the original equation, we get

$$y = x\frac{\partial\{y\}}{\partial x} + f\left(\frac{\partial\{y\}}{\partial x}\right)$$
$$\therefore ax + b = x \cdot a + f(a)$$
$$\therefore b = f(a)$$

Therefore

$$y = ax + f(a) \tag{101}$$

is a general solution with an arbitrary constant of a. Furthermore, when we take the differentiation of the equation with respect to a, we get

$$\frac{\partial\{y\}}{\partial a} = \frac{\partial\{ax + f(a)\}}{\partial a}$$
$$\therefore 0 = \frac{\partial\{ax\}}{\partial a} + \frac{\partial\{f(a)\}}{\partial a}$$
$$\therefore 0 = x + \frac{\partial\{f(a)\}}{\partial a}$$

We solve the equation for a. Let's assume $a = A(x)$ satisfies $x + \frac{\partial\{f(a)\}}{\partial a} = 0$. The resultant expression of a using x, which is $A(x)$ is put into $y = ax + f(a)$ to obtain a particular solution of Equation (102).

$$y = A(x) \cdot x + f(A(x)) \tag{102}$$

7) In order to solve second order differential equations

$$\frac{\partial^2 y}{\partial x^2} + v\frac{\partial\{y\}}{\partial x} + wy = r(x), \tag{103}$$

where v, w are the constant values,

a) Production of an auxiliary equation by forcing $r(x)$ to 0
By substituting

$$\frac{\partial^2 y}{\partial x^2} = \lambda^2, \frac{\partial\{y\}}{\partial x} = \lambda, y = \lambda^0 = 1 \tag{104}$$

into the original original equation, forcing $r(x)$ to zero, we solve the auxiliary equation of

$$\lambda^2 + v\lambda + w = 0 \tag{105}$$

and we obtain the answers $\lambda = \alpha$ and β.

b) Set complementary function as follows:

i) α and β are real and $\alpha \neq \beta$
Set the complementary function $Y_1(x)$ as

$$Y_1(x) = a\mathfrak{e}^{\alpha x} + b\mathfrak{e}^{\beta x} \tag{106}$$

where a, b are constant value which is found from the initial condition.

ii) α and β are real and $\alpha = \beta$
Set the complementary function $Y_1(x)$ as

$$Y_1(x) = a\mathfrak{e}^{\alpha x} + bx\mathfrak{e}^{\alpha x} \tag{107}$$

iii) α and β are complex numbers and $p \pm \jmath q$ (where p, q are real)
Set the complementary function $Y_1(x)$ as

$$Y_1(x) = \mathfrak{e}^{px}(a\cos qx + b\sin qx) \tag{108}$$

c) Check the characteristics of $r(x)$ and set the particular integral

i) $r(x)$ is proportional to $\mathfrak{e}^{cx}$, where c is a constant value

A) $\alpha \neq c$ and $\beta \neq c$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = g\mathfrak{e}^{cx} \tag{109}$$

where g is a constant value which is found from Equation (103).

B) $\alpha = c$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = gx^k\mathfrak{e}^{cx} \tag{110}$$

where k is 1 or 2 or 3 ...

ii) $r(x)$ is nth order polynomial

A) $\alpha \neq 0$ and $\beta \neq 0$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = \sum_{m=0}^{n} g_m x^m \tag{111}$$

where g_m is a constant value which is found from Equation (103).

B) $\alpha = 0$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = x^k\left(\sum_{m=0}^{n} g_m x^m\right) \tag{112}$$

where k is 1 or 2 or 3 ...

iii) $r(x)$ is in the form of $P(x)\mathfrak{e}^{cx}$ where $P(x)$ is the nth order polynomial.

A) $\alpha \neq c$ and $\beta \neq c$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = \mathfrak{e}^{cx}\left(\sum_{m=0}^{n} g_m x^m\right) \tag{113}$$

where g_m is a constant value which is found from Equation (103).

B) $\alpha = c$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = \mathfrak{e}^{cx} x^k \left(\sum_{m=0}^{n} g_m x^m\right) \tag{114}$$

where k is 1 or 2 or 3 ...

iv) $r(x)$ is the combination of $\cos \omega x$ and $\sin \omega x$

A) $\alpha \neq \pm \jmath\omega$ and $\beta \neq \pm \jmath\omega$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = g \cos \omega x + h \sin \omega x \tag{115}$$

where g and h are constant values which is found from Equation (103).

B) $\alpha = \pm \jmath\omega$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = x^k (g \cos \omega x + h \sin \omega x) \tag{116}$$

where k is 1 or 2 or 3 ...

v) $r(x)$ is the combination of $\mathfrak{e}^{cx} \cos \omega x$ and $\mathfrak{e}^{cx} \sin \omega x$

A) $\alpha \neq c \pm \jmath\omega$ and $\beta \neq c \pm \jmath\omega$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = \mathfrak{e}^{cx} (g \cos \omega x + h \sin \omega x) \tag{117}$$

where g and h are constant values which is found from Equation (103).

B) $\alpha = c \pm \jmath\omega$
Set the particular integral $Y_2(x)$ as

$$Y_2(x) = x^k \mathfrak{e}^{cx} (g \cos \omega x + h \sin \omega x) \tag{118}$$

where k is 1 or 2 or 3 ...

d) Find the constant values g and h by

$$\frac{\partial^2 Y_2(x)}{\partial x^2} + v\frac{\partial \{Y_2(x)\}}{\partial x} + wY_2(x) = r(x) \tag{119}$$

e) Get the general solution of The general solution is $y = Y_1(x) + Y_2(x)$ leaving a and b unknown.

f) Find the constant values a and b
Usually there are initial conditions for $y(0)$ and $\frac{\partial \{y\}}{\partial x}|_{x=0}$. Using these conditions, a and b are found.

g) The particular solution is $y = Y_1(x) + Y_2(x)$.

Summary Procedure of 2nd order ODE $\frac{\partial^2 y}{\partial x^2} + v\frac{\partial \{y\}}{\partial x} + wy = r(x)$

a) Produce and solve an auxiliary equation by setting $r(x) = 0$
b) Set the complementary function $Y_1(x)$ with the unknown variables a and b
c) Set particular integral $Y_2(x)$ with the unknown variables g and h
d) Find g and h from $\frac{\partial^2 Y_2(x)}{\partial x^2} + v\frac{\partial \{Y_2 x\}}{\partial x} + wY_2(x) = r(x)$
e) Get the general solution $y = Y_1(x) + Y_2(x)$ with unknown a and b
f) Find a and b using the initial conditions
g) Get the particular solution $y = Y_1(x) + Y_2(x)$ with known a and b

8) Lookup table for 2nd order ODE

$r(x)$	particular integral $Y_2(x)$
$\mathfrak{e}^{cx}, \alpha \neq c, \beta \neq c$	$g\mathfrak{e}^{cx}$
$\mathfrak{e}^{cx}, \alpha = c$	$gx^k\mathfrak{e}^{cx}$
$\sum_{m=0}^{n} \rho_m x^m, \alpha \neq 0, \beta \neq 0$	$\sum_{m=0}^{n} g_m x^m$
$\sum_{m=0}^{n} \rho_m x^m, \alpha = 0$	$x^k \left(\sum_{m=0}^{n} g_m x^m\right)$
$\mathfrak{e}^{cx} \sum_{m=0}^{n} \rho_m x^m, \alpha \neq c, \beta \neq c$	$\mathfrak{e}^{cx} \sum_{m=0}^{n} g_m x^m$
$\mathfrak{e}^{cx} \sum_{m=0}^{n} \rho_m x^m, \alpha = c$	$x^k\mathfrak{e}^{cx} \sum_{m=0}^{n} g_m x^m$
$\rho_1 \cos\omega x + \rho_2 \sin\omega x, \alpha \neq \pm\jmath\omega, \beta \neq \pm\jmath\omega$	$g\cos\omega x + h\sin\omega x$
$\rho_1 \cos\omega x + \rho_2 \sin\omega x, \alpha = \pm\jmath\omega$	$x^k(g\cos\omega x + h\sin\omega x)$
$\mathfrak{e}^{cx}(\rho_1 \cos\omega x + \rho_2 \sin\omega x), \alpha \neq c \pm \jmath\omega, \beta \neq c \pm \jmath\omega$	$\mathfrak{e}^{cx}(g\cos\omega x + h\sin\omega x)$
$\mathfrak{e}^{cx}(\rho_1 \cos\omega x + \rho_2 \sin\omega x), \alpha = c \pm \jmath\omega$	$x^k\mathfrak{e}^{cx}(g\cos\omega x + h\sin\omega x)$

TABLE I

PARTICULAR INTEGRAL FOR THE SECOND ORDER ODE

9) Summary for 1st order ODE

Equation type	Procedure to follow
$\frac{\partial\{y\}}{\partial x} = f(x)g(y)$	a) Allocate $f(x)$ and $g(x)$ b) Calculate $$\int \frac{1}{g(y)} dy = \int f(x)dx$$
$\frac{\partial\{y\}}{\partial x} = f(\frac{y}{x})$	a) Find $f(\frac{y}{x})$ b) Calculate $$\int \frac{dz}{f(z)-z} \triangleq g(z)$$ c) Set $\ln(x) + c = g(z)$ d) Replace z with $\frac{y}{x}$ so that $\ln(x) + c = g(\frac{y}{x})$ is the answer
$\frac{\partial\{y\}}{\partial x} = -\frac{f(x,y)}{g(x,y)}$	a) Allocate $f(x,y)$ and $g(x,y)$ b) Confirm $$\frac{\partial\{f(x,y)\}}{\partial y} = \frac{\partial\{g(x,y)\}}{\partial x}$$ c) Apply $\int_{x_0}^{x} f(x,y)dx + \int_{y_0}^{y} g(x_0,y)dy = c$ d) Merge all the terms which have x_0 and y_0
$\frac{\partial\{y\}}{\partial x} = -P(x)y + Q(x)$	a) Allocate $P(x)$ and $Q(x)$ b) Calculate $\int P(x)dx$ c) Calculate $\Phi(x) = \mathfrak{e}^{\int P(x)dx}$ d) Calculate $y = \frac{1}{\Phi(x)}\left[\int \Phi(x)Q(x)dx + c\right]$
$\frac{\partial\{y\}}{\partial x} = -p(x)y + q(x)y^{\alpha}$	a) allocate $p(x)$ and $q(x)$ b) identify the value of α c) allocate $P(x) = (1-\alpha)p(x)$ and $Q(x) = (1-\alpha)q(x)$ d) calculate $\int P(x)dx$ e) calculate $\Phi(x) = \mathfrak{e}^{\int P(x)dx}$ f) calculate $y^{1-\alpha} = \frac{1}{\Phi(x)}[\int \Phi(x)Q(x)dx + c]$
$\frac{\partial\{y\}}{\partial x} = \frac{y}{x} + \frac{1}{x} f\left(\frac{\partial\{y\}}{\partial x}\right)$	a) Allocate $f(\frac{\partial\{y\}}{\partial x})$ b) Write down the general solution of $$y = ax + f(a)$$ which is the answer!. State a is a constant value. c) Differentiate $$y = ax + f(a)$$ with respect to a d) Express a as a function of x, let's say $a = g(x)$ e) Insert $a = g(x)$ into the general solution to get a particular solution of $$y = x \cdot g(x) + f(g(x))$$