Partial Evidential Stable Models for Disjunctive Deductive Databases

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Abstract. In this paper we consider the basic semantics of *stable* and *partial stable models* for disjunctive deductive databases (with default negation), cf. [8,14]. It is well–known that there are disjunctive deductive databases where no stable or partial stable models exist, and these databases are called inconsistent w.r.t. the basic semantics.

We define a consistent variant of each class of models, which we call *evidential stable* and *partial evidential stable models*. It is shown that if a database is already consistent w.r.t. the basic semantics, then the class of evidential models coincides with the basic class of models. Otherwise, the set of evidential models is a subset of the set of minimal models of the database. This subset is non-empty, if the database is logically consistent. It is determined according to a suitable *preference relation*, whose underlying idea is to *minimize* the amount of *reasoning by contradiction*.

The technical ingredients for the construction of the new classes of models are two transformations of disjunctive deductive databases. First, the *evidential transformation* is used to realize the preference relation, and to define evidential stable models. Secondly, based on the `tu`–*transformation* the result is lifted to the three–valued case, that is, partial evidential stable models are defined.

Keywords

disjunctive logic programming, non–monotonic reasoning, stable and partial stable models, handling inconsistency, program transformations

1 Introduction

The semantics of *stable* and *partial stable models*, cf. Gelfond, Lifschitz [8,9] and Przymusinski [14], are among the most prominent semantics for disjunctive databases. Unfortunately, there are databases which are logically consistent, but are inconsistent w.r.t. these semantics. For normal databases, i.e. databases that may contain negation but do not contain disjunctions, however, the partial stable models semantics is always consistent and it is equivalent to the *well–founded semantics* of van Gelder, Ross and Schlipf [19].

For large databases, small inconsistent parts can prohibit the existence of stable models, and even of partial stable models. Thus, we will introduce two new variants of the stable model semantics, which are always consistent if the database is logically consistent: First, the two–valued semantics of *evidential stable models*, which is stronger than minimal model but weaker than stable model semantics. Secondly, a three–valued version, called *partial evidential stable models*, which for normal databases coincides with the well–founded semantics. For stratified–disjunctive databases both evidential semantics coincide with the perfect model semantics.

Consider the disjunctive database $\mathcal{P} = \{r\}$ consisting of one rule $r = q \leftarrow \mathit{not}\, a$. Among its two minimal models $M_1 = \{q\}$ and $M_2 = \{a\}$, the first model is *preferred* to the second. Intuitively, the reason is that in M_2 the truth of "a" has been derived by *contradiction*, i.e. r has been fulfilled by making its body *false*. In contrast, in M_1 the truth of "q" is derived constructively from the head of r. Thus, M_1 is the so–called *perfect model* of $\mathcal{P}$, and it is considered to be the intended model.

The *evidential transformation* $\mathcal{EP}$ is a positive–disjunctive database that is derived from $\mathcal{P}$ by moving default negated body literals to the rule heads and prefixing them with "$\mathcal{E}$". Thus, the rule r is translated to $q \vee \mathcal{E}a$. Additionally, rules relating atoms and evidential atoms are introduced: $\mathcal{E}q \leftarrow q$, $\mathcal{E}a \leftarrow a$. A similar construction has been used by Fernández et al., cf. [6], to characterize the stable models of $\mathcal{P}$. But our use of evidences has a different interpretation, and moreover we use additional *normality rules*, which are not needed in [6]. *Evidential stable models* are defined as minimal models M of $\mathcal{EP}$ which also minimize the set of atoms that are derived by contradiction solely: such atoms A are *false* in M, but $\mathcal{E}A$ is *true* in M. Then we call $\mathcal{E}A$ an $\mathcal{E}$–violation. In our example, the minimal models of $\mathcal{EP}$ are $M_1' = \{\, q, \mathcal{E}q \,\}$ and $M_2' = \{\, \mathcal{E}a \,\}$. In M_1' there is no $\mathcal{E}$–violation, whereas in M_2' there is the $\mathcal{E}$–violation "$\mathcal{E}a$". Thus, M_1' is the unique evidential stable model of $\mathcal{P}$. We will show, that for databases which have stable models the evidential stable models coincide with the stable models, when evidential atoms $\mathcal{E}A$ are interpreted as atoms A. Furthermore, evidential stable models always exist for logically consistent databases. E.g. the database $\mathcal{P}' = \{\, a \leftarrow \mathit{not}\, a \,\}$, which does not have any stable models, has the unique evidential stable model $M' = \{\mathcal{E}a\}$, which is interpreted as the model $M = \{a\}$ of $\mathcal{P}$.

The second type of transformation we use is the tu–*transformation* $\mathcal{P}^{\mathsf{tu}}$ of a disjunctive database $\mathcal{P}$, which suitably *annotates* the atoms in $\mathcal{P}$ by the two truth values *true* ("t") and *undefined* ("u"), cf. [17]. We state a *characterization* of the partial stable models of $\mathcal{P}$ in terms of the stable models of $\mathcal{P}^{\mathsf{tu}}$. Then, *partial evidential stable models* are defined based on the evidential stable models of $\mathcal{P}^{\mathsf{tu}}$, where the characterization for partial stable models motivates the new definition. As in the two–valued case, partial evidential stable models always exist for a logically consistent database. If there exist partial stable models of the database, then the partial evidential stable models coincide with the partial stable models, when evidential atoms are interpreted as atoms.

The paper is organized as follows: In Sections 2 and 3 we review the basic definitions and notation for disjunctive databases, partial Herbrand interpretations and partial stable models. In Section 4 we introduce the evidential transformation and the evidential stable models of a disjunctive database $\mathcal{P}$. In Section 5 we define the tu–transformation $\mathcal{P}^{\mathsf{tu}}$ of $\mathcal{P}$ and we state a characterization of the partial stable models of $\mathcal{P}$ in terms of the total stable models of $\mathcal{P}^{\mathsf{tu}}$. This motivates the definition of partial evidential stable models in Section 6. In Sections 7 and 8 we compare the new semantics with other approaches known from literature, and we briefly comment on some of their abstract properties.

2 Basic Definitions and Notations

Given a first order language $\mathcal{L}$, a *disjunctive database* $\mathcal{P}$ consists of logical inference rules of the form

$$r = A_1 \vee \ldots \vee A_k \leftarrow B_1 \wedge \ldots \wedge B_m \wedge not\, C_1 \wedge \ldots \wedge not\, C_n, \qquad (1)$$

where A_i, $1 \leq i \leq k$, B_i, $1 \leq i \leq m$, and C_i, $1 \leq i \leq n$, are atoms in the language $\mathcal{L}$; $k, m, n \in I\!N_0$, and *not* is the negation–by–default operator.[1] A rule is called a fact if $m = n = 0$. The set of all *ground instances* of the rules and facts in $\mathcal{P}$ is denoted by $gnd\,(\mathcal{P})$. A rule (or database) is called *positive–disjunctive* if it does not contain default negation (i.e. $n = 0$). A rule r of the form (1) above is denoted for short as:

$$r = \alpha \leftarrow \beta \wedge not \cdot \gamma, \qquad (2)$$

where $\alpha = A_1 \vee \ldots \vee A_k$, $\beta = B_1 \wedge \ldots \wedge B_m$, and $\gamma = C_1 \vee \ldots \vee C_n$.[2]

[1] By $I\!N_+$ we denote the set $\{1, 2, 3, \ldots\}$ of positive natural numbers, whereas $I\!N_0$ denotes the set $\{0, 1, 2, \ldots\}$ of all natural numbers.

[2] Note that γ is a disjunction, and, according to De Morgan's law, $not \cdot \gamma$ is taken to be a conjunction.

Herbrand Interpretations and Partial Herbrand Interpretations

The Herbrand base $\mathit{HB}_{\mathcal{P}}$ of a disjunctive database $\mathcal{P}$ contains all ground atoms over the language of $\mathcal{P}$. A *partial Herbrand interpretation* of $\mathcal{P}$ is given by a mapping I: $\mathit{HB}_{\mathcal{P}} \to \{\, \mathsf{t}, \mathsf{f}, \mathsf{u} \,\}$ that assigns a truth value "t" (*true*), "f" (*false*) or "u" (*undefined*) to each atom in $\mathit{HB}_{\mathcal{P}}$. Thus, partial Herbrand interpretations are also called three–valued Herbrand interpretations. I is called a *total* or *total Herbrand interpretation*, if all atoms A are mapped to classical truth values t or f.

Equivalently, a partial Herbrand interpretation I can be represented by using the concept of *annotated atoms*. Given an atom $A = p(t_1, \ldots, t_n)$ and a truth value $\mathsf{v} \in \{\, \mathsf{t}, \mathsf{f}, \mathsf{u} \,\}$, we define $A^{\mathsf{v}} = p^{\mathsf{v}}(t_1, \ldots, t_n)$, where p^{v} is taken to be a new predicate symbol. We will use two ways of representing I as a set of annotated atoms, either by specifying the *true* and *false* atoms or by specifying the *true* and *undefined* atoms:

tf–*Representation:* $\quad I^{\mathsf{tf}} = I^{\mathsf{t}} \cup I^{\mathsf{f}}$,
tu–*Representation:* $\quad I^{\mathsf{tu}} = I^{\mathsf{t}} \cup I^{\mathsf{u}}$,

where I^{t}, I^{u} and I^{f} are given by:

$$\begin{aligned} I^{\mathsf{t}} &= \{\, A^{\mathsf{t}} \mid A \in \mathit{HB}_{\mathcal{P}} \wedge I(A) = \mathsf{t} \,\}, \\ I^{\mathsf{f}} &= \{\, A^{\mathsf{f}} \mid A \in \mathit{HB}_{\mathcal{P}} \wedge I(A) = \mathsf{f} \,\}, \\ I^{\mathsf{u}} &= \{\, A^{\mathsf{u}} \mid A \in \mathit{HB}_{\mathcal{P}} \wedge (\, I(A) = \mathsf{t} \vee I(A) = \mathsf{u} \,) \,\}. \end{aligned}$$

Note that in the tu–representation every *true* atom A is recorded as A^{t} and as A^{u}, which will become important later. Note also that the tf–representation is essentially the same as the conventional representation of I as a *set of literals*, where A^{t} becomes the atom A itself and A^{f} becomes the negative literal $\neg A$. For a set $\mathcal{I}$ of partial Herbrand interpretations we will use the same notations for $\mathsf{v} \in \{\, \mathsf{tf}, \mathsf{tu} \,\}$: $\mathcal{I}^{\mathsf{v}} = \{\, I^{\mathsf{v}} \mid I \in \mathcal{I} \,\}$. By $\mathcal{I} =_{\mathsf{v}} \mathcal{J}$, we denote that $\mathcal{J} = \mathcal{I}^{\mathsf{v}}$ is the v–representation of $\mathcal{I}$.

Consider for instance the Herbrand base $\mathit{HB}_{\mathcal{P}} = \{\, a, b, c, d \,\}$. Then the partial Herbrand interpretation I with $I(a) = \mathsf{t}$, $I(b) = \mathsf{t}$, $I(c) = \mathsf{f}$, and $I(d) = \mathsf{u}$, is represented as follows:

$$I^{\mathsf{tf}} = \{\, a^{\mathsf{t}}, b^{\mathsf{t}}, c^{\mathsf{f}} \,\}, \quad I^{\mathsf{tu}} = \{\, a^{\mathsf{t}}, a^{\mathsf{u}}, b^{\mathsf{t}}, b^{\mathsf{u}}, d^{\mathsf{u}} \,\}.$$

Obviously, a total Herbrand interpretation I can simply be represented by the set $J = \{\, A \in \mathit{HB}_{\mathcal{P}} \mid I(A) = \mathsf{t} \,\}$ of *true* atoms. Conversely, any set $J \subseteq \mathit{HB}_{\mathcal{P}}$ of ground atoms *induces* a total Herbrand interpretation $J^{\diamond}$, where $J^{\diamond}(A) = \mathsf{t}$ iff $A \in J$. For a set $\mathcal{J}$ of sets of atoms, $\mathcal{J}^{\diamond} = \{\, J^{\diamond} \mid J \in \mathcal{J} \,\}$.

Truth Ordering and Knowledge Ordering

There are two common *partial orderings on truth values*, the truth ordering and the knowledge ordering, cf. Fitting [7], which are shown by Figure 1:

Truth Ordering $\leq_{\mathsf{t}}$: $\mathsf{f} \leq_{\mathsf{t}} \mathsf{u}$, $\mathsf{u} \leq_{\mathsf{t}} \mathsf{t}$,
Knowledge Ordering $\leq_{\mathsf{k}}$: $\mathsf{u} \leq_{\mathsf{k}} \mathsf{f}$, $\mathsf{u} \leq_{\mathsf{k}} \mathsf{t}$.

Given $\mathsf{v}_1, \mathsf{v}_2 \in \{\, \mathsf{t}, \mathsf{f}, \mathsf{u} \,\}$, by $\mathsf{v}_1 \geq_{\mathsf{x}} \mathsf{v}_2$ we denote the fact that $\mathsf{v}_2 \leq_{\mathsf{x}} \mathsf{v}_1$, for $\mathsf{x} \in \{\, \mathsf{t}, \mathsf{k} \,\}$.

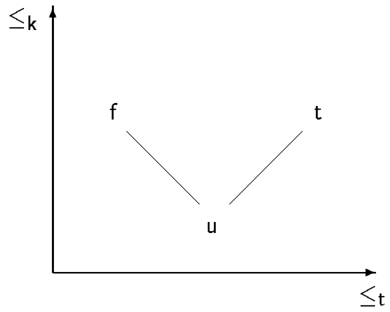

Fig. 1. Truth Ordering and Knowledge Ordering

These partial orderings have been generalized (pointwise) to partial Herbrand interpretations as follows. For $\mathsf{x} \in \{\, \mathsf{t}, \mathsf{k} \,\}$: $I_1 \leq_{\mathsf{x}} I_2$, iff $(\, \forall A \in HB_{\mathcal{P}} : I_1(A) \leq_{\mathsf{x}} I_2(A)\,)$. The truth ordering on partial Herbrand interpretations corresponds to the subset ordering on the tu–representations: $I_1 \leq_{\mathsf{t}} I_2$ iff $I_1^{\mathsf{tu}} \subseteq I_2^{\mathsf{tu}}$. The knowledge ordering corresponds to the subset ordering on the tf–representations: $I_1 \leq_{\mathsf{k}} I_2$ iff $I_1^{\mathsf{tf}} \subseteq I_2^{\mathsf{tf}}$.

$\wedge$	t	f	u
t	t	f	u
f	f	f	f
u	u	f	u

$\vee$	t	f	u
t	t	t	t
f	t	f	u
u	t	u	u

$\neg$	
t	f
f	t
u	u

Fig. 2. Boolean operations in three–valued logic

The *Boolean operations* "$\vee$", "$\wedge$" and "$\neg$" on truth values are defined based on the truth ordering, cf. Figure 2. The truth value of a disjunction

$v_1 \vee v_2$ and a conjunction $v_1 \wedge v_2$ of truth values are constructed by taking the *maximum* and the *minimum* of v_1 and v_2, respectively. "$\vee$" and "$\wedge$" both are commutative and associative, and thus can be generalized to disjunctions and conjunctions, respectively, of more than one truth value.

Models and Partial Models, Minimality

Let M be a partial Herbrand interpretation of a disjunctive database $\mathcal{P}$. For $A_i \in \mathit{HB}_{\mathcal{P}}$, $1 \leq i \leq k$, and a connective $\otimes \in \{ \vee, \wedge \}$ we define $M(A_1 \otimes \ldots \otimes A_k) = M(A_1) \otimes \ldots \otimes M(A_k)$. For $k = 0$, the empty disjunction (i.e. $\otimes = \vee$) evaluates to f, whereas the empty conjunction (i.e. $\otimes = \wedge$) evaluates to t. M is called *partial model* of a ground rule $r = \alpha \leftarrow \beta \wedge \mathit{not} \cdot \gamma$ if

$$M(\alpha) \geq_t M(\beta) \wedge \neg M(\gamma). \tag{3}$$

M is called *partial model* of $\mathcal{P}$ if M is a partial model of all ground instances $r \in \mathit{gnd}\,(\mathcal{P})$ of all rules of $\mathcal{P}$. This is denoted by $M \models_3 \mathcal{P}$.

Minimality of partial models is defined w.r.t. the truth ordering. M is called *partial minimal model* of $\mathcal{P}$ if M is a partial model of $\mathcal{P}$ and there is no other partial model I of $\mathcal{P}$ such that $I \leq_{\mathrm{t}} M$. The set of all partial minimal models of $\mathcal{P}$ is denoted by $\mathcal{MM}_3(\mathcal{P})$. A partial model M of a disjunctive database $\mathcal{P}$ that is total is called a *model* of $\mathcal{P}$. This is denoted by $M \models_2 \mathcal{P}$. A partial minimal model M of $\mathcal{P}$ that is total is called a *minimal model* of $\mathcal{P}$. The set of all minimal models of $\mathcal{P}$ is denoted by $\mathcal{MM}_2(\mathcal{P})$.

3 Stable and Partial Stable Models

The *Gelfond–Lifschitz transformation* (GL–transformation) of a disjunctive database $\mathcal{P}$ w.r.t. a partial Herbrand interpretation M is obtained from the ground instance $\mathit{gnd}\,(\mathcal{P})$ of $\mathcal{P}$ by replacing in every rule the negative body by its truth value $M(\mathit{not} \cdot \gamma) = \neg M(\gamma)$ w.r.t. M.[3]

Definition 1 (Gelfond–Lifschitz Transformation, [8,14]).
Let M be a partial Herbrand interpretation of a disjunctive database $\mathcal{P}$.

1. For $r = \alpha \leftarrow \beta \wedge \mathit{not} \cdot \gamma \in \mathit{gnd}\,(\mathcal{P})$ we define $r^M = \alpha \leftarrow \beta \wedge \neg M(\gamma)$.
2. The *Gelfond–Lifschitz transformation* of $\mathcal{P}$ is $\mathcal{P}^M = \{ r^M | r \in \mathit{gnd}\,(\mathcal{P}) \}$.

[3] If this truth value is "t", then "t" can be deleted from the body. If it is "f", then the whole rule can be deleted from $\mathcal{P}^M$.

The GL–transformation $\mathcal{P}^M$ is a ground positive–disjunctive database that has as additional atoms the truth values t, f and u. Note that these truth values must evaluate to themselves under all partial Herbrand interpretations I of $\mathcal{P}^M$.

Definition 2 (Partial Stable Models, Stable Models, [8,14]). Let M be a partial Herbrand interpretation of a disjunctive database $\mathcal{P}$.

1. M is called *partial stable model* of $\mathcal{P}$ if $M \in \mathcal{MM}_3(\mathcal{P}^M)$. The set of all partial stable models of $\mathcal{P}$ is denoted by $\mathcal{STABLE}_3(\mathcal{P})$.
2. A partial stable model M of $\mathcal{P}$ that is total is called *stable model* of $\mathcal{P}$. The set of all stable models of $\mathcal{P}$ is denoted by $\mathcal{STABLE}_2(\mathcal{P})$.

It can be shown that $\mathcal{STABLE}_2(\mathcal{P}) \subseteq \mathcal{STABLE}_3(\mathcal{P})$ for all disjunctive databases. That is, the semantics of stable models is always stronger than the semantics of partial stable models. The following databases will be used as running examples throughout the paper.

Example 3 (Partial Stable Models).

1. For the disjunctive database

$$\mathcal{P}_1 \;=\; \{\; a \vee b,\; q \leftarrow b \wedge not\, a,\; q \leftarrow a \wedge not\, b \;\},$$

we get the following set of partial stable models:

$$\mathcal{STABLE}_3(\mathcal{P}_1) \;=_{\mathsf{tf}}\; \{\, \{\, a^{\mathsf{t}}, q^{\mathsf{t}}, b^{\mathsf{f}} \,\},\; \{\, b^{\mathsf{t}}, q^{\mathsf{t}}, a^{\mathsf{f}} \,\} \,\}.$$

E.g. for $M^{\mathsf{tf}} = \{\, a^{\mathsf{t}}, q^{\mathsf{t}}, b^{\mathsf{f}} \,\}$ we get the GL–transformation $\mathcal{P}_1^M = \{\; a \vee b,\; q \leftarrow b \wedge \mathsf{f},\; q \leftarrow a \wedge \mathsf{t} \;\}$, and $\mathcal{MM}_3(\mathcal{P}_1^M) \;=_{\mathsf{tf}}\; \{\, M^{\mathsf{tf}}, N^{\mathsf{tf}} \,\}$, for $N^{\mathsf{tf}} = \{\, b^{\mathsf{t}}, a^{\mathsf{f}}, q^{\mathsf{f}} \,\}$. Here all partial stable models are also stable models, i.e. $\mathcal{STABLE}_3(\mathcal{P}_1) = \mathcal{STABLE}_2(\mathcal{P}_1)$. Since $\mathcal{P}_1$ is stratified, the stable models coincide with the *perfect models*.
2. For the disjunctive database

$$\mathcal{P}_2 \;=\; \{\; a \leftarrow not\, b,\; b \leftarrow not\, c,\; c \leftarrow not\, a \;\},$$

there is a unique partial stable model, which is not stable:

$$\mathcal{STABLE}_3(\mathcal{P}_2) \;=_{\mathsf{tf}}\; \{\, \emptyset \,\},\;\; \mathcal{STABLE}_2(\mathcal{P}_2) \;=_{\mathsf{tf}}\; \emptyset.$$

3. The disjunctive database $\mathcal{P}_3 \;=\; \mathcal{P}_2 \;\cup\; \{\; a \vee b \vee c \;\}$, cf. also [14], is inconsistent w.r.t. the semantics of stable and partial stable models, i.e. $\mathcal{STABLE}_3(\mathcal{P}_3) = \mathcal{STABLE}_2(\mathcal{P}_3) = \emptyset$.

4 Evidential Stable Models

Given an atom $A = p(t_1, \ldots, t_n)$, we define the *evidential atom* $\mathcal{E}A = \mathcal{E}p(t_1, \ldots, t_n)$, where $\mathcal{E}p$ is taken to be a new predicate symbol. For a disjunction $\alpha = A_1 \vee \ldots \vee A_k$ and a conjunction $\beta = B_1 \wedge \ldots \wedge B_m$ of atoms we define $\mathcal{E}\alpha = \mathcal{E}A_1 \vee \ldots \vee \mathcal{E}A_k$ and $\mathcal{E}\beta = \mathcal{E}B_1 \wedge \ldots \wedge \mathcal{E}B_m$.

Definition 4 (Evidential Transformation).
Let $\mathcal{P}$ be a disjunctive database.

1. For a rule $r = \alpha \leftarrow \beta \wedge not \cdot \gamma \in \mathcal{P}$ we define

$$\mathcal{E}r = \alpha \vee \mathcal{E}\gamma \leftarrow \beta, \quad \mathcal{E}^2 r = \mathcal{E}\alpha \vee \mathcal{E}\gamma \leftarrow \mathcal{E}\beta.$$

2. The *evidential transformation* of $\mathcal{P}$ is

$$\mathcal{E}\mathcal{P} = \{\, \mathcal{E}r \mid r \in \mathcal{P} \,\} \cup \{\, \mathcal{E}^2 r \mid r \in \mathcal{P} \,\} \cup \{\, \mathcal{E}A \leftarrow A \mid A \in H_{B_\mathcal{P}} \,\}.$$

A rule $\mathcal{E}r$ describes that, if the positive body β of r is *true*, then this gives rise to deriving either the head α "constructively" or an *evidence* for γ "by contradiction". The rules $\mathcal{E}^2 r$ could be compared with the *normality rules* from the *autoepistemic logic of beliefs*, cf. [15], and the rules $\mathcal{E}A \leftarrow A$ with the *necessitation rules*. For an implementation, $\mathcal{E}\mathcal{P}$ can be optimized: facts $\mathcal{E}^2 r = \mathcal{E}\alpha \vee \mathcal{E}\gamma$ obtained from rules $r = \alpha \leftarrow not \cdot \gamma \in \mathcal{P}$ with an empty positive body are *redundant*, since they are implied by $\mathcal{E}r = \alpha \vee \mathcal{E}\gamma$ and the necessitation rules.

Example 5 (Evidential Transformation).
For the disjunctive database $\mathcal{P}_1$ of Example 3 we get the following $\mathcal{E}\mathcal{P}_1$, where the fact $\mathcal{E}a \vee \mathcal{E}b$ is redundant:

$$\begin{aligned}\mathcal{E}\mathcal{P}_1 = \{\ & a \vee b,\ q \vee \mathcal{E}a \leftarrow b,\ q \vee \mathcal{E}b \leftarrow a\ \} \cup \\ \{\ & \mathcal{E}a \vee \mathcal{E}b,\ \mathcal{E}q \vee \mathcal{E}a \leftarrow \mathcal{E}b,\ \mathcal{E}q \vee \mathcal{E}b \leftarrow \mathcal{E}a\ \} \cup \\ \{\ & \mathcal{E}a \leftarrow a,\ \mathcal{E}b \leftarrow b,\ \mathcal{E}q \leftarrow q\ \}.\end{aligned}$$

Every pair of total Herbrand interpretations J and K of $\mathcal{P}$ induces a total Herbrand interpretation I of $\mathcal{E}\mathcal{P}$, denoted by $J \cup \mathcal{E}K$, where for $A \in H_{B_\mathcal{P}}$:

$$(J \cup \mathcal{E}K)(A) = J(A), \quad (J \cup \mathcal{E}K)(\mathcal{E}A) = K(A).$$

Conversely, every total Herbrand interpretation I of $\mathcal{E}\mathcal{P}$ can be represented as $J \cup \mathcal{E}K$. The total Herbrand interpretation K of $\mathcal{P}$, that determines I on evidential atoms, will be denoted by $\mathcal{K}(I)$, i.e. $\mathcal{K}(J \cup \mathcal{E}K) = K$.

$\mathcal{K}(I)$ will be considered to be the total Herbrand interpretation of $\mathcal{P}$ that corresponds to I. It ignores the part J and interprets evidential atoms as (regular) atoms. For a set $\mathcal{I}$ of total Herbrand interpretations of $\mathcal{EP}$ we define $\mathcal{K}(\mathcal{I}) = \{\, \mathcal{K}(I) \mid I \in \mathcal{I}\,\}$.

Based on a similar transformation $\mathcal{FP} = \{\mathcal{E}r \mid r \in \mathcal{P}\} \cup \{\mathcal{E}A \leftarrow A \mid A \in \mathit{HB}_{\mathcal{P}}\}$, which is a subset of $\mathcal{EP}$, and the set $\mathcal{CP} = \{\leftarrow \mathcal{E}A \wedge \mathit{not}\, A \mid A \in \mathit{HB}_{\mathcal{P}}\}$ of *test constraints*, a characterization of stable models has been given by Fernández et al.:

Theorem 6 (Characterization of Stable Models, [6]).
Given a disjunctive database $\mathcal{P}$, then it holds

$$\mathcal{STABLE}_2(\mathcal{P}) \;=\; \mathcal{K}(\,\{\, I \in \mathcal{MM}_2(\mathcal{FP}) \mid I \models_2 \mathcal{CP}\,\}\,).$$

This characterization of stable models can also be proven for $\mathcal{EP}$ instead of $\mathcal{FP}$. It does not refer to the "normality rules" $\mathcal{E}^2 r$, since they are fulfilled automatically, if I strictly fulfills all of the test constraints in $\mathcal{CP}$. In our approach, however, they will be needed to guarantee that $\mathcal{K}(I)$ is a model of $\mathcal{P}$ if I is a model of $\mathcal{EP}$.

We propose the new concept of *evidential stable models*, which are minimal Herbrand models I of $\mathcal{EP}$, such that $\mathcal{K}(I) \in \mathcal{MM}_2(\mathcal{P})$. The strict requirement given by $\mathcal{CP}$ is relaxed to a *preference relation*: I' is preferred to I, if $\mathcal{V}(I') \subsetneq \mathcal{V}(I)$, where $\mathcal{V}(I)$ denotes the set of *violations* of test constraints.

Definition 7 (Evidential Stable Models).
Given a disjunctive database $\mathcal{P}$ and a set $\mathcal{I}$ of total Herbrand interpretations of $\mathcal{EP}$.

1. The set of *$\mathcal{E}$–violations* of $I \in \mathcal{I}$ is given by

   $$\mathcal{V}(I) = \{\, \mathcal{E}A \mid I \models_2 \mathcal{E}A \text{ and } I \not\models_2 A\,\},$$

   and $\mathit{min}_{\mathcal{V}}(\mathcal{I}) = \{\, I \in \mathcal{I} \mid \nexists I' \in \mathcal{I} : I \neq I' \wedge \mathcal{V}(I') \subsetneq \mathcal{V}(I)\,\}$ denotes the set of *$\mathcal{V}$–minimal* interpretations in $\mathcal{I}$.
2. The set of *evidential stable models* of $\mathcal{P}$ is

   $$\mathcal{ESTABLE}_2(\mathcal{P}) = \mathit{min}_{\mathcal{V}}(\,\{\, I \in \mathcal{MM}_2(\mathcal{EP}) \mid \mathcal{K}(I) \in \mathcal{MM}_2(\mathcal{P})\,\}\,).$$

The name evidential stable models has been chosen, since an evidential stable model I contains evidential atoms, and it can be shown that $\mathcal{K}(I)$ is a stable model of a suitably, minimally transformed database, where all atoms A, such that $\mathcal{E}A$ is an $\mathcal{E}$–violation in I, are moved from negative rule bodies to rule heads (see $\mathcal{P}'_2$ below).

Example 8 (Evidential Stable Models).

1. For the disjunctive database $\mathcal{P}_1$ we get

$$\mathcal{STABLE}_2(\mathcal{P}_1) = \mathcal{K}(\mathcal{ESTABLE}_2(\mathcal{P}_1)) \subsetneq \mathcal{MM}_2(\mathcal{P}_1).$$

2. For the disjunctive database $\mathcal{P}_2$ we get the following $\mathcal{EP}_2$, where redundant facts have been left out already:

$$\mathcal{EP}_2 \;=\; \{\, a \vee \mathcal{E}b,\; b \vee \mathcal{E}c,\; c \vee \mathcal{E}a \,\} \cup \{\, \mathcal{E}a \leftarrow a,\; \mathcal{E}b \leftarrow b,\; \mathcal{E}c \leftarrow c \,\}.$$

From $\mathcal{MM}_2(\mathcal{EP}_2)$, the first three models are $\mathcal{V}$–minimal:

$$\mathcal{MM}_2(\mathcal{EP}_2) \;=\; \{\, \{\, a, \mathcal{E}a, \mathcal{E}c \,\},\; \{\, b, \mathcal{E}b, \mathcal{E}a \,\},\; \{\, c, \mathcal{E}c, \mathcal{E}b \,\},\; \{\, \mathcal{E}a, \mathcal{E}b, \mathcal{E}c \,\} \,\}^{\diamond}.$$

E.g. for $I_1 = \{\, a,\ \mathcal{E}a,\ \mathcal{E}c \,\}^{\diamond}$ and $I_2 = \{\, \mathcal{E}a, \mathcal{E}b, \mathcal{E}c \,\}^{\diamond}$ we get $\mathcal{V}(I_1) = \{\, \mathcal{E}c \,\} \subsetneq \mathcal{V}(I_2) = \{\, \mathcal{E}a, \mathcal{E}b, \mathcal{E}c \,\}$. The meaning of I_1 is that "a" is *true*, but there is only an evidence that "c" is *true*, i.e. "c" has been derived by contradiction:

$$\mathcal{K}(\mathcal{ESTABLE}_2(\mathcal{P}_2)) \;=\; \{\, \{\, a, c \,\}, \{\, a, b \,\}, \{\, b, c \,\} \,\}^{\diamond} \;=\; \mathcal{MM}_2(\mathcal{P}_2).$$

Finally, $\mathcal{K}(I_1) = \{\, a, c \,\}^{\diamond}$ is a stable model of the suitably, minimally transformed database $\mathcal{P}_2' \;=\; \{\, a \leftarrow \mathit{not}\, b,\ b \vee c,\ c \leftarrow \mathit{not}\, a \,\}$ for I_1.
3. For the disjunctive database $\mathcal{P}_3$ we get $\mathcal{ESTABLE}_2(\mathcal{P}_3) = \mathcal{ESTABLE}_2(\mathcal{P}_2)$.

Evidential stable models provide more information than just truth, namely the information of whether an atom was derived constructively or solely by contradiction. In the models $\mathcal{K}(I)$, however, this information is ignored. The following theorem relates the evidential stable models of a disjunctive database to the minimal and the stable models.

Theorem 9 (Characterization of Evidential Stable Models).
Given a disjunctive database $\mathcal{P}$, then it holds:

1. *If $\mathcal{MM}_2(\mathcal{P}) \neq \emptyset$, then $\mathcal{K}(\mathcal{ESTABLE}_2(\mathcal{P})) \neq \emptyset$.*
2. *If $\mathcal{STABLE}_2(\mathcal{P}) \neq \emptyset$, then $\mathcal{K}(\mathcal{ESTABLE}_2(\mathcal{P})) = \mathcal{STABLE}_2(\mathcal{P})$.*
3. *$\mathcal{STABLE}_2(\mathcal{P}) \subseteq \mathcal{K}(\mathcal{ESTABLE}_2(\mathcal{P})) \subseteq \mathcal{MM}_2(\mathcal{P})$.*

Proof.

1. Assume $\mathcal{MM}_2(\mathcal{P}) \neq \emptyset$. Every minimal model $M \in \mathcal{MM}_2(\mathcal{P})$ of $\mathcal{P}$ induces a Herbrand interpretation $M \cup \mathcal{E}M$, which obviously is a model of $\mathcal{EP}$. Thus, there exists a minimal model $I \in \mathcal{MM}_2(\mathcal{EP})$, such that

$I = J \cup \mathcal{E}K \subseteq M \cup \mathcal{E}M$.[4] Since $\mathcal{E}K$ must be a model of $\mathcal{P}' = \{\, \mathcal{E}^2 r \,|\, r \in \mathcal{P} \,\}$, and $\mathcal{E}M$ is a minimal model of $\mathcal{P}'$, and $\mathcal{E}K \subseteq \mathcal{E}M$, we get that $K = M$. Thus, $I = J \cup \mathcal{E}M \in \mathcal{MM}_2(\mathcal{EP})$ and $\mathcal{K}(I) = M \in \mathcal{MM}_2(\mathcal{P})$. This means that the set of interpretations which we minimize is not empty, i.e. $\mathcal{K}(\mathcal{ESTABLE}_2(\mathcal{P})) \neq \emptyset$.

2. The test condition (A) $I \models_2 \mathcal{CP}$ is equivalent to (B) $\mathcal{V}(I) = \emptyset$. Thus, for a stable model K of $\mathcal{P}$, the Herbrand interpretation $I = K \cup \mathcal{E}K$ of $\mathcal{EP}$ always is minimal w.r.t. violation, and thus I is evidential stable. Moreover, if there exists any stable model of $\mathcal{P}$, then all evidential stable models I of $\mathcal{P}$ must fulfill (B), i.e. they are of the form $I = K \cup \mathcal{E}K$, such that $K \in \mathcal{STABLE}_2(\mathcal{P})$.
3. First, the inclusion $\mathcal{K}(\mathcal{ESTABLE}_2(\mathcal{P})) \subseteq \mathcal{MM}_2(\mathcal{P})$ holds by definition. Secondly, the inclusion $\mathcal{STABLE}_2(\mathcal{P}) \subseteq \mathcal{K}(\mathcal{ESTABLE}_2(\mathcal{P}))$ is an immediate consequence of (ii).

$\square$

Note that the concept of evidential stable models cannot be lifted to the three–valued case by simply taking partial minimal models of $\mathcal{EP}$. The reason is that for positive–disjunctive databases (without default negation), such as $\mathcal{EP}$, the partial minimal models coincide with the minimal models.

5 Annotation of Databases and Partial Stable Models

We will use a special concept of annotating disjunctive rules, which encodes the condition that partial Herbrand models have to fulfill in terms of their tu–representation. Given a truth value $\mathsf{v} \in \{\, \mathsf{t}, \mathsf{u} \,\}$, for a disjunction $\alpha = A_1 \vee \ldots \vee A_k$ and a conjunction $\beta = B_1 \wedge \ldots \wedge B_m$ of atoms we define $\alpha^{\mathsf{v}} = A_1^{\mathsf{v}} \vee \ldots \vee A_k^{\mathsf{v}}$ and $\beta^{\mathsf{v}} = B_1^{\mathsf{v}} \wedge \ldots \wedge B_m^{\mathsf{v}}$.

Definition 10 (Annotation of Databases).

1. For a disjunctive rule $r = \alpha \leftarrow \beta \wedge not \cdot \gamma$ we define the *annotated rules*
$$r^{\mathsf{t}} = \alpha^{\mathsf{t}} \leftarrow \beta^{\mathsf{t}} \wedge not \cdot \gamma^{\mathsf{u}}, \quad r^{\mathsf{u}} = \alpha^{\mathsf{u}} \leftarrow \beta^{\mathsf{u}} \wedge not \cdot \gamma^{\mathsf{t}}.$$
2. For a disjunctive database $\mathcal{P}$ we define $\mathcal{P}^{\mathsf{t}} = \{r^{\mathsf{t}} | r \in \mathcal{P}\}$, $\mathcal{P}^{\mathsf{u}} = \{r^{\mathsf{u}} | r \in \mathcal{P}\}$, and the *annotated database* $\mathcal{P}^{\mathsf{tu}} = \mathcal{P}^{\mathsf{t}} \cup \mathcal{P}^{\mathsf{u}} \cup \{A^{\mathsf{u}} \leftarrow A^{\mathsf{t}} | A \in H_{B_\mathcal{P}}\}$.

[4] Note that in this proof, total Herbrand interpretations are treated as their sets of *true* atoms, and they are compared like sets.

Example 11 (Annotation of Databases).
For the disjunctive database $\mathcal{P}_1$ of Example 3 we get

$$\begin{aligned}\mathcal{P}_1^{\mathsf{t}} &= \{\ a^{\mathsf{t}} \vee b^{\mathsf{t}},\ q^{\mathsf{t}} \leftarrow b^{\mathsf{t}} \wedge not\, a^{\mathsf{u}},\ q^{\mathsf{t}} \leftarrow a^{\mathsf{t}} \wedge not\, b^{\mathsf{u}}\ \},\\ \mathcal{P}_1^{\mathsf{u}} &= \{\ a^{\mathsf{u}} \vee b^{\mathsf{u}},\ q^{\mathsf{u}} \leftarrow b^{\mathsf{u}} \wedge not\, a^{\mathsf{t}},\ a^{\mathsf{u}} \leftarrow q^{\mathsf{u}} \wedge not\, b^{\mathsf{t}}\ \},\\ \mathcal{P}_1^{\mathsf{tu}} &= \mathcal{P}_1^{\mathsf{t}} \cup \mathcal{P}_1^{\mathsf{u}} \cup \{\ a^{\mathsf{u}} \leftarrow a^{\mathsf{t}},\ b^{\mathsf{u}} \leftarrow b^{\mathsf{t}},\ q^{\mathsf{u}} \leftarrow q^{\mathsf{t}}\ \}.\end{aligned}$$

The construction of $\mathcal{P}^{\mathsf{tu}}$ is motivated by condition (3), which every partial Herbrand model M of a ground rule $r = \alpha \leftarrow \beta \wedge not \cdot \gamma$ must fulfill. Condition (3) can be encoded in the annotated rules r^{t} and r^{u}, since it is equivalent to the following:

$$\begin{aligned}&(\ (\ M(\beta) \geq_{\mathsf{t}} \mathsf{t}\ \wedge\ \neg(M(\gamma) \geq_{\mathsf{t}} \mathsf{u}\ )\ \Longrightarrow\ M(\alpha) \geq_{\mathsf{t}} \mathsf{t}\ )\ \wedge\\ &(\ (\ M(\beta) \geq_{\mathsf{t}} \mathsf{u}\ \wedge\ \neg(M(\gamma) \geq_{\mathsf{t}} \mathsf{t}\ )\ \Longrightarrow\ M(\alpha) \geq_{\mathsf{t}} \mathsf{u}\ ).\end{aligned}$$

The rules $A^{\mathsf{u}} \leftarrow A^{\mathsf{t}}$ are due to the fact that we want to perceive the Herbrand models of $\mathcal{P}^{\mathsf{tu}}$ as Herbrand models of $\mathcal{P}$ in tu–representation.

Properties of the Annotated Database

It can be shown that annotation preserves stratification: Given a disjunctive database $\mathcal{P}$, the annotated disjunctive database $\mathcal{P}^{\mathsf{tu}}$ is *stratified* if and only if $\mathcal{P}$ is stratified. Based on this, one can give an alternative proof of the well–known fact (see [14]) that the partial stable models of a stratified–disjunctive database $\mathcal{P}$ coincide with the perfect models of $\mathcal{P}$. This fact implies in particular that the partial stable models of a stratified–disjunctive database $\mathcal{P}$ are total.

The annotated database $\mathcal{P}^{\mathsf{tu}}$ can be represented as a database over two predicate symbols "t" and "u". Then, annotated atoms A^{t} and A^{u} in rules can be represented by atoms $\mathsf{t}(A)$ and $\mathsf{u}(A)$, respectively, where "A" is seen as a term now. In this representation the (possibly infinite) set $\{\ A^{\mathsf{u}} \leftarrow A^{\mathsf{t}} \mid A \in H\!B_{\mathcal{P}}\ \}$ of rules can simply be represented by one rule $\mathsf{u}(X) \leftarrow \mathsf{t}(X)$, where "$X$" is a variable symbol for atoms. Then $\mathcal{P}^{\mathsf{tu}}$ has the size of $2 \cdot n + 1$ rules if $\mathcal{P}$ consists of n rules. This compact representation has been used for an implementation dealing with $\mathcal{P}^{\mathsf{tu}}$.

Characterization of Partial Minimal and Partial Stable Models

The following theorem shows that the partial stable models of a disjunctive database $\mathcal{P}$ correspond to the total stable models of the annotated database $\mathcal{P}^{\mathsf{tu}}$. For any total Herbrand interpretation I of $\mathcal{P}^{\mathsf{tu}}$ we introduce the notation I^{∇} for the partial Herbrand interpretation of $\mathcal{P}$ that is induced by I, i.e. for $A \in H\!B_{\mathcal{P}}$

$$I^{\nabla}(A) = \begin{cases} \mathsf{t} \text{ if } I(A^{\mathsf{t}}) = \mathsf{t} \\ \mathsf{u} \text{ if } I(A^{\mathsf{u}}) = \mathsf{t} \text{ and } I(A^{\mathsf{t}}) = \mathsf{f} \\ \mathsf{f} \text{ if } I(A^{\mathsf{u}}) = \mathsf{f} \end{cases}$$

For a collection $\mathcal{I}$ of total Herbrand interpretations of $\mathcal{P}^{\mathsf{tu}}$, let $\mathcal{I}^{\nabla} = \{\, I^{\nabla} \mid I \in \mathcal{I} \,\}$.

Theorem 12 (Partial Minimal and Partial Stable Models, [17]). *Given a disjunctive database $\mathcal{P}$, then*

$$\mathcal{STABLE}_3(\mathcal{P}) = \mathcal{STABLE}_2(\mathcal{P}^{\mathsf{tu}})^{\nabla}, \quad \textit{and} \quad \mathcal{MM}_3(\mathcal{P}) = \mathcal{MM}_2(\mathcal{P}^{\mathsf{tu}})^{\nabla}.$$

Example 13 (Partial Stable Models).

1. For the disjunctive database $\mathcal{P}_1$ of Example 3, whose annotated database $\mathcal{P}_1^{\mathsf{tu}}$ has been given in Example 11, we get

$$\mathcal{STABLE}_3(\mathcal{P}_1) \;=_{\mathsf{tu}}\; \{\, \{\, a^{\mathsf{t}}, a^{\mathsf{u}}, q^{\mathsf{t}}, q^{\mathsf{u}} \,\},\ \{\, b^{\mathsf{t}}, b^{\mathsf{u}}, q^{\mathsf{t}}, q^{\mathsf{u}} \,\} \,\}.$$

2. For the disjunctive database $\mathcal{P}_2$ of Example 3, we get the annotated database

$$\begin{aligned} \mathcal{P}_2^{\mathsf{tu}} = \{\ & a^{\mathsf{t}} \leftarrow \mathit{not}\, b^{\mathsf{u}},\ b^{\mathsf{t}} \leftarrow \mathit{not}\, c^{\mathsf{u}},\ c^{\mathsf{t}} \leftarrow \mathit{not}\, a^{\mathsf{u}}, \\ & a^{\mathsf{u}} \leftarrow \mathit{not}\, b^{\mathsf{t}},\ b^{\mathsf{u}} \leftarrow \mathit{not}\, c^{\mathsf{t}},\ c^{\mathsf{u}} \leftarrow \mathit{not}\, a^{\mathsf{t}}, \\ & a^{\mathsf{u}} \leftarrow a^{\mathsf{t}},\ b^{\mathsf{u}} \leftarrow b^{\mathsf{t}},\ c^{\mathsf{u}} \leftarrow c^{\mathsf{t}}\ \}. \end{aligned}$$

   Thus, we get

$$\mathcal{STABLE}_3(\mathcal{P}_2) \;=_{\mathsf{tu}}\; \{\, \{\, a^{\mathsf{u}}, b^{\mathsf{u}}, c^{\mathsf{u}} \,\} \,\}.$$

6 Partial Evidential Stable Models

For defining partial evidential stable models we can use the techniques described in the previous two sections. Partial evidential stable models are defined based on the evidential stable models of the tu–transformation of the database, i.e. Theorem 12 for partial stable models motivates the following definition.

Definition 14 (Partial Evidential Stable Models).
The set of *partial evidential stable models* of a disjunctive database $\mathcal{P}$ is

$$\mathcal{ESTABLE}_3(\mathcal{P}) = \mathcal{ESTABLE}_2(\mathcal{P}^{\mathsf{tu}})^{\nabla}.$$

Thus, for constructing partial evidential stable models we need the evidential transformation $\mathcal{E}(\mathcal{P}^{\mathsf{tu}})$ of the tu–transformation of $\mathcal{P}$. Here, we identify evidential atoms $\mathcal{E}(A^{\mathsf{v}})$ of $\mathcal{E}(\mathcal{P}^{\mathsf{tu}})$ with annotated atoms $(\mathcal{E}A)^{\mathsf{v}}$ of $(\mathcal{E}\mathcal{P})^{\mathsf{tu}}$. [5] Note that in an implementation, $\mathcal{E}(\mathcal{P}^{\mathsf{tu}})$ can be represented compactly as a disjunctive database over four predicate symbols "t, u, $\mathcal{E}\mathsf{t}$, $\mathcal{E}\mathsf{u}$".

Example 15 (Partial Evidential Stable Models).

1. For the disjunctive database $\mathcal{P}_2$ of Example 3 we get

$$\mathcal{K}(\mathcal{ESTABLE}_3(\mathcal{P}_2)) = \mathcal{STABLE}_3(\mathcal{P}_2).$$

2. For the disjunctive database $\mathcal{P}_3$ of Example 3 we get $\mathcal{E}(\mathcal{P}_3^{\mathsf{tu}})$, where the redundant facts $\mathcal{E}^2 r$ for rules $r \in \mathcal{P}_3^{\mathsf{tu}}$ with empty positive bodies have been left out:

$$\begin{aligned} \mathcal{E}(\mathcal{P}_3^{\mathsf{tu}}) = \{\ & a^{\mathsf{t}} \vee b^{\mathsf{t}} \vee c^{\mathsf{t}},\ a^{\mathsf{t}} \vee \mathcal{E}b^{\mathsf{u}},\ b^{\mathsf{t}} \vee \mathcal{E}c^{\mathsf{u}},\ c^{\mathsf{t}} \vee \mathcal{E}a^{\mathsf{u}}\ \} \cup \\ & \{\ a^{\mathsf{u}} \vee b^{\mathsf{u}} \vee c^{\mathsf{u}},\ a^{\mathsf{u}} \vee \mathcal{E}b^{\mathsf{t}},\ b^{\mathsf{u}} \vee \mathcal{E}c^{\mathsf{t}},\ c^{\mathsf{u}} \vee \mathcal{E}a^{\mathsf{t}}\ \} \cup \\ & \{\ A^{\mathsf{u}} \leftarrow A^{\mathsf{t}},\ \mathcal{E}A^{\mathsf{u}} \leftarrow \mathcal{E}A^{\mathsf{t}},\ \mathcal{E}A^{\mathsf{t}} \leftarrow A^{\mathsf{t}},\ \mathcal{E}A^{\mathsf{u}} \leftarrow A^{\mathsf{u}} \mid A \in H_{B_{\mathcal{P}_3}}\ \}. \end{aligned}$$

We get the set $\mathcal{MM}_2(\mathcal{E}(\mathcal{P}_3^{\mathsf{tu}})) = \{I_1(a,b,c), I_2(a,b,c), I_1(c,a,b), I_2(c,a,b), I_1(b,c,a), I_2(b,c,a)\}$ of minimal models, where for $A, B, C \in H_{B_{\mathcal{P}_3}}$:

$$\begin{aligned} I_1(A,B,C) &= \{\ A^{\mathsf{t}},\ \mathcal{E}A^{\mathsf{t}},\ A^{\mathsf{u}},\ \mathcal{E}A^{\mathsf{u}},\ B^{\mathsf{u}},\ \mathcal{E}B^{\mathsf{u}},\ \mathcal{E}C^{\mathsf{u}}\ \}^{\diamond}, \\ I_2(A,B,C) &= \{\ A^{\mathsf{t}},\ \mathcal{E}A^{\mathsf{t}},\ A^{\mathsf{u}},\ \mathcal{E}A^{\mathsf{u}},\ \mathcal{E}C^{\mathsf{t}},\ \mathcal{E}C^{\mathsf{u}}\ \}^{\diamond}. \end{aligned}$$

Here, $\mathcal{V}(\ I_1(A,B,C)\ ) = \{\ \mathcal{E}C^{\mathsf{u}}\ \} \subsetneq \mathcal{V}(\ I_2(A,B,C)\ ) = \{\ \mathcal{E}C^{\mathsf{t}},\ \mathcal{E}C^{\mathsf{u}}\ \}$. Thus, we get

$$\mathcal{K}(\mathcal{ESTABLE}_3(\mathcal{P}_3)) =_{\mathsf{tu}} \{\{\ a^{\mathsf{t}}, a^{\mathsf{u}}, b^{\mathsf{u}}, c^{\mathsf{u}}\}, \{\ c^{\mathsf{t}}, c^{\mathsf{u}}, a^{\mathsf{u}}, b^{\mathsf{u}}\}, \{\ b^{\mathsf{t}}, b^{\mathsf{u}}, c^{\mathsf{u}}, a^{\mathsf{u}}\}\}.$$

The following theorem relates the partial evidential stable models of a disjunctive database to the partial minimal and the partial stable models. It is a consequence of Theorems 9 and 12, and Definition 14.

Theorem 16 (Characterization of Partial Evidential Stable Models). *Given a disjunctive database $\mathcal{P}$, then it holds:*

1. *If $\mathcal{MM}_3(\mathcal{P}) \neq \emptyset$, then $\mathcal{K}(\mathcal{ESTABLE}_3(\mathcal{P})) \neq \emptyset$.*
2. *If $\mathcal{STABLE}_3(\mathcal{P}) \neq \emptyset$, then $\mathcal{K}(\mathcal{ESTABLE}_3(\mathcal{P})) = \mathcal{STABLE}_3(\mathcal{P})$.*
3. *$\mathcal{STABLE}_3(\mathcal{P}) \subseteq \mathcal{K}(\mathcal{ESTABLE}_3(\mathcal{P})) \subseteq \mathcal{MM}_3(\mathcal{P})$.*

[5] But note that even with this identification, $\mathcal{E}(\mathcal{P}^{\mathsf{tu}})$ and $(\mathcal{E}\mathcal{P})^{\mathsf{tu}}$ are different if there is negation in $\mathcal{P}$.

Partial evidential stable models provide a "consistent extension" of the well–founded semantics from normal databases to disjunctive databases $\mathcal{P}$, namely the set $\mathcal{K}(\mathcal{ESTABLE}_3(\mathcal{P}))$ of partial Herbrand interpretations. For stratified–disjunctive databases, this extension coincides with the perfect models if there exist perfect models of $\mathcal{P}$ (which is for instance guaranteed for databases without denial rules).

7 Comparison with Other Approaches

Regular Models

The semantics of *regular models* has been introduced by You and Yuan [21,22]: A regular model M is a justifiable model[6] which has a minimal set M^{u} of undefined atoms.

For a large class of disjunctive databases including all examples considered so far in this paper, the partial evidential stable models coincide with the regular models. But there are example databases for showing that the regular models do not always coincide with the (partial) evidential stable models, neither with the (partial) stable models [5].

L–Stable and M–Stable Models

Eiter, Leone and Sacca [5] have investigated several interesting subsets of the set of partial stable models, like the *least undefined* and the *maximal partial stable models*, which they call *L–stable* and *M–stable models*, respectively.[7] For normal databases (without disjunctions), the M–stable models coincide with the regular models of You and Yuan.

Since L–stable and M–stable models always are partial stable models, they do not give a consistent interpretation for databases without partial stable models (like $\mathcal{P}_3$), while (partial) evidential stable models do so (if a database is logically consistent).

It tuns out that the concepts of minimizing undefinedness and maximizing knowledge can be combined with our concept of partial evidential stable models. That is, since $\mathcal{K}(\mathcal{ESTABLE}_3(\mathcal{P}))$ is a set of partial Herbrand interpretations, it makes sense to look for the least undefined elements in that set, and also for the elements with maximal knowledge.

[6] A justifiable model M is a minimal model of a variant of the three–valued Gelfond–Lifschitz transformation, where only those rules are selected whose negative bodies are *true* w.r.t. M (rather than *undefined* or *false*).

[7] Within the set of all partial stable models, an L–stable model M must have a minimal set M^{u} of undefined atoms, whereas an M–stable model must have a maximal set M^{tf} of knowledge.

Disjunctive Well–Founded Semantics

For achieving a consistent interpretation of disjunctive databases, also several types of *well–founded semantics* have been proposed, cf. [1,2,12]. It seems that the semantics of evidential stable models are stronger than the semantics D-WFS of Brass and Dix [2], and still they are consistent.

8 Abstract Properties of the Evidential Semantics

In the following we will give a brief analysis of the two evidential semantics according to several *abstract properties* of semantics [3]. A summary is given by Figure 3.

First, both evidential semantics have the property of *independence*. They even have the stronger property of *modularity*. This means that if a database can be decomposed into separate components that do not have any atoms in common, then the (partial) evidential stable models can be computed on the components separately. As a consequence, only on those parts of a disjunctive database that are inconsistent w.r.t. (partial) stable models we have to compute (partial) evidential stable models. On the consistent part of a database – which usually will be the main part – we can compute the basic (partial) stable model semantics.

Secondly, evidential stable model semantics is *supraclassical*, i.e. it derives more consequences – by sceptical reasoning – than classical logic, since evidential stable models are also minimal models, cf. Theorems 9. On the other hand, partial evidential stable model semantics is not supraclassical, since for normal databases it is equivalent to the well–founded semantics, which is not supraclassical.

Thirdly, both evidential semantics allow for the *elimination of tautologies*. The semantics of evidential stable models also allows for the *elimination of contradictions*, whereas partial evidential stable models do not. This well matches with the conjecture of [3] that elimination of contradictions should be given up for (three–valued) semantics of general – i.e. non–stratified – disjunctive databases.

Fourth, both evidential semantics do not satisfy the *generalized property of partial evaluation* (GPPE). For partial evidential stable models this can be shown by an example that originally was used for showing that partial stable models do not satisfy GPPE [17]. Given the fact that evidential stable models satisfy elimination of tautologies and elimination of contradictions, using a theorem of [3] it can be concluded that evidential stable models cannot satisfy GPPE – the reason is that otherwise the set of evidential stable models always would have to be a subset of the set of stable models.

	Taut.	*Contr.*	*GPPE*	*Indep.*	*Supra.*
$\mathcal{STABLE}_2$	+	+	+	—	+
$\mathcal{STABLE}_3$	+	—	—	—	—
$\mathcal{ESTABLE}_2$	+	+	—	+	+
$\mathcal{ESTABLE}_3$	+	—	—	+	—
D-WFS	+	—	+	+	—
Regular	+	+	—	—	—

Fig. 3. Abstract properties of semantics

Finally, note that as a consequence of Theorem 12, the technique of partial evaluation can still be applied to the tu–transformation of a database – rather than the database itself – for computing its partial stable models, and thus also the superset of partial evidential stable models.

9 Conclusions

The evidential semantics presented in this paper can be seen as a special case of the *general framework for revising non–monotonic theories* that was introduced by Witteveen and van der Hoek in [20]. In that case, the *intended models* would be the (partial) stable models, and the *backup models* – from among which the models are chosen if there exist no intended models – would be the (partial) minimal models, cf. Theorems 9 and 16.

The computation of (partial) evidential stable model semantics has been implemented within the system DisLog for efficient reasoning in disjunctive databases, cf. [18].

The detailed investigation of the properties and possible implementations of evidential stable models and partial evidential stable models will be the subject of future work.

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