CS 3172 Advanced Algorithms

(2nd semester 2002/2003)

Solutions

Jürgen Dix

16. March, 2003

1. Suppose there is a function $f : \mathbb{N} \to \mathbb{N}$ mapping integers of length k onto integers of length k such that

   (a) $f \in \mathbf{P}$,

   (b) $f^{-1} \notin \mathbf{P}$.

   Show that
$$\{\langle x,y\rangle : \ f^{-1}(x) < y\} \in (\mathbf{NP} \cap \textbf{co-NP}) \setminus \mathbf{P}$$

   Solution: Let us denote the set $\{\langle x,y\rangle : \ f^{-1}(x) < y\}$ by M. We have to show the following three things:

   $M \in$ **NP:** We are given a pair $\langle x,y\rangle$ and have to come up with a NDTM running in polynomial time. *We cannot compute $f^{-1}(x)$ and check whether the result is less than y, because this is not polynomial!* Instead, we guess a natural number z and check whether $f(z) = x$. This check runs in polynomial time (because $f \in \mathbf{P}$). We then check whether $z < y$.

   $M \in$ **co-NP:** Exactly as above. We guess z and check whether $f(z) = x$. If so, we check whether $z \geq y$ (this is the complement).

   $M \notin$ **P:** Suppose $M \in \mathbf{P}$. We claim that then $f^{-1} \in \mathbf{P}$, contradicting the assumption above. Given x, how can we compute $f^{-1}(x)$ by making use of M? By the above assumption, f maps integers of length k to integers of length k. Thus we only need to check for all integers y of length identical to $\log(x)$, whether $f^{-1}(x) < y$. This check is polynomial. How often do we have to do this check? Only $\log(x)$ many times[1]! Thus the whole check would be still polynomial.

2. Suppose we have a DTM M that decides a language $\mathcal{L}$. If $x \in \mathcal{L}$, M says yes in polynomial time. If $x \notin \mathcal{L}$, M says no in exponential time. Is $\mathcal{L} \in \mathbf{P}$?

   Solution: At first sight, it seems the answer is "no": after all we have to wait exponentially long for a "no" answer. But what does it mean that it says yes in polynomial time? It means that there is a fixed polynomial $p(x)$, such that for all inputs of size n: if the answer is "yes" it is produced after $p(n)$ steps. Therefore if we are not getting a "yes" answer after that time, we can conclude the answer is "no". Thus $\mathcal{L} \in \mathbf{P}$.

[1] Remember the problem of finding out a number between 1 and x by just asking yes/no questions. You can do that by asking at most $\log(x)$ questions (if you are smart enough).

3. Classify the following problems to **t** (true), **f** (false), or **u** (unknown to science, i.e. depending on the P/NP problem).

   - SAT $\in$ **P**.
     Solution: We know that SAT $\in$ **NP**. If **P** =**NP** then the answer is "true". If **P** $\neq$ **NP** then the answer is "false", because SAT is **NP**-complete. Thus this statement is **unknown to science**.

   - If a problem in **NP** can be solved in polynomial time, then **P** $=$ **NP**.
     Solution: This is a bit tricky. Note that this is an implication, which is already true if the consequence is true (independently of the assumption). The assumption is certainly true (because **NP** contains **P** as a subset).

     If **P** =**NP** then the answer is "true" (because the consequence is true). If **P** $\neq$ **NP** then the answer is "false". Thus this statement is **unknown to science**.

   - If a problem in **NP** can be shown not to be solvable in polynomial time, then **P** $\neq$ **NP**.
     Solution: Obviously "true".

   - SAT can be polynomially reduced to 1-CNF.
     Solution: 1-CNF can be trivially solved in polynomial time (linear even, you just have to check for a pair of complementary literals).

     If **P** =**NP** then the answer is "true" (because then SAT itself would be in **P**). If **P** $\neq$ **NP** then the answer is "false", because SAT is **NP** complete and cannot be reduced to a polynomial problem. Thus this statement is **unknown to science**.

   - All problems in **P** can be polynomially reduced to each other.
     Solution: True, by the very definition of a reduction.

   - Some problems in **NP** cannot be polynomially reduced to each other.
     Solution: If **P** =**NP** then the answer is "false". If **P** $\neq$ **NP** then the answer is "true", because SAT is **NP** complete and cannot be reduced to a polynomial problem. Thus this statement is **unknown to science**.