The Relationship between KLM and MAK Models for Nonmonotonic Inference Operations

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Abstract

The purpose of this note is to make quite clear the relationship between two variants of the general notion of a preferential model for nonmonotonic inference: the models of Kraus, Lehmann and Magidor (KLM models) and those of Makinson (MAK models).

On the one hand, we introduce the notion of the *core* of a KLM model, which suffices to fully determine the associated nonmonotonic inference relation. On the other hand, we slightly *amplify* MAK models with a monotonic consequence operation as additional ingredient.

We give two equivalent characterizations of the cores of KLM models: they are precisely the amplified MAK models whose satisfaction relation

- may be expressed as the intersection of some non-empty family of satisfaction relations each of which is classically well-behaved,

or

- satisfies certain syntactic conditions.

This gives corollary characterizations of certain particular classes of KLM models, notably those that are (in their terminology) *cumulative* and more specifically those they call *preferential*.

1 Introduction

Preferential models were introduced by Shoham ([Sho86], [Sho88]) as a generalization of the constructions used in circumscription. Given a first-order language and a partial ordering $\prec$ of all the classical models of that language, one may define an inference relation $\mathrel{|\!\sim}_{\prec}$ between sets Φ of formulae and individual formulae ϕ by the rule:

$\Phi \mathrel{|\!\sim}_{\prec} \phi$ iff ϕ is satisfied in every classical model of Φ that is minimal (under $\prec$) among the models that satisfy Φ.

This notion was presented in rather more general fashion by [Mak89] and [KLM90]. Instead of restricting attention to classical models, they consider arbitrary objects (which, to avoid confusion, are usually called *states*). The ordering relation $\prec$ holds between such states and is taken as arbitrary, eventually satisfying certain conditions weaker than partial order.

Whilst the formulations of [Mak89] and of [KLM90] are evidently very close to each other, they are not quite the same. The most conspicuous difference is that the latter makes use of *worlds* as well as *states*, and a *labelling function* that associates a non-empty set of worlds with each state.

Our main idea is to show that one can swap KLM's family of labelled sets of worlds for a background consequence operation and a single satisfaction relation on states, both satisfying appropriate constraints.

We begin in section 2 by recalling the definition of a Makinson model (MAK model), and adding a suitably constrained background consequence operation into its structure to form an *amplified* MAK model. We then recall in section 3 the definition of a Kraus, Lehmann and Magidor model (KLM model), and the special subclasses of *cumulative* and *preferential* KLM models. We also define the *core* of any such model, which suffices to determine the induced nonmonotonic operation.

With this background, we are then able to prove in section 4 our principal results, which may be stated as follows:

Theorem
Consider any language $\mathcal{L}$, closed under classical propositional connectives, and any structure $\mathcal{S} = (S, \models\!\equiv, \prec, Cn)$. Then the following three conditions are equivalent:

(1) $\mathcal{S}$ is the core of a KLM model for $\mathcal{L}$;

(2) $\mathcal{S}$ is an amplified MAK model whose satisfaction relation $\models\!\equiv$ may be expressed as the intersection of a non-empty family of satisfaction relations $\models\!\equiv_i$, each of which is classically well-behaved;

(3) $\mathcal{S}$ is an amplified MAK model whose satisfaction relation $\models\!\equiv$ satisfies the following conditions for all $s \in S$:

(a) $s \models\!\equiv \alpha$ and $s \models\!\equiv \beta$ implies $s \models\!\equiv \alpha \wedge \beta$,

(b) There are formulae α and β with $s \models\!\equiv \alpha$ and not $s \models\!\equiv \beta$.

Corollary
Consider any language $\mathcal{L}$, closed under classical propositional connectives, and any structure $\mathcal{S} = (S, \models\!\equiv, \prec, Cn)$. Then the following two conditions are equivalent:

1. *$\mathcal{S}$ is the core of a preferential KLM model for $\mathcal{L}$;*

2. *$\mathcal{S}$ is an amplified MAK model whose satisfaction relation $\models\!\equiv$ is classically well-behaved and whose preference relation is transitive and irreflexive.*

Our presentation is self-contained, but some familiarity with [Mak89] or [KLM90] will help to motivate the reader.

2 MAK and Amplified MAK Models

MAK models are defined for any language $\mathcal{L}$ regardless of whether it is closed under classical propositional connectives or not. We write Fml for the set of all formulae of the language $\mathcal{L}$.

Definition 2.1 (MAK model)
A MAK model $\mathcal{S}$ for the language $\mathcal{L}$ is a triple $\mathcal{S} = (S, \models\!\equiv, \prec)$ where:

1. *S is an arbitrary set, called the set of* states*;*

2. *$\models\!\equiv$ is any subset of $S \times Fml$, called the* satisfaction relation*;*

3. *$\prec$ is any binary relation on $\mathcal{S}$, called the* preference relation.

For any state s and any formula $\phi \in Fml$, we say that s preferentially satisfies ϕ, *and we write:*

$$s \models\!\equiv_{\prec} \phi, \text{ iff } s \models\!\equiv \phi \text{ and there is no } s' \in S \text{ with } s' \prec s \text{ and } s' \models\!\equiv \phi.$$

Note that the elements of S are arbitrary objects. Likewise $\prec$ is an arbitrary relation on S; no conditions are imposed, at least initially. Again, $\models\!\equiv$ is an arbitrary relation between S and Fml. Even if Fml is closed under classical propositional connectives, it is not required that $\models\!\equiv$ respects them, i.e. that $s \models\!\equiv \alpha \wedge \beta$ iff ($s \models\!\equiv \alpha$ and $s \models\!\equiv \beta$) and $s \models\!\equiv \neg\alpha$ iff not $s \models\!\equiv \alpha$. We use the notation $\models\!\equiv$ with three bars rather than the more familiar $\models$ because we shall shortly need to make a distinction. We write $s \models\!\equiv \Phi$ to mean that $s \models\!\equiv \phi$ for all $\phi \in \Phi$.

Each MAK model then generates a nonmonotonic inference operation by using Shoham's rule, as follows:

Definition 2.2 (Induced nonmonotonic inference operation $\mathrm{C}_{\mathcal{S}}$)
For any MAK model $\mathcal{S}$ we define $C_{\mathcal{S}}$ by the rule:

$$\psi \in C_{\mathcal{S}}(\Phi) \text{ iff } s \models\!\equiv \psi \text{ for all } s \text{ in } S \text{ with } s \models\!\equiv_{\prec} \Phi.$$

When there is no danger of confusion, we shall sometimes drop the subscript from $\mathrm{C}_{\mathcal{S}}$, writing simply $\psi \in C(\Phi)$. It is often convenient to express C as a relation $\mid\!\sim$, writing

$$\Phi \mid\!\sim \psi \text{ iff } \psi \in C(\Phi).$$

We recall that a consequence operation (in the sense of Tarski) is an operation

$$Cn : 2^{Fml} \longrightarrow 2^{Fml} \text{ with } \Phi \subseteq Cn(\Phi) = Cn(Cn(\Phi)) \subseteq Cn(\Phi')$$

for all $\Phi \subseteq \Phi' \subseteq Fml$. Each MAK model $\mathcal{S}$ also generates such a consequence operation $Cn_{\mathcal{S}}$ in a natural way, by putting

$$\psi \in Cn_{\mathcal{S}}(\Phi) \text{ iff } s \models\!\equiv \psi \text{ for every } s \in S \text{ with } s \models\!\equiv \Phi.$$

Noting that $\models\!\equiv_{\prec}$ is a subrelation of $\models\!\equiv$, we clearly have $Cn_{\mathcal{S}}(\Phi) \subseteq C_{\mathcal{S}}(\Phi)$ for all $\Phi \subseteq Fml$, i.e. writing it more briefly, $Cn_{\mathcal{S}} \leq C_{\mathcal{S}}$.

When the language $\mathcal{L}$ contains, and is closed under classical propositional connectives, so that the operation of classical consequence, denoted by Cn_0, may be applied to it, it is useful to add to the structure of a MAK model an additional ingredient:

Definition 2.3 (Amplified MAK model)
An amplified MAK model for a language $\mathcal{L}$ closed under classical propositional connectives is a quadruple $\mathcal{S} = (S, \models\!\equiv, \prec, Cn)$ where:

1. *$(S, \models\!\equiv, \prec)$ is a MAK model;*
2. *Cn is a consequence operation such that*
   - (α) *Cn is compact, i.e. whenever $\psi \in Cn(\Phi)$ then $\psi \in Cn(\Phi_0)$ for some finite $\Phi_0 \subseteq \Phi$,*
   - (β) *$Cn_0 \leq Cn \leq Cn_{\mathcal{S}}$,*
   - (γ) *Cn satisfies "disjunction in the antecedent", i.e. for all $\Phi \cup \{\alpha, \beta, \gamma\} \subseteq Fml$:*

$$\text{If } \gamma \in Cn(\Phi \cup \{\alpha\}) \cap Cn(\Phi \cup \{\beta\}) \text{ then } \gamma \in Cn(\Phi \cup \{\alpha \vee \beta\}).$$

Of course, in the particular case that Cn is chosen to be Cn_0, conditions (α) and (γ) are automatically satisfied. Clearly, a MAK model $\mathcal{S} = (S, \models\!\equiv, \prec)$ has an amplification $(S, \models\!\equiv, \prec)$ iff $Cn_0 \leq Cn_{\mathcal{S}}$: amplifications of $\mathcal{S}$ need not in general be unique.

The following well-known lemma will be useful for the proof of our theorem:

Lemma 2.4
Let Cn be any compact consequence operation with $Cn_0 \leq Cn$. Then:

(a) Every Cn-consistent set can be extended to one which is maximally Cn-consistent;

(b) The following two conditions are equivalent:

(i) Cn satisfies "disjunction in the antecedent",

(ii) If $\alpha \notin Cn(\Phi)$ then there is a Cn-consistent extension of Φ containing $\neg\alpha$.

We recall the easy proof. For (a) the compactness assumption on Cn implies, by Zorn's Lemma, that any Cn-consistent set can be extended to a maximally Cn-consistent one. To prove (b), let Cn satisfy "disjunction in the antecedent". If there is no Cn-consistent extension of Φ containing $\neg\alpha$, then $Cn(\Phi \cup \{\neg\alpha\})$ is inconsistent, thus $\alpha \in Cn(\Phi \cup \{\neg\alpha\})$ (because $Cn_0 \leq Cn$). Since obviously $\alpha \in Cn(\Phi \cup \{\alpha\})$ (again because $Cn_0 \leq Cn$) we get by disjunction in the antecedent

$$\alpha \in Cn(\Phi \cup \{\alpha \vee \neg\alpha\}) = Cn(\Phi)$$

and we are done.

For the converse, let $\gamma \in Cn(\Phi \cup \{\alpha\}) \cap Cn(\Phi \cup \{\beta\})$. If $\gamma \notin Cn(\Phi \cup \{\alpha \vee \beta\})$ then there is a Cn-consistent extension of $\Phi \cup \{\alpha \vee \beta\}$ containing $\neg\gamma$. This extension can be extended (using (a)) to a maximal Cn-consistent set M_{max}. Since M_{max} is maximally Cn-consistent with $Cn_0 \leq Cn$, and $\alpha \vee \beta \in M_{max}$, we have using (ii) that $\alpha \in M_{max}$ or $\beta \in M_{max}$. In both cases we have $\gamma \in M_{max}$ which is a contradiction.

3 KLM models and their Cores

KLM models, due to [KLM90], are defined only for languages $\mathcal{L}$ that are closed under classical propositional connectives:

Definition 3.1 (KLM model)
A KLM model is a quadruple $\mathcal{S} = (S, W, l, \prec)$ where:

1. *S is an arbitrary set, called the set of* states.

2. *W is a non-empty set of classical propositional models for $\mathcal{L}$, called the set of* worlds. *These worlds can equivalently be viewed as subsets of Fml, maximally consistent under Cn_0. It is not assumed that all the classical models for $\mathcal{L}$ are in W. However it is assumed that W is* compact *in the sense that there is a $w \in W$ with $\Phi \subseteq w$ whenever for every finite $\Phi_0 \subseteq \Phi$ there is a $w_0 \in W$ with $\Phi_0 \subseteq w_0$.*

3. *$l : S \longrightarrow 2^W$ is a function associating a non-empty set of worlds with each state. It is called the* labelling function.

4. *$\prec$ is any binary relation over S, called the* preference relation.

Note, that neither the set W of worlds *nor the labelling function l occur in MAK models. We write $\models$ for the satisfaction relation determined by W, i.e. $w \models \phi$ for $w(\phi) = 1$ or $\phi \in w$ according to the representation of a world that is used.*

From the satisfaction relation $\models$ on worlds, KLM define a derived *satisfaction $\models\!\!\!\equiv$ on states by the rule*

$$s \models\!\!\!\equiv \phi \quad \textit{iff} \quad w \models \phi \textit{ for all } w \textit{ in } l(s).$$

Given this $\models\!\!\!\equiv$, they define $s \models\!\!\!\equiv_{\prec} \phi$ in exactly the same way as for MAK models, namely

$$s \models\!\!\!\equiv_{\prec} \phi \quad \textit{iff} \quad s \models\!\!\!\equiv \phi \textit{ and there is no } s' \in S \textit{ with } s' \prec s \textit{ and } s' \models\!\!\!\equiv \phi.$$

The induced nonmonotonic inference operation is also defined as in the context of MAK models (definition 2.2), eg in the $\mid\sim$ -formulation:

Definition 3.2 (Induced nonmonotonic inference operation $\mid\sim_{\mathcal{S}}$)
For any KLM model $\mathcal{S}$ we define $\mid\sim_{\mathcal{S}}$ by the rule:

$$\Phi \mid\sim_{\mathcal{S}} \psi \quad \textit{iff} \quad s \models\!\!\equiv \psi \textit{ for all } s \textit{ in } S \textit{ with } s \models\!\!\equiv_{\prec} \Phi.$$

Clearly then, once the derived satisfaction relation $\models\!\!\equiv$ is given, along with the set S of states and the preference relation $\prec$, neither W nor l plays any further role in determining the induced nonmonotonic inference operation. W does however play a part in determining a natural consequence operation Cn_W, which is used by Kraus, Lehmann and Magidor to formulate certain properties of interaction with C. We therefore define:

Definition 3.3 (KLM core)
Any KLM model $\mathcal{S} = (S, W, l, \prec)$ induces a structure $(S', \models\!\!\equiv, \prec', Cn_W)$, called its core, *by putting*

1. $S' = S$,
2. $\models\!\!\equiv$ *is the derived satisfaction relation from $\mathcal{S}$,*
3. $\prec' = \prec$,
4. $Cn_W(\Phi) = \{\psi : \ w \models \psi$ *for all* $w \in W$ *with* $w \models \Phi\}$.

Evidently, if two KLM models have the same core then they induce the same nonmonotonic inference operation.

We note in passing that if in a KLM model one expands W to include more classical models, then so long as one leaves unchanged the set S of states and the labelling function l, the induced nonmonotonic inference operation C will be unaffected. On the other hand, the consequence operation Cn_W determined by W will be made smaller by such a manoeuvre. This means that whilst the *intrinsic* properties of C will not be affected by enlarging W, other properties describing its interaction with Cn_W may be affected. For example, the property of "consistency preservation", that whenever $C(\Phi) = Fml$ then $Cn_W(\Phi) = Fml$, may hold for a given KLM model but fail when W is enlarged.

We are now in a position to prove our theorem identifying the cores of KLM models with certain amplified MAK models. But before we do so, some remarks on the literature may help avoid confusion.

The most general models actually considered by Kraus, Lehmann and Magidor are those KLM models, as defined above, in which the preference relation $\prec$ satisfies a certain condition of *smoothness* - also known as *stoppering* in the infinitary version of [Mak89]. This is the condition that for every formula α and every $s \in S$, if $s \models\!\!\equiv \alpha$ then either $s \models\!\!\equiv_{\prec} \alpha$ or there is a $s' \in S$ with $s' \prec s$ and $s' \models\!\!\equiv_{\prec} \alpha$. It serves to guarantee that the induced nonmonotonic inference operation is cumulative, and for this reason Kraus, Lehmann and Magidor call models satisfying the smoothness condition *cumulative* (definition 3.10). They reserve the term *preferential models* for a yet more restricted class, consisting of those cumulative models in which $\prec$ is irreflexive and transitive and $l(s)$ is a singleton for all $s \in S$ (definition 5.6).

On the other hand, Shoham's original preferential models are even more specific. They may be identified with those preferential models in the sense of Kraus, Lehmann and Magidor in which W is the set of *all* classical models for $\mathcal{L}$, $S = W$, and the labelling function l is the injection $w \to \{w\}$.

The use of the term *preferential* is thus still rather anarchic at the edges. As well as the above two uses of Shoham and of Kraus, Lehmann and Magidor, there is also that of [Mak89], who employed the term *preferential model structure* for what in this note are for clarity called MAK models.

4 Proof of the Theorem

(1) implies (2)

Let $(S, \models\!\!\equiv, \prec, Cn_W)$ be the core of a KLM model $\mathcal{S} = (S, W, l, \prec)$. First we need to show that it is an amplified MAK model, for which it suffices to check that Cn_W is a compact consequence operation with $Cn_0 \le Cn_W \le Cn_{\mathcal{S}}$, satisfying in addition "disjunction in the antecedent". That Cn_W is a consequence operation is clear from its definition. That it is compact follows immediately from the hypothesized compactness of W. That Cn_W satisfies disjunction in the antecedent is immediate from the assumption that the elements of W are classical models. That $Cn_0 \le Cn_W$ is also immediate from the same assumption (the converse inclusion may fail when some classical models are missing from W).

Finally, that $Cn_W \le Cn_{\mathcal{S}}$ may easily be verified as follows. Suppose $\alpha \in Cn_W(\Phi)$ and suppose $s \models\!\!\equiv \Phi$; we need to show that $s \models\!\!\equiv \alpha$. Since $s \models\!\!\equiv \Phi$, we have $w \models \Phi$ for all $w \in l(s)$, so since $\alpha \in Cn_W(\Phi)$ we have $w \models \alpha$ for all $w \in l(s)$, so that $s \models\!\!\equiv \alpha$ as desired.

It remains to show that $\models\!\!\equiv$ may be expressed as the intersection of a non-empty set of satisfaction relations $\models\!\!\equiv_i$, each of which respects classical connectives. Let $\mathcal{I}$ be the set of all choice functions $i : S \longrightarrow W$ with $i(s) \in l(s)$. Recalling that $l(s)$ is always non-empty, the axiom of choice implies that $\mathcal{I}$ is non-empty.

For each $i \in \mathcal{I}$ define $\models\!\!\equiv_i$ by putting

$$s \models\!\!\equiv_i \alpha \quad \text{iff} \quad i(s) \models \alpha.$$

Since the satisfaction relation $\models$ is classically well-behaved, so is each $\models\!\!\equiv_i$. Also we have $\models\!\!\equiv \subseteq \models\!\!\equiv_i$ for each $i \in \mathcal{I}$. For if $s \models\!\!\equiv \alpha$ and $w = i(s) \in l(s)$ we have by the definition of the derived satisfaction relation $\models\!\!\equiv$ that $w \models \alpha$ so $s \models\!\!\equiv_i \alpha$.

It remains to show that

$$\bigcap \{\models\!\!\equiv_i : i \in \mathcal{I}\} \subseteq \models\!\!\equiv.$$

Suppose not $s \models\!\!\equiv \alpha$. Then by the definition of $\models\!\!\equiv$ there is a $w \in l(s)$ with not $w \models \alpha$. By the axiom of choice, there is a choice function $i : S \longrightarrow W$ with $i(s) = w$. Since not $i(s) \models \alpha$ this gives us not $s \models\!\!\equiv_i \alpha$ and we are done.

(2) implies (3)

Let $(S, \models\!\!\equiv, \prec, Cn)$ be an amplified MAK model such that $\models\!\!\equiv$ may be expressed as the intersection of a non-empty set of satisfaction relations each of which respects classical connectives. We need to check that $\models\!\!\equiv$ satisfies the two listed syntactic conditions. The verifications are straightforward, as follows:

(a) Suppose $s \models\!\!\equiv \alpha$ and $s \models\!\!\equiv \beta$. Then for all $i \in I$: $s \models\!\!\equiv_i \alpha$ and $s \models\!\!\equiv_i \beta$, so since each $\models\!\!\equiv_i$ is classically well-behaved, we have that for all $i \in I$, $s \models\!\!\equiv_i \alpha \wedge \beta$ so that $s \models\!\!\equiv \alpha \wedge \beta$ as desired.

(b) Let $s \in S$. Since each $\models\!\!\equiv_i$ is classically well-behaved, we have $s \models\!\!\equiv_i \phi \vee \neg\phi$ so $s \models\!\!\equiv \phi \vee \neg\phi$. Moreover, we know that I is non-empty; choose some $i \in I$. Then $\models\!\!\equiv_i$ is classically well-behaved, so not $s \models\!\!\equiv_i \phi \wedge \neg\phi$ so not $s \models\!\!\equiv \phi \wedge \neg\phi$.

(3) implies (1)

Let $(S, \models\!\!\equiv, \prec, Cn)$ be an amplified MAK model satisfying the conditions (a) and (b). We want to show that it is the core of some KLM model $(S, W, l, \prec)$. We need to define W, l, and verify the required properties.

First note that *every* amplified MAK model satisfies the following condition:

$$(*) \quad \text{whenever } s \models\!\!\equiv \alpha \text{ and } \beta \in Cn(\{\alpha\}) \text{ then } s \models\!\!\equiv \beta.$$

For suppose $s \models\!\!\equiv \alpha$ and $\beta \in Cn(\{\alpha\})$. Since $Cn \le Cn_{\mathcal{S}}$, we have $\beta \in Cn_{\mathcal{S}}(\{\alpha\})$, so, by the definition of $Cn_{\mathcal{S}}$, since $s \models\!\!\equiv \alpha$, we have $s \models\!\!\equiv \beta$ as desired.

Let W be the set of all maximal Cn-consistent sets of formulae in Fml. Using part (b) of our lemma 2.4, it is immediate that W consists of classical models.

For each $s \in S$ put $l(s)$ to be the set of all maximal Cn-consistent sets w of formulae such that $\{\delta :\ s \models\!\equiv \delta\} \subseteq w$.

Note that for all $s \in S$

$$Cn(\{\delta :\ s \models\!\equiv \delta\}) = \{\delta :\ s \models\!\equiv \delta\} \neq Fml.$$

The equality follows easily from the compactness of Cn, $Cn_0 \leq Cn$, the hypothesized property (a) and fact (*). The inequality is given directly by hypothesis (b). Using part (a) of lemma 2.4, this proves $l(s)$, and thus also W, to be non-empty.

The compactness of W is immediate from that of Cn. It remains to show that $(S, \models\!\equiv, \prec, Cn)$ is indeed the core of $(S, W, l, \prec)$. For Cn we have to show

$$\alpha \in Cn(\Phi) \quad \text{iff} \quad w \models \alpha \ \text{ for every maximal } Cn\text{-consistent } w \text{ that includes } \Phi.$$

From left to right is immediate (Cn is a monotone consequence operation). The converse is exactly (the contraposition of) condition (b) (ii) of our lemma 2.4.

For $\models\!\equiv$, we need to show that

$$s \models\!\equiv \alpha \quad \text{iff} \quad w \models \alpha \text{ for all } w \in l(s).$$

But since

$$s \models\!\equiv \alpha \ \text{ iff } \alpha \in \{\delta : s \models\!\equiv \delta\} = Cn(\{\delta : s \models\!\equiv \delta\})$$

then, applying the definition of $l(s)$, this is merely a particular case of what we have just shown about Cn.

■

5 Applications to Particular Cases

Note that the definition of *smoothness* (at the end of section 3) concerns only the ingredients S, $\prec$, $\models\!\equiv$ (and not W, l). Hence the theorem continues to hold when *KLM model* is restricted to *cumulative KLM model* and *MAK model* to *smooth MAK model*.

The definition of *preferential* KLM models (also at the end of section 3) involves S, $\prec$ and the labelling function l. To verify the corollary of our theorem (stated in section 1) we need only run through the proof again, making two small additions. In the verification "(1) implies (2)" we note that if each $l(s)$ is a singleton, then $\mathcal{I}$ contains exactly one choice function, so that $\models\!\equiv$ is classically well-behaved. In the verification of "(3) implies (1)" note that since $\models\!\equiv$ is classically well-behaved, the set $\{\delta :\ s \models\!\equiv \delta\} = Cn(\{\delta :\ s \models\!\equiv \delta\}) \neq \text{Fml}$ is maximally Cn_0-consistent. Since $Cn_0 \leq Cn$, this set is also maximally Cn-consistent, so that $l(s)$ is a singleton.

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