Exercises on Advanced Algorithms II.

1. In this question, by graphs we mean finite, undirected graphs.

    (a) Explain clearly what is meant by a depth-first search (DFS) of a graph. Using suitable notation, write a program to perform DFS of graphs.

    [7 marks]

    (b) Explain how a DFS of a graph separates the edges of the graph into several classes.

    [3 marks]

    (c) What is meant by an articulation point of a graph?

    [3 marks]

    (d) Explain clearly how DFS can be used to calculate the articulation points of a graph (you may either write a program - which you should explain - or give a clear explanation of how a DFS can be used to determine articulation points).

    [7 marks]

2. (a) Explain what is meant by an Eulerian circuit in a finite undirected graph.

    What simple property of the graph corresponds exactly to the existence of Eulerian circuits?

    Explain clearly how to compute an Eulerian circuit in such graphs (you need not write a program, but whatever algorithm you choose must be clearly explained).

    [10 marks]

    (b) For finite undirected graphs, explain what is meant by a colouring of a graph.

    Show that if a graph can be coloured with two colours then there are no cycles of odd length.

    Show conversely that if there are no cycles of odd length then the graph can be coloured with two colours.

    [10 marks]

3. This question is about showing a problem on finite, undirected graphs is NP-complete.

   An independent set of nodes in a graph is a set of nodes no two of which are linked by an edge. The problem is to determine, for any graph, whether or not it has an independent set of nodes of size k.

   Show firstly that the problem is in NP i.e. that we can test in polynomial time whether a possible solution is a solution.

   [4 marks]

   Recall that the problem of determining whether a graph has a clique of k nodes in NP complete.

   Now show that the problem is NP-complete by reducing the problem of finding cliques to the problem of finding independent sets of nodes. Hint: consider the complement of a graph G, ie the graph which has the same set of nodes as G but edges only between pairs of nodes which are not linked by an edge in G.

   [16 marks]

4. This is another question about showing that a graph problem is NP-complete.

   We state as a fact that the problem of finding whether or not there is a Hamiltonian circuit in a directed graph is NP-complete.

   You are then asked to show that the problem of determining whether or not an undirected graph has a Hamiltonian circuit is NP-complete, as follows:

   Show firstly that the problem for undirected graphs is in NP i.e. that we can test in polynomial time whether a possible solution is a solution. [4 marks]

   Now show how to translate the problem for directed graphs into that for undirected graphs. Hint: consider the following construction of an undirected graph G' from a directed graph G. For each node n in G, there are three nodes n-IN, n-MID, and n-OUT in G'. For an edge from u to v in G, there is an indirected edge between u-OUT and v-IN in G'. In addition there are edges from n-IN to n-MID and from n-MID to n-OUT in G' for each node n in G. Show that G has a Hamiltonian circuit exactly when G' does. [16 marks]