Herbrand's Theorem for Nonstandard Consequence Operations

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Abstract

Recently, H. Herre developed a framework for nonmonotonic inference relations using closure operators $\mathcal{I}$ that associate with every theory Φ a subset $\mathcal{I}(\Phi)$ of its ordinary models. He asked to what extent Herbrand's Theorem is valid for these model-based operators. In particular, he asked if it is true for $\models_{min}$, $\models_{r.e.}$ and $\models_{rec}$: truth in all *minimal*, in all *recursively enumerable*, or in all *recursive* Herbrand models, respectively. We answer these questions negatively and discuss Herbrand's Theorem also for well-known *nonmonotonic* theories (the *closed world assumption* CWA, Reiter's *default logic* DL), Minker's *weak generalized closed world assumption* WGCWA and Clark's *completion* $comp(P)$, as well as for *monotonic* theories, e.g. for $\models_{\mathbf{d}}$ where $\mathbf{d}$ is a *degree of unsolvability*, and for theories where $\mathcal{I}$ is restricted to be a subset of the Herbrand models. In case of Clark's $comp(P)$, Minker's WGCWA and $\models_{\mathbf{d}}$ we get some interesting positive results. We also consider the *Compactness*-property and its relation to Herbrand's Theorem in the more general setting of nonmonotonic relations $\mid\!\sim$ that need not fit into Herre's framework.

Keywords: Herbrand's Theorem, nonmonotonic logic, recursion-theoretic model theory, semantics of logic programs

1 Introduction

H. Herre studied in [5] various operators $\mathcal{I}$ associating with every theory Φ (i.e. a set of sentences, not necessarily deductively closed) a subset of its ordinary models: $\mathcal{I}(\Phi) \subseteq Mod(\Phi)$. The corresponding model based inference-operation $\models_{\mathcal{I}}$ (viewed as a relation between sets of formulae) is given by $\Phi \models_{\mathcal{I}} \psi$ iff $\mathcal{I}(\Phi) \models \psi$. He asked to what extent Herbrand's Theorem is valid for these model based operators:

> *Is Herbrand's Theorem true for $\mathcal{I}_{min}$, defined by all minimal Herbrand models, for $\mathcal{I}_{r.e.}$, defined by all recursively enumerable Herbrand models, or for $\mathcal{I}_{rec.}$, given by all recursive Herbrand models?*

We show that Herbrand's Theorem is not satisfied by giving (suitable) counterexamples. We also obtain various *positive* results, e.g. in case of Clark's *completion*, Minker's *weak generalized closed world assumption* and $\models_{\mathbf{d}}$. In addition, we prove that under very weak assumptions, *Herbrand's Theorem* is implied by a suitable form of *Compactness* (as is the case for classical predicate logic).

Let us fix some standard notation. We assume that our language $\mathcal{L}$ is first-order predicate logic *without* equality and that it contains at least one constant symbol c (to ensure that the Herbrand universe is not empty). Fml is the set of all $\mathcal{L}$-sentences. A *positive* formula is one which is equivalent to a formula built up solely from *positive* literals (atoms) using $\vee, \wedge, \forall$ and $\exists$. Similarly, a *negative* formula is one which is equivalent to a formula built up solely from *negative* literals (negated atoms) using $\vee, \wedge, \forall$ and $\exists$. Equivalently, a negative formula can be defined as (being equivalent to) the *negation* of a positive one. We denote by Th the classical first-order consequence relation $\models$: $Th(\Phi) = \{\psi : \Phi \models \psi\}$. Th_{herb}, or equivalently $\models_{herb}$, stands for *"truth in all Herbrand models"*. Note that Herbrand's Theorem holds for $\models_{herb}$: this is a corollary of the usual (classical) proof of Herbrand's Theorem.

Generalizing Herre's framework we are also considering nonmonotonic relations $\mid\!\sim\ \subseteq Fml \times Fml$ and use the notation $C_{\mid\sim}(\Phi)$:

$$C_{\mid\sim}(\Phi) := \{\psi : \Phi \mid\!\sim \psi\}.$$

If we take the classical predicate logic consequence-relation $\models$ for $\mid\!\sim$, we get a *monotonic* relation: $\Phi \subseteq \Psi$ implies $C_{\models}(\Phi) \subseteq C_{\models}(\Psi)$. As usual we call a set $\Phi \subseteq Fml$ $\mid\!\sim$ -consistent if $C_{\mid\sim}(\Phi) \neq Fml$.

We now formulate the fundamental notions for this paper. These are immediate generalizations of the classical notions of *Compactness* and *Herbrand's Theorem*.

Definition 1 (Compactness and Herbrand's Theorem for arbitrary $\mid\!\sim$) Let a first-order language $\mathcal{L}$ together with a consequence relation $\mid\!\sim\ \subseteq 2^{Fml} \times Fml$ (possibly nonmonotonic) be given.

We say that

a) $\mid\!\sim$ satisfies *Herbrand's Theorem*, if for every universal theory $\Phi_\forall$ and every existential sentence $\delta_\exists = \exists x \psi(x)$:

$$\Phi_\forall \mid\!\sim \exists x \psi(x) \implies \text{there are terms } t_1, \ldots, t_n \text{ such that: } \Phi_\forall \mid\!\sim \psi(t_1) \vee \cdots \vee \psi(t_n)$$

If the disjunction consists of a single disjunct $\psi(t_1)$, we say that Herbrand's Theorem is true *in the strong sense*.

b) $\mid\!\sim$ has the *Compactness property*, if for every $\Phi \subseteq Fml$:

Fin 1: every finite Φ_{fin} with $\Phi_{fin} \subseteq \Phi$ is $\mid\!\sim$-consistent. $\implies$ Φ is $\mid\!\sim$-consistent

Another (usually stronger) version is: for every $\Phi \subseteq Fml$:

Fin 2: $\Phi \mid\!\sim \psi \implies$ there is a finite Φ_{fin} with $\Phi_{fin} \subseteq \Phi$ and $\Phi_{fin} \mid\!\sim \psi$.

The classical consequence relation $\models$ satisfies Fin 1 and Fin 2, as well as Herbrand's Theorem. Indeed, the latter property follows from Fin 1 (or Fin 2) by a standard *Compactness*-argument, while the converse implication *"Herbrand implies Compactness"* is not true, even for *monotonic* consequence relations: choose for example $\models_{herb}$, truth in all Herbrand models. $\models_{herb}$ satisfies Herbrand's Theorem (as we noted above) but obviously does not satisfy Compactness. Note that while Fin 1 and Fin 2 are equivalent for classical $\models$, they differ for general $\mid\!\sim$ as will be explained in Section 5.

The paper is organized as follows. We first collect in Section 2 some fundamental examples showing the failure of Herbrand's Theorem and of the Compactness-property. These examples will be used in Sections 3 and 4 to motivate some *positive* results. In Section 3, we take a look at (variants of) the Closed World Assumption CWA, at Reiter's Default Logic DL and also consider (suitable generalizations of) semantics for logic programs, e.g. Clark's completion $comp(P)$, Gelfond/Lifschitz' stable model semantics STABLE and Minker's weak generalized closed world assumption WGCWA. For all these theories, we discuss the question for which classes of formulae $\delta_\exists$ Herbrand's Theorem is satisfied or not. Section 4 considers *monotonic* consequence relations based on *effective* Herbrand models. We give a nice characterization of the validity of Herbrand's Theorem for $\models_{\mathbf{d}}$ where $\mathbf{d}$ is a *degree*. Finally, in Section 5 we consider in a general setting inference relations $\mid\!\sim$ and state a rather general theorem showing under which circumstances *Compactness* implies *Herbrand's Theorem*.

2 Some fundamental examples

Although all three examples in this section show the failure of Herbrand's Theorem for certain theories, they are fundamental in that they will be used in the following sections to motivate interesting and nontrivial *positive* results:

- Example 2 leads us to consider a weaker version of $\models_{min}$ known in the context of *disjunctive* logic programs (see Section 3.2),
- Example 3 leads to a result that is closely connected to the well-known *completeness of the Negation-as-Finite-Failure Rule* in Logic Programming (see Section 3.2.2), and
- Example 4 finally leads to a necessary and sufficient condition for $\models_{\mathbf{d}}$ to satisfy Herbrand's Theorem. This condition is a very famous one and was investigated deeply in the work of Jockusch and Soare ([6]).

We answer Herre's question about the validity of Herbrand's Theorem for the case where $\mathcal{I}_{min}(\Phi)$ is given by Min-Mod(Φ), the set of all minimal Herbrand models of Φ. (*Minimal* in the sense that the extensions of the predicates are minimal.) The answer is negative:

Example 2 (Failure of Herbrand's Theorem for $\mathcal{I}_{min}$ and CIRC) Fix the signature $\mathcal{L} = \{0,\ s(),\ P()\}$, where s is a unary function symbol, and P is a unary predicate symbol. As usual, we identify $s^i(0)$ with $i \in I\!N$. Let

- $\Phi_\forall := \{P(i) \vee P(j) :\ i \neq j \in I\!N\}$
- $\delta_\exists := \exists x \neg P(x)$.

We have Min-Mod($\Phi_\forall$)=$\{\mathcal{M}_i : i \in I\!N\}$ with $P^{\mathcal{M}_i} = I\!N \setminus \{i\}$. Thus

$$\Phi_\forall \models_{min} \exists x \neg P(x),$$

but there is certainly no finite number of terms, such that the corresponding disjunction

$$\neg P(t_1) \vee \neg P(t_2) \vee \cdots \vee \neg P(t_n)$$

is valid.

By introducing a further binary predicate symbol $E(x, y)$ (representing equality) together with the well-known axioms for equality, $\Phi_\forall$ can be chosen finitely axiomatizable.

If we consider CIRC(Φ), i.e. the set of *all* minimal models of Φ, not only the Herbrand models, then we have to enlarge $\Phi_\forall$ by the axiom $\neg 0 = s(0)$. This modified version still serves as a counterexample.

Both the negation in $\delta_\exists$ and the disjunction in $\Phi_\forall$ is important:

- For *positive* $\delta_\exists$ and arbitrary universal $\Phi_\forall$: "$\Phi_\forall \models_{min} \delta_\exists \Longrightarrow \Phi_\forall \models \delta_\exists$" (see [16]). Thus, Herbrand's Theorem is true, since the *classical* version applies to the universal theory $\Phi_\forall$.
- For a *definite* program $\Phi_\forall$ and a *negative* $\delta_\exists$, Herbrand's Theorem (w.r.t. $\models_{min}$) is also true, since $\delta_\exists$ is true in $\Phi_\forall$'s *least* Herbrand model $\mathrm{M}_{\Phi_\forall}$.

Let us now turn to more specific theories: positive disjunctive programs. In this context, CIRC was originally introduced in [11] under the name of GCWA. A weaker version, called WGCWA, restricting the derivation of negative literals, has also been defined ([14]). We will see in Section 3.2 that this semantics satisfies Herbrand's Theorem in the strong sense for *negative* $\delta_\exists$.

In Logic Programming, the completion semantics of Clark (see [9] or [2]) plays a fundamentol rôle. This semantics defines the set of formulae derivable from an arbitrary logic program. It turns out that even for *definite* programs, Herbrand's Theorem is not true under *comp*(P) for *arbitrary* existential sentences $\delta_\exists$:

Example 3 (Failure of HT for *comp*(P)**, definite** P**, and arbitrary** $\delta_\exists$**)** Let P be the program

$$\begin{array}{c} A(x) \leftarrow A(x) \\ B(s(c)) \leftarrow A(y). \end{array}$$

Clark's completion *comp*(P) is given by Clark's equational theory CET together with

$$\begin{array}{c} \forall x \ \ (A(x) \ \leftrightarrow \ A(x)) \\ \forall y \ \ (B(y) \ \leftrightarrow \ (y = s(c) \ \wedge \ \exists z \ A(z))). \end{array}$$

So we obviously have *comp*$(P) \models \exists x(\neg B(s(c)) \vee A(x))$ but there is no finite disjunctions of terms. The same applies literally to *comp*$(P) \models_{herb} \exists x(\neg B(s(c)) \vee A(x))$.

To give some positive results, we prove that Herbrand's Theorem is true for *definite* programs under the completion semantics if $\delta_\exists$ is *positive* or *negative* (both for Mod(*comp*(P)) and Mod_{herb}(*comp*(P)) if $\delta_\exists$:

- $\delta_\exists$ *positive:* in this case, $comp(P) \models \delta_\exists \Longrightarrow P \models \delta_\exists$ (see [16, Theorem 15]) and we can apply Herbrand's Theorem in its original form. The same argument in the proof there also shows that $comp(P) \models_{herb} \delta_\exists \Longrightarrow P \models \delta_\exists$.
- $\delta_\exists$ *negative:* here, Herbrand's Theorem is true for $\text{Mod}_{herb}(comp(P))$ even in the strong sense. The Herbrand models of $comp(P)$ are exactly the fixpoints of the immediate consequence operator T_P (see [9] or [2]). As the set of all Herbrand interpretations of P forms a complete lattice and T_P is monotone, $\text{lfp}(T_P)$ and $\text{gfp}(T_P)$ exist. Thus, if $comp(P) \models_{herb} \delta_\exists$, there is a term t such that $\text{gfp}(T_P) \models \psi(t)$ which implies $comp(P) \models_{herb} \psi(t)$ since ψ is negative and $\text{gfp}(T_P)$ is the *greatest* fixpoint.

Unfortunately, the last argument only applies to $\text{Mod}_{herb}(comp(P))$ and not to its superset $\text{Mod}(comp(P))$. We will consider this much more difficult case in the next section (Section 3.2.2) and show how it relates to a classical result in Logic Programming.

Before we consider our third example, we note that our results cannot be generalized to *normal* programs, i.e. programs where negations are allowed. Herbrand's Theorem is neither true for *positive* nor for *negative* existential $\delta_\exists$, as can be seen by slightly modifying the above program P: Let P' be the program consisting of the two clauses

$$A(x) \leftarrow A(x) \text{ and } B(s(c)) \leftarrow \neg A(y).$$

Clark's completion $comp(P')$ is given by

$$\forall x\ (A(x) \leftrightarrow A(x)) \quad \wedge \quad \forall y\ (B(y) \ \leftrightarrow \ (y = s(c) \ \wedge \ \exists z \neg A(z))).$$

The negative existential sentence $\exists x(\neg A(x) \vee \neg B(s(c)))$ is derivable from $comp(P)$, but no finite disjunction $\bigvee_{i=1}^{n}(\neg A(t_i) \vee \neg B(s(c))$ is derivable. Adding two clauses

$$A'(x) \leftarrow \neg A(x),\ B'(s(c)) \leftarrow \neg B(s(c))$$

and setting $\delta_\exists := \exists x(A'(x) \vee B'(s(c)))$, we obtain the same result for *positive* existential sentences. The same applies literally to $\text{Mod}_{herb}(comp(P))$.

We end this section with consequence relations restricted to Herbrand models. We know that Herbrand's Theorem is trivially satisfied if we consider "truth in all Herbrand models". But what happens if we restrict to Herbrand models of a special kind, i.e. *recursive* or *recursive enumerable* Herbrand models? Again, using the well-known notion of *recursively inseparable* set (see [13, II.2.4]), we can show that the answer is negative:

Example 4 (Failure of Herbrand's Theorem for $\models_{rec}$ and $\models_{r.e.}$) Fix two *recursively inseparable* r.e. sets A and B. Let $\mathcal{L} = \{0,\ s(),\ A'(),\ B'()\}$, where s is a unary function symbol, and A', B' are unary predicates. We identify $s^i(0)$ with $i \in I\!N$.

a) Let

$$\Phi_\forall := \{A'(i) :\ i \in A\} \cup \{B'(j) : j \in B\}.$$

Obviously we have in any *recursive* Herbrand model $\mathcal{H}_{rec}$ of $\Phi_\forall$:

$$\mathcal{H}_{rec} \models \exists x(A'(x) \wedge B'(x))$$

(the extension of A' is a recursive superset of A, thus it intersects B) but there is no finite disjunction uniform for all such models.

b) For *recursively enumerable* Herbrand models, a similar counterexample is obtained by enlarging the signature with a unary predicate R and defining

$$\Phi_\forall = \{A'(i): \; i \in A\} \cup \{B'(j) : j \in B\} \cup \{\neg R(x) \Leftrightarrow A'(x)\}.$$

Again, in any *recursively enumerable* Herbrand model $\mathcal{H}_{r.e.}$ of $\Phi_\forall$:

$$\mathcal{H}_{r.e.} \models \exists x (A'(x) \wedge B'(x)).$$

This is, because the extension of A' is recursive (its complement is r.e.). Obviously, there is no finite disjunction.

Note that $\Phi_\forall$ is r.e. In Section 4 we allow models recursive in any fixed Turing degree **d**. Utilizing the above counterexample we characterize the degrees **d** such that Herbrand's Theorem *holds* for all r.e. $\Phi_\forall$.

3 Herbrand's Theorem in nonmonotonic reasoning

Example 2 shows that Herbrand's Theorem is not true for CIRC. In the first subsection, we show that it can also be used for Reiter's Default Logic (DL). We further consider two variants of the closed world assumption (CWA) to illustrate what happens in general for nonmonotonic consequence relatons.

The second subsection deals with the most important semantics for logic programs. We consider the stable semantics STABLE of Gelfond/Lifschitz, and we show that Herbrand's Theorem is true with respect to positive or negative $\delta_\exists$ both for definite programs under Clark's completion and for positive disjunctive programs under Minker's WGCWA.

3.1 The classical nonmonotonic systems

Example 2 applies also to Reiter's Default Logic DL (see [3] for the exact definitions of DL). We consider the default theory $(\mathcal{D}, \Phi_\forall)$ where

- $\mathcal{D} := \{(\emptyset : \neg P(i))/ \neg P(i) \; : \; i \in I\!N\}$,
- $\Phi_\forall := \{P(i) \vee P(j) : \; i \neq j \in I\!N\}$.

The extensions of this default theory are given by

$$E_i := Th(\{P(j) : \; i \neq j \in I\!N\} \cup \{\neg P(i)\})$$

so that

$$\exists x \neg P(x) \;\in\; \bigcap_{i \in I\!N} E_i \text{ and therefore } (\mathcal{D}, \Phi_\forall) \;\mid\!\sim\; \exists x \neg P(x)$$

but there is certainly no finite disjunction. The inference relation $\mid\!\sim$ used above is defined by *truth in all extensions*.

This example applies equally well to Autoepistemic Logic AEL by using the well-known equivalences of DL-extensions with AEL-expansions (see [7] and [10]). Note in particular that our counterexample is of a very special form: all defaults are *prerequisite-free* and *normal*.

We now consider the classical closed world assumption:

$$\mathrm{CWA}(\Phi) := \mathrm{Th}(\Phi \cup \{\neg A : A \text{ a ground atom with } \Phi \not\models A\})$$

and a variant of it that will serve as a counterexample:

$$\mathrm{CWA}_{herb}(\Phi) := \mathrm{Th}_{herb}(\Phi \cup \{\neg A : A \text{ a ground atom with } \Phi \not\models_{herb} A\})$$

The corresponding nonmonotonic inference relations $\mid\!\sim_{\mathbf{CWA}}$ are given by

$$\Phi \mid\!\sim_{\mathbf{CWA}} \psi \quad \underline{:\text{iff}} \quad \mathrm{CWA}(\Phi) \models \psi.$$

Obviously, CWA and CWA_{herb} both satisfy Herbrand's Theorem while CWA also satisfies Fin 1 and Fin 2. The following example shows that CWA_{herb} does satisfy neither Fin 1 nor Fin 2.

Example 5 (Failure of Fin 1 and Fin 2 for $\mathbf{CWA}_{herb}$) Let $\Phi := \{P(i) : i \in I\!N\} \cup \{\forall x P(x) \rightarrow Q(0)\}$. We have:

- $\Phi \vdash_{herb} Q(0)$, and thus $\mathrm{CWA}_{herb}(\Phi) \models Q(0)$, but
- for all $\Phi_{fin} \subseteq \Phi$: $\Phi_{fin} \not\models_{herb} Q(0)$, and thus $\mathrm{CWA}_{herb}(\Phi_{fin}) \models \neg Q(0)$ and $\mathrm{CWA}_{herb}(\Phi_{fin}) \not\models Q(0)$.

Even worse: it may happen for nonmonotonic relations that any finite subset is inconsistent, while the whole set is consistent.

Example 6 (Failure of the converse of Fin 1) Let $\Phi_\forall := \{P(i) \wedge (P(i+1) \vee P(i+2)) : i \in I\!N\}$. Every finite subset of $\Phi_\forall$ is inconsistent with respect to CWA_{herb}, while $\Phi_\forall$ itself is consistent:

$$\mathrm{CWA}_{herb}(\Phi_\forall) = \mathrm{Th}(\{P(i) : i \in I\!N\}).$$

The converse of Fin 2 also fails.

3.2 Semantics for logic programs: $\mathcal{I}_{wmin}$ and $comp(P)$

We are now investigating various nonmonotonic theories that evolved in the context of Logic Programming. The reader should note that in this area usually only *ground* literals are defined to be derivable. In order to prove or disprove Herbrand's Theorem, we need a more general notion of derivability: at least a definition of deriving arbitrary *existential* sentences. So, we may have to slightly modify and enlarge a given definition. For example CIRC is the natural generalization of Minker's GCWA that was defined originally for positive disjunctive logic programs (see [12]). CIRC extends GCWA in that it allows us to derive *arbitrary* formulae.

Remark 7 Any semantics based on a *single* model, like the least-model semantics for definite programs P (based on the *least* Herbrand model M_P), the *supported* semantics of stratified programs (based on the *supported* model M_P^{supp}, see [1]) or the *well-founded* semantics (based on a three-valued model) trivially satisfies Herbrand's Theorem, even in the strong sense.

Let us first note that Example 2 also applies to the stable model semantics STABLE (see [4]) by modifying the set $\Phi_\forall := \{P(i) \vee P(j) :\ i \neq j \in I\!N\}$ appropriately:

$$\Phi_\forall := \{P(i) \leftarrow \neg P(j) :\ i \neq j \in I\!N\}.$$

The stable models of $\Phi_\forall$ are exactly the minimal models of $\Phi_\forall$. Therefore, Herbrand's Theorem is not satisfied for STABLE.

3.2.1 Minker's WGCWA

In the following, we consider Minker's weaker version of GCWA: WGCWA. The reader is referred to [14] where to any *positive disjunctive* program Φ a *definite* program Φ^* and hence the least Herbrand model M_{Φ^*} was associated. Using this construction the original definition of WGCWA reads

$$\begin{aligned} WGCWA(\Phi) \;\; = & \quad \{\alpha :\ \alpha \text{ an atom with Min-Mod}(\Phi) \models \alpha\ \} \\ \cup & \quad \{\neg\beta :\ \beta \text{ an atom with } M_{\Phi^*} \models \neg\beta\} \end{aligned}$$

Let us note two useful properties:

(1) For *ground* atoms α: Min-Mod$(\Phi)\models \alpha \Longrightarrow M_{\Phi^*} \models \alpha$.
This follows from the above mentioned implication "Min-Mod$(\Phi)\models \delta_\exists$ implies $\Phi \models \delta_\exists$" and the fact that M_{Φ^*} is a model of Φ.

(2) For *ground* atoms α: $M_{\Phi^*} \models \neg\alpha \Longrightarrow$ Min-Mod$(\Phi)\models \neg\alpha$
This is easily proved by contraposition and by induction on the construction of a minimal model.

This gives us the following generalization of WGCWA in terms of an operator $\mathcal{I}$:

$$\mathcal{I}_{wmin}(\Phi) := \text{Min-Mod}(\Phi)\ \cup\ \{M_{\Phi^*}\}$$

We now have a notion of deriving *arbitrary* formulae, not only ground literals.

Lemma 8 ($\mathcal{I}_{wmin}$ and Herbrand's Theorem)

a) For positive $\delta_\exists$: $\mathcal{I}_{wmin}$ satisfies Herbrand's Theorem.

b) For negative $\delta_\exists$: $\mathcal{I}_{wmin}$ satisfies Herbrand's Theorem in the strong sense.

c) For arbitrary $\delta_\exists$, $\mathcal{I}_{wmin}$ does not satisfy Herbrand's Theorem.

Proof For *positive* existential formulae $\delta_\exists$, we can apply Herbrand's Theorem in the classical case by using "Min-Mod$(\Phi)\models \delta_\exists$ implies $\Phi \models \delta_\exists$": we obtain a disjunction $\bigvee_{i=0}^n \psi(t_i)$ that follows from $\Phi_\forall$ and, by (1) above, is also true in $M_{\Phi_\forall^*}$.

Let $\delta_\exists = \exists x\psi(x)$ where ψ is negative. $\mathcal{I}_{wmin} \models \exists x\psi(x)$ means in particular that $M_{\Phi_\forall^*} \models \exists x\psi(x)$. Thus, there exists a term t such that $M_{\Phi_\forall^*} \models \psi(t)$. By transforming $\psi(t)$ into disjunctive normal form, we can assume w.l.o.g. that $\psi(t)$ is a conjunction of negative ground atoms. Using (2) above, we are done.

To give a counterexample for the case where $\delta_\exists$ is neither positive nor negative, we modify Example 2. Let $\Phi_\forall := \{P(i) \vee P(j) :\ i \neq j \in I\!N\}$ and $\delta_\exists := \exists x(\neg P(x)\ \vee P(s(x)))$.

In the next section we show that an analogous result holds for definite programs under $comp(P)$.

3.2.2 Clark's *comp*(P)

As mentioned in Example 3 even for a *definite* program, Herbrand's Theorem is not true under the completion semantics. But we noted in Section 2 (after Example 3) that Herbrand's Theorem is true for $\mathrm{Mod}_{herb}(comp(P))$ and *positive* or *negative* formulae $\delta_{\exists}$. To obtain the same result for $\mathrm{Mod}(comp(P))$, we have to slightly modify and generalize the well-known proof of the *completeness of the negation-as-failure rule* due to Wolfram, Maher and Lassez. We refer the reader to [9, Theorem 16.1] or, alternatively, to [2, Lemma 5.29] for the necessary notions and definitions. Our aim is to prove the strong form of Herbrand's Theorem via contraposition. This means that we need a model of $comp(P) \cup \{\forall x p(x)\}$ and not only a model of $comp(P) \cup \{\exists x p(x)\}$, which would follow by the usual completeness of negation-as-failure.

The following lemma solves this problem. There is, however, a technical problem with the infinite goal G needed below. This is related to the so called *standardising variables apart*. When we build the SLD-tree, we need *variants* of the original program clauses. But since our goal below contains *all* terms, we have no fresh variables (variables that are disjoint with all variables in G) available. This can be easily repaired: in the course of constructing the tree, we only need to choose variables *different from the finitely many on the current branch*.

Lemma 9 (Modified "Completeness of Negation-as-Failure") Let P be a definite program, p a predicate symbol and G:=$\{p(t)$: t an arbitrary (not necessarily ground) term over $Var\}$.
If $P \cup G$ has a fair SLD-tree which is not finitely failed then $comp(P) \cup \{\forall x p(x)\}$ has a model (not necessarily a Herbrand model).

Proof The proof exactly parallels the proof of the *completeness of the negation-as-failure rule* as in [9, Theorem 16.1] or in [2, Lemma 5.29]. Note that we are considering a goal consisting of *infinitely* many atoms: all the usual definitions (SLD-tree, finitely-failed, non-failed branch) are well defined for infinite goals. Obviously, "successful finite branches" do not exist, but this does not matter, as we are only interested in non-failing branches.

Let BR be any non-failed branch in the fair SLD-tree for $\mathrm{P} \cup G$. BR is a sequence of goals $\mathrm{G}=\mathrm{G}_0$, G_1, ... G_n, ... with mgu's θ_0, θ_1, ... θ_n, ... and input clauses C_0, C_1, ... , C_n, ... The model of $comp(P) \cup \{\forall x p(x)\}$ is constructed as in the original proof.

A finitely-failed SLD-tree for an infinite (arbitrary) goal G may well exist: this is the case, if and only if there is a finite subset G_0 of G, such that there exists a finitely-failed SLD-tree for G_0. More importantly, if the goal G is of the special form G:=$\{p(t)$: t an arbitrary term$\}$, we have:

Lemma 10 Let P be a definite program. We assume that *any variable occuring in a clause of P already occurs in the head of the clause*. Let G:=$\{p(t)$: t an arbitrary (not necessarily ground) term$\}$.
There exists a finitely-failed SLD-tree for $P \cup G$ iff there exists a ground term t_0, such that $P \cup \{p(t_0)\}$ has a finitely-failed SLD-tree.

Proof If there exists a finitely-failed SLD-tree for $P \cup G$, then only finitely many atoms $G_0 \subseteq G$ are used. By instantiating all variables, we get a finitely-failed SLD-tree for $P \cup G_0\theta$ (see [2, Lemma 5.7]): here we use our assumption. We can now apply a well-known lemma (see [2, Lemma 5.8]) to obtain a finitely-failed SLD-tree for $P \cup \{p(t_0)\}$ where $p(t_0)$ is from $G_0\theta$.

The converse implication is trivial, since G contains $p(t)$ for *every* ground term t.

We strongly believe that the above lemma is also true without our assumption. The proof, however, seems to be rather tedious: we have to take a close look to the unification-machinery used in [2, Lemma 5.7]: there are many technical subtleties.

Using the last two lemmas, we are now in a position to summarize our results about the validity of Herbrand's Theorem for $comp(P)$ with respect to *definite* programs P:

Theorem 11 (Herbrand's Theorem for $comp(P)$ with definite P) Let P be a definite program.

a) For positive $\delta_\exists$, both $\mathrm{Mod}_{herb}(comp(P))$ and $\mathrm{Mod}(comp(P))$ satisfy Herbrand's Theorem.

b) For negative $\delta_\exists$, both $\mathrm{Mod}_{herb}(comp(P))$ and $\mathrm{Mod}(comp(P))$ satisfy Herbrand's Theorem in the strong sense.

c) For arbitrary $\delta_\exists$, neither $\mathrm{Mod}_{herb}(comp(P))$ nor $\mathrm{Mod}(comp(P))$ satisfy Herbrand's Theorem.

In **b)** we assume for $\mathrm{Mod}(comp(P))$ that *any variable occuring in a clause of P already occurs in the head of the clause.*

Proof a) and c) have been already proved in Section 2. For b), it remains to consider $\mathrm{Mod}(comp(P))$. We prove the following statement, which is the contraposition of the strong form of Herbrand's Theorem, by induction on the construction of Ψ: for any *positive* formula $\Psi(x)$, we have

"for all *ground* terms t: $comp(P) \cup \{\Psi(t)\}$ has a model"

$$\Rightarrow$$

"$comp(P) \cup \{\forall x\, \Psi(x)\}$ has a model".

If $\Psi(x)$ is a positive atom $p(x)$, the implication follows from Lemma 9 and Lemma 10 by using the *soundness of the negation-as-failure rule*: if $\mathrm{P} \cup \{p(t_0)\}$ has a finitely-failed SLD-tree, then $comp(P) \models \forall x\, \neg p(t_0)$, which is a contradiction. Therefore, our assumption implies that no fair SLD-tree for $P \cup \{p(t) : \text{t an arbitrary term}\}$ is finitely-failed, and Lemma 9 can be applied.

If $\Psi(x)$ is a disjunction $\Psi_1(x) \vee \Psi_2(x)$, we consider the program

$$P' := P \cup \{\Psi(x) \leftarrow \Psi_1(x), \quad \Psi(x) \leftarrow \Psi_2(x)\},$$

where Ψ is a *new* predicate symbol. One easily verifies that

$$comp(P') = comp(P) \cup \{\, \forall x\, (\Psi(x) \leftrightarrow (\Psi_1(x) \vee \Psi_2(x)))\}.$$

This implies that "for all ground terms t: $comp(P) \cup \{\Psi_1(t) \vee \Psi_2(t)\}$ has a model" is equivalent to "for all ground terms t: $comp(P') \cup \{\Psi(t)\}$ has a model". Therefore, this case is reduced to the first one.

If $\Psi(x)$ is a conjunction $\Psi_1(x) \wedge \Psi_2(x)$, we consider the program $P' := P \cup \{\Psi(x) \leftarrow \Psi_1(x), \Psi(x)\}$. It is easily verified that $comp(P') = comp(P) \cup \{\forall x\, (\Psi(x) \leftrightarrow (\Psi_1(x) \wedge \Psi_2(x)))\}$ and therefore this case also reduces to the first one.

As already noticed after Lemma 10, we believe that our assumption about the occurrence of the variables in P is not needed. Without it, our proof can be slightly modified and proves that Herbrand's Theorem is true for $\mathrm{Mod}(comp(P))$ (the validity of the strong form of Herbrand's Theorem for $\mathrm{Mod}_{herb}(comp(P))$ is not affected).

4 Herbrand's Theorem for effective models

We already know that Herbrand's Theorem does not hold for recursive Herbrand models. In this section we consider more general models whose Turing degree is bounded by some fixed Turing degree $\mathbf{d}$. We assume that the signature is finite. The degree $\deg(\mathcal{A})$ of a model $\mathcal{A}$ whose universe is a subset of $I\!N$ is defined as the Turing degree of its atomic diagram ([17]).

Let $\mathbf{d}$ be a Turing degree. We say that:

- $\mathcal{A}$ is *recursive* in $\mathbf{d}$:iff $\deg(\mathcal{A}) \leq_T \mathbf{d}$ ($\leq_T$ denotes Turing reduction),
- $\mathcal{A}$ is *r.e. in* $\mathbf{d}$:iff the extension of each predicate is recursively enumerable relative to $\mathbf{d}$, and each function is recursive in $\mathbf{d}$.

In particular, $\mathcal{A}$ is recursive :iff $\deg(\mathcal{A}) = \mathbf{0}$, and $\mathcal{A}$ is r.e. :iff $\mathcal{A}$ is r.e. in $\mathbf{0}$. ($\mathbf{0}$ is the Turing degree of the recursive sets.)

We may identify via a Gödelnumbering the Herbrand universe with a recursive subset of $I\!N$. The following property holds in our setting by the usual proof:

> If $\Phi_\forall$ has a model of degree less than or equal $\mathbf{d}$ (r.e. in $\mathbf{d}$) then it also has a Herbrand model of degree less than or equal $\mathbf{d}$ (r.e. in $\mathbf{d}$).

Let $\models_{\mathbf{d}}$ ($\models_{r.e.\ in\ \mathbf{d}}$) denote the model based inference operation with respect to models recursive (r.e.) in $\mathbf{d}$. By the previous remark these operations coincide when applied to universal theories (on the left hand side) and existential sentences (on the right hand side) with the operations defined with respect to Herband models of degree less than or equal to $\mathbf{d}$ (r.e. in $\mathbf{d}$). In this case we get $\models_{rec} = \models_{\mathbf{0}}$, $\models_{r.e.} = \models_{r.e.\ in\ \mathbf{0}}$.

By relativizing the concept of recursively inseparable sets to a degree $\mathbf{d}$ we see from Example 4 that Herbrand's Theorem fails for $\models_{\mathbf{d}}$ and $\models_{r.e.\ in\ \mathbf{d}}$. In these counterexamples $\Phi_\forall$ is arbitrary complex. We now consider the case when $\Phi_\forall$ is required to be r.e. and obtain a nontrivial positive result.

Notation 12 For the following we require $\Phi_\forall$ to be recursively enumerable. By *complete* we mean complete and consistent.

Theorem 13 Herbrand's Theorem is true for $\models_{\mathbf{d}}$ iff $\mathbf{d}$ is the degree of a complete extension of Peano Arithmetic.

Proof Let $\mathbf{d}$ be the degree of a complete extension of Peano Artihmetic. By a result of Scott ([15], see also [13, V.5.35]), any consistent r.e. set of formulae has a complete extension of degree less than or equal to $\mathbf{d}$. By a well-known theorem, every recursive complete theory has a recursive model (see [17]). This fact relativizes, thus any consistent r.e. set of formulae has a model of degree less than or equal to $\mathbf{d}$. Therefore, for any r.e. Φ and any formula ψ, $\Phi \models \psi$ iff $\Phi \models_{\mathbf{d}} \psi$. It follows immediately that Herbrand's Theorem (for $\Phi_\forall$ r.e.) is satisfied.

For the converse, suppose that Herbrand's Theorem is true for $\mathbf{d}$. Let A, B be a pair of effectively inseparable r.e. sets ([13, III.6.24]). Consider $\Phi_\forall$ as in Example 4, (a). There is a Herbrand model $\mathcal{A}$ with $\deg(\mathcal{A}) \leq_T \mathbf{d}$ and $\mathcal{A} \models \neg\exists x(A'(x) \wedge B'(x))$. It follows that there exists a set C recursive in $\mathbf{d}$ which separates A and B. Jockusch and Soare ([6, Proposition 6.1]) have shown that there exists a complete extension of Peano Arithmetic which is recursive in C. Solovay (see [13, V.5.36]) has shown that the set of degrees of the complete extensions of Peano Arithmetic is closed upwards. Thus $\mathbf{d}$ contains a complete extension of Peano Arithmetic.

The proof shows the surprising fact that, with respect to r.e. Φ, Herbrand's Theorem is true for $\models_{\mathbf{d}}$ *only if*

$$C_{\models_{\mathbf{d}}}(\Phi) = C_{\models}(\Phi).$$

Using example Example 4, (b) in the above proof we obtain the same characterization for $\models_{r.e.\ in\ \mathbf{d}}$:

Corollary 14 Herbrand's Theorem is true for $\models_{r.e.\ in\ \mathbf{d}}$ iff $\mathbf{d}$ is the degree of a complete extension of Peano Arithmetic.

Proof By Theorem 13 and the remark above, if $\mathbf{d}$ is the degree of a complete extension of Peano Arithmetic then $C_{\models_{\mathbf{d}}}(\Phi) = C_{\models}(\Phi)$. As every model recursive in $\mathbf{d}$ is in particular r.e. in $\mathbf{d}$ we get $C_{\models_{r.e.\ in\ \mathbf{d}}}(\Phi) = C_{\models}(\Phi)$, so Herbrand's Theorem is true for $\models_{r.e.\ in\ \mathbf{d}}$.

For the converse suppose that Herbrand's Theorem is true for $\models_{r.e.\ in\ \mathbf{d}}$. Let A, B be a pair of effectively inseparable r.e. sets. By Example 4, (b) there exists a set C recursive in $\mathbf{d}$ which separates A and B. By the proof of Theorem 4.2, $\mathbf{d}$ contains a complete extension of Peano Arithmetic.

The following theorem summarizes the results.

Theorem 15 (Herbrand's Theorem for $\models_{\mathbf{d}}$) The following statements are equivalent (for r.e. $\Phi_\forall$):

- Herbrand's Theorem is true for $\models_{\mathbf{d}}$.
- Herbrand's Theorem is true for $\models_{r.e.\ in\ \mathbf{d}}$.
- $C_{\models_{\mathbf{d}}}(\Phi) = C_{\models}(\Phi)$ for r.e. Φ.
- $\mathbf{d}$ is the degree of a complete extension of Peano Arithmetic.

Theorem 15 also holds if we require $\Phi_\forall$ to be finite, as Example 4 can be finitely axiomatized (within an enlarged signature) using, say, the representability of r.e. sets by logic programs (see [2, Corollary 4.5]).

5 Herbrand's Theorem in a general setting

In this section we formulate a very general theorem for *arbitrary* inference relations $\mid\!\sim$. We show under which conditions the *Compactness*-property implies Herbrand's Theorem.

While in general nonmonotonic reasoning a variety of different consequence relations $\mid\!\sim$ are investigated, some minimal requirements for $\mid\!\sim$ are often used. Among them the following three very weak assumptions (see [8]): let $\Phi \cup \{\beta\} \subseteq Fml$

Cut: $\beta \in C_{\mid\sim}(\Phi) \implies C_{\mid\sim}(\Phi \cup \{\beta\}) \subseteq C_{\mid\sim}(\Phi)$,

Reflexivity: $\Phi \subseteq C_{\mid\sim}(\Phi)$,

Left Absorption: $Th(C_{\mid\sim}(\Phi)) = C_{\mid\sim}(\Phi)$.

There is another interesting property needed in the sequel (for all sentences β):

Red: $\beta \notin C_{\mid\sim}(\Phi) \implies (\Phi \cup \{\neg\beta\})$ $\mid\!\sim$ -consistent.

Red is important for our general theorem relating Compactness and Herbrand's Theorem. It is a form of *reductio ad absurdum* and can be seen as a weaker version of the hard half of the deduction-theorem

$$\Phi \cup \{\beta\} \mid\!\sim \gamma \implies \Phi \mid\!\sim \beta \rightarrow \gamma$$

from which it easily follows. As an example, Reiter's default logic DL has this property (see [3, Proposition 6.2.18]). *Red* also follows from the system $\mathcal{P}$ of *preferential* reasoning (see [8] for the definition of $\mathcal{P}$).

In the classical proof of Herbrand's Theorem (using Compactness), we need a property relating universal theories and the use of the quantifier $\exists$:

(*) If a universal theory $\Phi_\forall$ has a model then it also has a Herbrand model.

In our general $\mid\!\sim$ -setting, we have no model theory available (most nonmonotonic theories do not even possess an acceptable notion of a model) so that we must look for an appropriate *syntactic* version of (*). We consider the following condition Cond $_{\mid\sim}$ as the right notion. It formalizes the minimal intuition that one has about universal theories and the validity of existential sentences:

Cond $_{\mid\sim}$: If $\Phi_\forall$ is universal, then $\Phi_\forall \cup \{\neg\psi(t) : t \text{ a ground term }\} \cup \{\exists x \psi(x)\}$ is $\mid\!\sim$ -inconsistent.

Let us put together the following observations, the proofs of which are straightforward:

(α) For any *monotonic* $\mid\!\sim$:

- the converse of Fin 1 and the converse of Fin 2 are satisfied,
- if $\mid\!\sim$ satisfies *Red*: Fin 1 $\Longrightarrow$ Fin 2.

(β) For any $\mid\!\sim$:

- Fin 2 $\Longrightarrow$ Fin 1. Also, the converse of Fin 2 $\Longrightarrow$ the converse of Fin 1.

(here we are using *Left Absorption*)

In order to state the next theorem, we need a slightly stronger version of Compactness:

Compactness: Let Φ be $\mid\!\sim$ -consistent and Ψ be a set of formulae. For every finite $\Psi_{fin} \subseteq \Psi$: $\Phi \cup \Psi_{fin}$ is $\mid\!\sim$-consistent $\implies$ $\Phi \cup \Psi$ is $\mid\!\sim$-consistent

We are now in a position to formulate our general theorem:

Theorem 16 (Compactness implies Herbrand's Theorem) Let $\mid\!\sim$ be an inference relation satisfying *Cut*, *Reflexivity*, *Left Absorption* and *Red*. Then:

$$\text{Compactness and Cond}_{\mid\sim} \implies \text{Herbrand's Theorem}$$

Proof Assume that $\Phi_\forall \mid\!\sim \exists\psi(x)$. If $\Phi_\forall$ is $\mid\!\sim$ -inconsistent, we are done (using *Reflexivity* and *Left Absorption*). Otherwise, $\Phi_\forall \cup \{\exists x\psi(x)\}$ is $\mid\!\sim$ -consistent by *Cut* and *Left Absorption*. $\text{Cond}_{\mid\sim}$ ensures that the set

$$\Phi_\forall \cup \{\exists x\psi(x)\} \cup \{\neg\psi(t) : \ t \text{ a term}\}$$

is $\mid\!\sim$ -inconsistent. Our definition of *Compactness* implies the existence of a finite set $\Psi_{fin} = \{\neg\psi(t_1), \cdots, \neg\psi(t_n)\}$ with

$$\Phi_\forall \cup \{\exists x\psi(x)\} \cup \Psi_{fin} \text{ is } \mid\!\sim \text{-inconsistent.}$$

Red implies

$$\Phi_\forall \cup \{\exists x\psi(x)\} \mid\!\sim \psi(t_1) \vee \cdots \vee \psi(t_n)$$

and finally, using *Cut*, we are done.

The following example illustrates the need of $\text{Cond}_{\mid\sim}$:

Example 17 (Fin 2 does not imply Herbrand's Theorem)
For a fixed sentence $\alpha \in \mathcal{L}$, we define a nonmonotonic inference operation $\mid\!\sim_\alpha$ as follows:

$$\Phi \mid\!\sim_\alpha \psi \quad \underline{:\!iff} \quad \begin{cases} \psi \in Th(\Phi \cup \{\alpha\}) & \text{if } \Phi \not\models \neg\alpha \\ \psi \in Th(\Phi) & \text{otherwise} \end{cases}$$

The relation $\mid\!\sim_\alpha$ satisfies obviously Fin 2 and Compactness, but fails to satisfy Herbrand's Theorem: the theory $\Phi_\forall := \{\neg P(i) : \ i \in I\!N\}$, $\delta_\exists := \exists x P(x)$ and $\alpha := \delta_\exists$ serves as a counterexample.

Similar examples can be easily constructed and used to show that neither Fin 1 and Fin 2 nor the converse of Fin 1 and the converse of Fin 2 are equivalent in general. All these examples satisfy the rules of *Reflexivity*, *Left Absorption*, *Cut* and *Red*.

Conclusions

Solving a particular question about the truth of Herbrand's Theorem for consequence relations representable in Herre's framework, we collected three fundamental counterexamples. We then used these examples systematically to get both *positive* and *negative* results for a variety of inference relations $\mid\!\sim$ discussed in the literature:

- We showed the failure of Herbrand's Theorem for some well-known nonmonotonic logics (CWA, CIRC, DL, AEL) as well as for some monotonic logics ($\models_{r.e.}$, $\models_{rec.}$) obtained by restricting the underlying models to be certain natural Herbrand models.

- We also got some interesting positive results in the case of semantics of logic programs (*comp*(P), WGCWA) for particular classes of formulae $\delta_{\exists}$, and a natural characterization of those degrees $\mathbf{d}$ for which $\models_{\mathbf{d}}$ satisfies Herbrand's Theorem.

We completed our investigation with a general theorem concerning *arbitrary* inference relations $\mid\!\sim$: it states precisely under which additional assumptions *Compactness* implies Herbrand's Theorem.

We think that the validity of Herbrand's Theorem for an inference relation $\mid\!\sim$ is an indication of the constructive and regular character of $\mid\!\sim$. This is confirmed in particular by our positive results concerning *comp*(P) and WGCWA.

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