Previously ...

- propositional clause logic:
  atoms, literals (positive & negative), clauses (= multi-sets)
- calculus Res:
  resolution & positive factoring
- conversion to clause form:
  - optimisation using structural transformation

Ground Expressions, Ground Instances

- Ground terms are terms with no occurrences of variables.
- Ground atoms are atoms with no occurrences of variables.
- Ground literals, ground clauses, ground formulae are defined similarly.
- A ground instance of an expression (atom, literal, clause, formula) is obtained by uniformly instantiating the variables in it with ground terms.

Herbrand Universe

- Notation: Let Σ denote our language.
- We assume there is at least one constant in the language Σ.
- The Herbrand universe (over Σ), denoted T_Σ, is the set of all ground terms over Σ.
- **Example:** Suppose the language has one binary function symbol f and two constants a and b. The following are in T_Σ:

$$a,\ b,\ f(a,a),\ f(a,b),\ f(b,a),\ f(b,b),\ f(a,f(a,a)),\ \ldots$$

- If Σ contains non-constant function symbols then T_Σ is infinite.

Exercise

- Suppose Σ is the language with one unary function symbol f and one constant a.
  Write down the elements of the Herbrand universe T_Σ.

Exercise

- Suppose Σ is the language with one unary function symbol f and one constant a.
  Write down the elements of the Herbrand universe T_Σ.

$$\mathrm{T}_\Sigma = \{a, f(a), f(f(a)), \ldots, f^n(a), \ldots\}$$

Herbrand Interpretations

- A Herbrand interpretation (over Σ), denoted I, is a set of ground atoms over Σ.
- Truth in I of ground formulae is defined inductively by:

$$I \models A \text{ iff } A \in I, \text{ for any ground atom } A$$

$$I \models \top \qquad I \not\models \bot$$

$$I \models \neg F \text{ iff } I \not\models F$$

$$I \models F \wedge G \text{ iff } I \models F \text{ and } I \models G$$

$$I \models F \vee G \text{ iff } I \models F \text{ or } I \models G$$

- Note: A ground atom A is true in I if $A \in I$ and false otherwise.
  This implies:

$$I \models \neg A \text{ iff } I \not\models A \text{ iff } A \notin I$$

Herbrand Interpretations (cont'd)

- Truth in I of any quantifier-free formula F with free variables $x_1, \ldots, x_n$ is defined by:

$$I \models F(x_1, \ldots, x_n) \text{ iff } I \models F(t_1, \ldots, t_n), \text{ for every } t_i \in \mathrm{T}_\Sigma$$

- Truth in I of any set N of clauses/formulae is defined by:

$$I \models N \text{ iff } I \models C, \text{ for each } C \in N$$

- We say a formula F (or a set N of formulae) is satisfiable if there is an interpretation I such that $I \models F$ (or $I \models N$).
- A Herbrand interpretation I is called a Herbrand model of F (N), if $I \models F$ ($I \models N$).

Relation to standard interpretations (optional)

- In a Herbrand interpretation values are fixed to be ground terms and functions are fixed to be the (Skolem) functions in Σ.
- I is an interpretation $\mathcal{M}$ where the values are given by:
  constant a: $a^{\mathcal{M},s} = a$
  function f: $(f(t_1, \ldots, t_n))^{\mathcal{M},s} = f(t_1^{\mathcal{M},s}, \ldots, t_n^{\mathcal{M},s})$
- Only predicate symbols p may be freely interpreted as relations $p^{\mathcal{M}} \subseteq \mathrm{T}_\Sigma^n$.

Property 6

Every Herbrand interpretation (set of ground atoms) I uniquely determines an interpretation $\mathcal{M}$ via

$$(s_1, \ldots, s_n) \in p^{\mathcal{M}} \text{ iff } p(s_1, \ldots, s_n) \in I$$

Exercise

- Suppose Σ is the language with one unary function symbol f, one constant a and one predicate symbol p.
  Which of the following are Herbrand interpretations over Σ?
  1. $I_1 = \{p(a)\}$
  2. $I_2 = \{p(a),\ p(f(a))\}$
  3. $I_3 = \{p(a),\ \neg p(f(a))\}$
- For I_2 determine whether the following is true?
  1. $I_2 \models p(a)$
  2. $I_2 \models \neg p(a)$
  3. $I_2 \models \neg p(f(a))$
  4. $I_2 \models p(a) \wedge p(f(a))$
  5. $I_2 \models p(x)$

Exercise

- Suppose Σ is the language with one unary function symbol f, one constant a and one predicate symbol p.
  Which of the following are Herbrand interpretations over Σ?
  1. $I_1 = \{p(a)\}$ yes
  2. $I_2 = \{p(a),\ p(f(a))\}$ yes
  3. $I_3 = \{p(a),\ \neg p(f(a))\}$ no; only pos. atoms can belong to I
- For I_2 determine whether the following is true?
  1. $I_2 \models p(a)$ yes
  2. $I_2 \models \neg p(a)$ no
  3. $I_2 \models \neg p(f(a))$ no
  4. $I_2 \models p(a) \wedge p(f(a))$ yes
  5. $I_2 \models p(x)$ no

Exercise

- Suppose Σ is the language with one unary function symbol f, one constant a and one predicate symbol p.
  Which of the following are Herbrand interpretations over Σ?
  1. $I_1 = \{p(a)\}$ yes
  2. $I_2 = \{p(a),\ p(f(a))\}$ yes
  3. $I_3 = \{p(a),\ \neg p(f(a))\}$ no; only pos. atoms can belong to I
- For I_2 determine whether the following is true?
  1. $I_2 \models p(a)$
  2. $I_2 \models \neg p(a)$
  3. $I_2 \models \neg p(f(a))$
  4. $I_2 \models p(a) \wedge p(f(a))$
  5. $I_2 \models p(x)$

Examples of Herbrand Interpretations

- Suppose $\Sigma_{\mathbb{N}}$ has constant 1, binary function $+$, unary predicate symbol p.
- Notation: $n + m$ instead of $+(n, m)$
- Herbrand interpretation over $\Sigma_{\mathbb{N}}$:

$$I = \{\ p(1),\ p(1+1),\ p(1+1+1),\ p(1+1+1+1),\ \ldots\}$$

- Is model of: $p(x)$ representing that x is a natural number
  Not model of: $p(x)$ representing that x is an odd number

Examples of Herbrand Interpretations (cont'd)

- Suppose Σ is any signature. Let I be a Herbrand interpretation over Σ.
  - $I \models r(x,x)$ iff $r(t,t) \in I$ for every ground term t in T_Σ.
  - $I \models \neg r(x,x)$ iff $r(t,t) \notin I$ for every ground term t in T_Σ.
  - $I \models \neg r(x,y) \vee r(y,x)$ iff when $r(s,t) \in I$ then $r(t,s) \in I$, for any ground terms $s,t \in T_\Sigma$.
  - $I \models \neg r(x,y) \vee \neg r(y,z) \vee r(x,z)$ iff when $r(s,t) \in I$ and $r(t,u) \in I$ then $r(s,u) \in I$, for any ground terms $s,t,u \in T_\Sigma$.

Existence of Herbrand Models

- Let $G_\Sigma(N)$ denote the set of all ground instances of the clauses in N over the language Σ. I.e. $G_\Sigma(N) = \{C\sigma \mid C \in N, \sigma : X \longrightarrow T_\Sigma \text{ a ground substitution}\}$, where X denotes the set of variables in the language.

Property 7 (Herbrand)
Let N be a set of Σ-clauses.

N satisfiable iff N has a Herbrand model (over Σ)
iff $G_\Sigma(N)$ has a Herbrand model (over Σ)

(See optional, unassessed material – on completeness of general resolution – from course website for a proof.)

Examples

- Consider

$$N = \{p(x),\ q(f(y)) \vee r(y)\}$$
$$T_\Sigma = \{a, f(a), f(f(a)), \ldots\}$$

- $p(a)$ and $p(f(a))$ are both ground instances of the first clause $p(x)$ in N.
- **Exercise**: Give examples of ground instances of the second clause in N.

Example of a set of ground instances $G_\Sigma(N)$

- For $\Sigma_{\mathbb{N}}$ one obtains for

$$N = \{p(1),\ \neg p(x) \vee p(x+1)\}$$

the following ground instances:

$$G_\Sigma(N) = \{\ p(1),\ \neg p(1) \vee p(1+1),\ \neg p(1+1) \vee p(1+1+1),\ \neg p(1+1+1) \vee p(1+1+1+1),\ \ldots\ \}$$

Exercise

- Write down a Herbrand model of $G_\Sigma(N)$ (i.e. a Herbrand interpretation in which all clauses of $G_\Sigma(N)$ are true).
- Is N satisfiable?

Exercise

- Write down a Herbrand model of $G_\Sigma(N)$ (i.e. a Herbrand interpretation in which all clauses of $G_\Sigma(N)$ are true).

$$I = \{p(1), p(1+1), p(1+1+1), \ldots\}$$

- Is N satisfiable? Answer: Yes. Why?
  By Herbrand's theorem.
  In fact, I is also a model of N.

Exercise

- Write down a Herbrand model of $G_\Sigma(N)$ (i.e. a Herbrand interpretation in which all clauses of $G_\Sigma(N)$ are true).

$$I = \{p(1), p(1+1), p(1+1+1), \ldots\}$$

- Is N satisfiable?

Summary

- ground expressions, ground instances
- Herbrand universe
- Herbrand interpretation, Herbrand model
- satisfiability in I: $\models$
- Herbrand's Theorem

COMP60121: Automated Reasoning II

Lecture 4

Recap: Soundness and Completeness

- $N \vdash_{Cal} C$ means there exists a *Cal*-derivation of C from N.
  $N \vdash_{Res} C$ means there exists a *Res*-derivation of C from N.
  $N \models C$ means any model which satisfies N also satisfies C.
  C is true in each model of N; C follows semantically from N
  $N \models \bot$ means N is unsatisfiable, i.e. N has no model.
- *Cal* is said to be sound iff

$$N \vdash_{Cal} C \Rightarrow N \models C.$$

- *Cal* is said to be refutationally complete iff

$$N \models \bot \Rightarrow N \vdash_{Cal} \bot.$$

- In general, a calculus *Cal* is complete iff $N \models F \Rightarrow N \vdash_{Cal} F$.
- We prove that *Res* is sound and refutationally complete.

Previously . . .

- ground terms, ground atoms, ground literals, ground clauses
- Herbrand universe = set of ground terms
- Herbrand interpretation I = set of ground atoms
  - For atoms: $A \in I$ iff $I \models A$
    $A \notin I$ iff $I \not\models A$ iff $I \models \neg A$
- Herbrand's Theorem

Recap: Sound Inference Rule

- An inference rule

$$\frac{F_1 \ \ldots \ F_n}{F}$$

is called sound, if $F_1, \ldots, F_n \models F$,
i.e., if F is a semantic/logical consequence of $F_1 \wedge \ldots \wedge F_n$.

Soundness of Resolution

Property 8

The propositional resolution calculus, *Res*, (resolution on ground clauses) is sound.

Proof: We have to show: $N \vdash_{Res} C \;\Rightarrow\; N \models C$,
It suffices to show that every rule is sound, i.e. for every rule $\frac{C_1 \;\ldots\; C_n}{D}$ we have $C_1, \ldots, C_n \models D$.
For resolution, assume $I \models C \vee A$, $I \models \neg A \vee D$ and show $I \models C \vee D$.
(a) Case $I \models A$: Then $I \models D$, for else $I \not\models \neg A \vee D$. Hence $I \models C \vee D$. (b) Case $I \not\models A$: Then $I \models \neg A$. Since $I \models C \vee A$, $I \models C$ and consequently $I \models C \vee D$.
For factoring, assume $I \models C \vee A \vee A$ and show $I \models C \vee A$. Exercise.

Refutational Completeness of Resolution

- How to show refutational completeness of ground resolution?
- We have to show: $N \models \bot \;\Rightarrow\; N \vdash_{Res} \bot$,
  or equivalently: $N \not\vdash_{Res} \bot \;\Rightarrow\; N$ has a model.
- **Idea:**
  - Suppose that we have computed possibly infinitely many inferences from N (and not derived $\bot$).
  - Order the clauses in the derivation according to some appropriate ordering, inspect the clauses in ascending order, and construct a series of Herbrand interpretations.
  - The limit Herbrand interpretation can be shown to be a model of N.

Defining Clause Orderings

1. We assume that $\succ$ is any fixed ordering on ground atoms that is total and well-founded. (There exist many such orderings, e.g., the length-based ordering on atoms when these are viewed as words over a suitable alphabet.)
2. Extend $\succ$ to an ordering $\succ_L$ on ground literals:

   $$[\neg]A \succ_L [\neg]B, \quad \text{if } A \succ B$$
   $$\neg A \succ_L A$$

   (These are 5 conditions!)
3. Extend $\succ_L$ to an ordering $\succ_C$ on ground clauses:
   Let $\succ_C = (\succ_L)_{\text{mul}}$, the multi-set extension of $\succ_L$.

Notation: $\succ$ also for $\succ_L$ and $\succ_C$.

Example

- Suppose $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$.
- Then:

  $$\neg A_5 \succ A_5 \succ \neg A_4 \succ A_4 \succ \ldots \succ \neg A_0 \succ A_0$$
- And:

  $$\begin{array}{cc} & A_0 \vee A_1 \\ \prec & A_1 \vee A_2 \\ \prec & \neg A_1 \vee A_2 \\ \prec & \neg A_1 \vee A_4 \vee A_3 \\ \prec & \neg A_1 \vee \neg A_4 \vee A_3 \\ \prec & \neg A_5 \vee A_5 \end{array}$$

Exercise

- Suppose $A_4 \succ A_3 \succ A_2 \succ A_1$
- How are these clauses ordered by $\succ_C$?

1. $\neg A_3 \vee A_4$
2. $A_3 \vee A_1 \vee A_1$
3. $\neg A_4 \vee A_2$
4. $A_3 \vee A_1$

Exercise

- Suppose $A_4 \succ A_3 \succ A_2 \succ A_1$
- How are these clauses ordered by $\succ_C$?

1. $\neg A_3 \vee A_4$
2. $A_3 \vee A_1 \vee A_1$
3. $\neg A_4 \vee A_2$
4. $A_3 \vee A_1$

- Ordering of literals:

$$\neg A_4 \succ_L A_4 \succ_L \neg A_3 \succ_L A_3 \succ_L \neg A_2 \succ_L A_2 \succ_L \neg A_1 \succ_L A_1$$

- Ordering of clauses: $3 \succ_C 1 \succ_C 2 \succ_C 4$
  $(4 \prec_C 2 \prec_C 1 \prec_C 3)$

Exercise

- Suppose $A_4 \succ A_3 \succ A_2 \succ A_1$
- How are these clauses ordered by $\succ_C$?

1. $\neg A_3 \vee A_4$
2. $A_3 \vee A_1 \vee A_1$
3. $\neg A_4 \vee A_2$
4. $A_3 \vee A_1$

- Ordering of literals:

$$\neg A_4 \succ_L A_4 \succ_L \neg A_3 \succ_L A_3 \succ_L \neg A_2 \succ_L A_2 \succ_L \neg A_1 \succ_L A_1$$

Properties of Clause Orderings

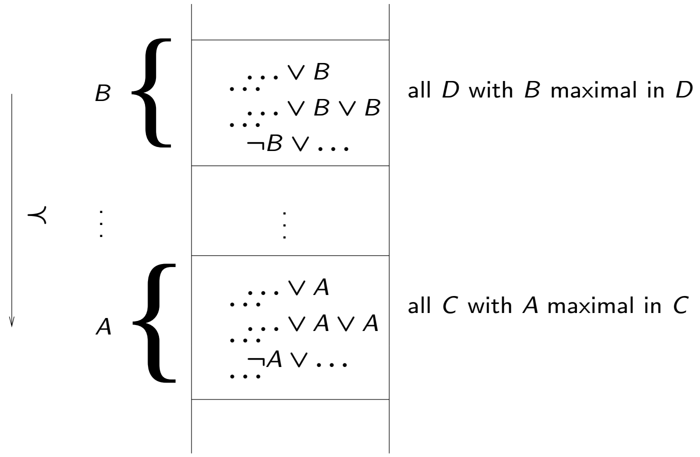
Property 9

1. The orderings ($\succ_L$ and $\succ_C$) on (ground) literals and clauses are total and well-founded.
2. Let C and D be clauses with A an occurrence of a maximal atom in C and B an occurrence of a maximal atom in D.
   (i) If $A \succ B$ then $C \succ D$.
   (ii) If $A = B$ and A occurs negatively in C but only positively in D, then $C \succ D$.

Note: in 2. A and B may be negated or unnegated occurrences.

Stratified Structure of Clause Sets

Let $A \succ B$. Clause sets are then stratified in this form:



Clauses in A-cluster are larger than clauses in B-cluster

Method of Computing All Possible Conclusions

- Define the operators Res, Res^n, Res^*:

$$Res(N) = \{C \mid C \text{ is the conclusion of applying a rule in } Res \text{ to premises in } N\}$$

  Define Res^n inductively by:

$$Res^0(N) = N$$
$$Res^{n+1}(N) = Res^n(N) \cup Res(Res^n(N)), \text{ for } n \geq 0$$
$$Res^*(N) = \bigcup_{n \geq 0} Res^n(N)$$

- $Res(N)$ is the set of 'immediate' resolvents and factors of N (all premises are in N). $Res^*(N)$ is the set of all possible resolvents and factors of N.

Saturation of Clause Sets under *Res*

- N is called saturated (wrt. resolution), if $Res(N) \subseteq N$.
- Method of level saturation (on the previous slide) computes the saturation of a set N as the closure of N which is given by $Res^*(N)$.

Property 10

(i) $Res^*(N)$ is saturated.

(ii) Res is (sound and) refutationally complete iff for each set N of ground clauses:

$$N \models \bot \text{ iff } \bot \in Res^*(N)$$

- Intuition: $\bot \in Res^*(N)$ implies $N \vdash_{Res} \bot$

Summary

- soundness and refutational completeness
- sound rule
- soundness of *Res*
- lifting ordering on ground atoms to ground literals and to ground clauses
- properties of $\succ_L$ and $\succ_C$
- properties of ordered sets of clauses, stratification
- saturated clause set, level saturation

COMP60121: Automated Reasoning II

Lecture 5

Previously ...

- soundness and refutational completeness
- sound rule
- soundness of *Res*
- literal ordering, clause ordering
- properties of ordered clause sets, stratification
- saturated clause set, level saturation

Construction of Herbrand Interpretations

- Our aim is to show the equivalence, where N is any set of ground clauses:

$$N \models \bot \ \text{ iff } \ \bot \in Res^*(N)$$

- The soundness result (Property 8) implies the "$\Leftarrow$" direction.
- We now show the "$\Rightarrow$" direction (i.e. refutational completeness), by showing

  If $\bot \notin Res^*(N)$, then N has a model.

- **Given**: set N of ground clauses, atom ordering $\succ$.
- **Wanted**: Herbrand interpretation I such that
  - "many" clauses from N are true in I, and
  - $I \models N$, if N is saturated and $\bot \notin N$.

Example

Let $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1 \succ A_0$ (strictly maximal literals in *red*)

	clauses C in N	I_C	Δ_C	Remarks
1	$\underline{\neg A_0}$	$\emptyset$	$\emptyset$	true in I_C
2	$A_0 \vee \underline{A_1}$	$\emptyset$	$\{A_1\}$	A_1 str. maximal
3	$A_1 \vee \underline{A_2}$	$\{A_1\}$	$\emptyset$	true in I_C
4	$\neg A_1 \vee \underline{A_2}$	$\{A_1\}$	$\{A_2\}$	A_2 str. maximal
5	$\neg A_1 \vee \underline{A_4} \vee A_3 \vee A_0$	$\{A_1, A_2\}$	$\{A_4\}$	A_4 str. maximal
6	$\neg A_1 \vee \underline{\neg A_4} \vee A_3$	$\{A_1, A_2, A_4\}$	$\emptyset$	A_3 not str. max.
				min. exception
7	$\neg A_1 \vee \underline{A_5}$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	A_5 str. maximal

$I = \{A_1, A_2, A_4, A_5\}$ is not a model of the clause set because there exists an exception (unfulfilled) clause, clause 6.

By definition, an exception clause for I is a clause that is not true in I.

Main Ideas of the Construction

- Approximate (!) description: Define I inductively by:
  - Starting with a minimal clause C in N.
    (Since in the ground case the ordering is total, there is a smallest clause and we start in fact with this clause.)
  - Consider the largest atom in C and attempt to define (in a certain way) $I_C \cup \Delta_C$ (!) as the minimal extension of the partial interpretation constructed so far (I_C) so that C becomes true.
  - Iterate for $N \backslash \{C\}$, and so forth.
- I.e. clauses are considered in the order given by $\prec$.
- When considering C, one already has a partial interpretation I_C available (initially $I_C = \emptyset$).

Main Ideas of the Construction (cont'd)

- If C is true in the partial interpretation I_C, nothing is done ($\Delta_C = \emptyset$).
- If C is false, change I_C such that C becomes true.
- Changes should, however, be monotone. One never deletes anything from I_C and the truth value of any clause smaller than C should be maintained the way it was in I_C.
- Hence, one chooses $\Delta_C = \{A\}$ iff C is false in I_C, and when both
  (i) A occurs positively in C, and
  (ii) this occurrence of A in C is strictly maximal (i.e. largest) in the ordering on literals.
- Note: (i) $\Rightarrow$ adding A will make C become true.
  (ii) $\Rightarrow$ changing the truth value of A has no effect on smaller clauses.

Resolution Reduces Exceptions

$$\frac{\neg A_1 \vee \underline{A_4} \vee A_3 \vee A_0 \qquad \neg A_1 \vee \underline{\neg A_4} \vee A_3}{\neg A_1 \vee \neg A_1 \vee A_3 \vee A_3 \vee A_0}$$

Construction of I for the extended clause set:

clauses C	I_C	Δ_C	Remarks
$\underline{\neg A_0}$	$\emptyset$	$\emptyset$	
$A_0 \vee \underline{A_1}$	$\emptyset$	$\{A_1\}$	
$A_1 \vee \underline{A_2}$	$\{A_1\}$	$\emptyset$	
$\neg A_1 \vee \underline{A_2}$	$\{A_1\}$	$\{A_2\}$	
$\neg A_1 \vee \neg A_1 \vee \underline{A_3} \vee \underline{A_3} \vee A_0$	$\{A_1, A_2\}$	$\emptyset$	A_3 occurs twice *min. exception*
$\neg A_1 \vee \underline{A_4} \vee A_3 \vee A_0$	$\{A_1, A_2\}$	$\{A_4\}$	
$\neg A_1 \vee \underline{\neg A_4} \vee A_3$	$\{A_1, A_2, A_4\}$	$\emptyset$	exception
$\neg A_1 \vee \underline{A_5}$	$\{A_1, A_2, A_4\}$	$\{A_5\}$	

The same I, but smaller exception, hence some progress was made.

Factoring Reduces Exceptions

$$\frac{\neg A_1 \vee \neg A_1 \vee \underline{A_3} \vee \underline{A_3} \vee A_0}{\neg A_1 \vee \neg A_1 \vee \underline{A_3} \vee A_0}$$

Construction of I for the extended clause set:

clauses C	I_C	Δ_C	Remarks
$\underline{\neg A_0}$	$\emptyset$	$\emptyset$	
$A_0 \vee \underline{A_1}$	$\emptyset$	$\{A_1\}$	
$A_1 \vee \underline{A_2}$	$\{A_1\}$	$\emptyset$	
$\neg A_1 \vee \underline{A_2}$	$\{A_1\}$	$\{A_2\}$	
$\neg A_1 \vee \neg A_1 \vee \underline{A_3} \vee A_0$	$\{A_1, A_2\}$	$\{A_3\}$	
$\neg A_1 \vee \neg A_1 \vee \underline{A_3} \vee \underline{A_3} \vee A_0$	$\{A_1, A_2, A_3\}$	$\emptyset$	true in I_C
$\neg A_1 \vee \underline{A_4} \vee A_3 \vee A_0$	$\{A_1, A_2, A_3\}$	$\emptyset$	
$\neg A_1 \vee \underline{\neg A_4} \vee A_3$	$\{A_1, A_2, A_3\}$	$\emptyset$	true in I_C
$\neg A_3 \vee \underline{A_5}$	$\{A_1, A_2, A_3\}$	$\{A_5\}$	

The resulting $I = \{A_1, A_2, A_3, A_5\}$ is a model of the clause set.

Construction of Candidate Models Formally

- Let $N, \succ$ be given. Guided by $\succ$, we define sets I_C and Δ_C for all ground clauses C over the given signature inductively by:

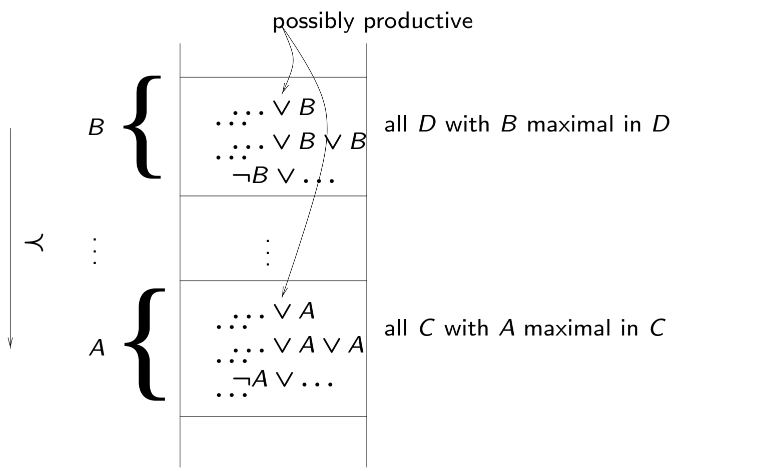
$$I_C := \bigcup_{C \succ D} \Delta_D$$

$$\Delta_C := \begin{cases} \{A\}, & \text{if } C \in N, \quad C = C' \vee A, \\ & A \succ C' \text{ and } I_C \not\models C \\ \emptyset, & \text{otherwise} \end{cases}$$

- We say, C produces A, or just C is productive, if $\Delta_C = \{A\}$.
- The candidate model for N (wrt. $\succ$) is given as $I_N^{\succ} := \bigcup_{C \in N} \Delta_C$.
- We also simply write I_N, or I, for $I_N^{\succ}$, if $\succ$ is either irrelevant or known from the context.

Structure of $(N, \succ)$

Let $A \succ B$; producing a new atom does not affect smaller clauses.



The smallest clauses in each cluster are possibly productive; but not necessarily (particularly if they are already true in I_C).

Some Properties of the Construction

Property 11

(i) $C = \neg A \vee C' \;\Rightarrow\;$ no D s.t. $D \succeq C$ produces A.

(ii) C productive $\;\Rightarrow\; I_C \cup \Delta_C \models C$ and $I_N \models C$.

(iii) Let $D' \succ D \succeq C$. Then

$$I_D \cup \Delta_D \models C \;\Rightarrow\; I_{D'} \cup \Delta_{D'} \models C \text{ and } I_N \models C.$$

If, in addition, $C \in N$ or $B \succ A$, where B and A are maximal atoms in D and C, respectively, then

$$I_D \cup \Delta_D \not\models C \;\Rightarrow\; I_{D'} \cup \Delta_{D'} \not\models C \text{ and } I_N \not\models C.$$

Some Properties of the Construction (cont'd)

(iv) Let $D' \succ D \succ C$. Then

$$I_D \models C \;\Rightarrow\; I_{D'} \models C \text{ and } I_N \models C.$$

If, in addition, $C \in N$ or $B \succ A$, where B and A are maximal atoms in D and C, respectively, then

$$I_D \not\models C \;\Rightarrow\; I_{D'} \not\models C \text{ and } I_N \not\models C.$$

(v) $C = C' \vee A$ produces $A \;\Rightarrow\; I_N \not\models C'$.

Model Existence Theorem

Property 12 (Bachmair, Ganzinger 1990)
Let $\succ$ be a clause ordering, let N be saturated wrt. *Res*, and suppose that $\bot \notin N$. Then

$$I_N^{\succ} \models N.$$

Corollary 13
Let N be saturated wrt. *Res*. Then

$$N \models \bot \text{ iff } \bot \in N.$$

Corollary 14
Res is refutationally complete.

Model Existence Theorem (cont'd)

Proof of Property 12:
Suppose $\bot \notin N$, but $I_N \not\models N$. (NB: $I_N = I_N^{\succ}$)
Let $C \in N$ be minimal (wrt. $\succ$) such that $I_N \not\models C$.
Since C is false in I_N, C is not productive.
As $C \neq \bot$, there exists a maximal atom A in C.

Case 1: $C = \neg A \vee C'$ (i.e., the maximal atom occurs negatively)
$\Rightarrow$ $I_N \not\models \neg A$ and $I_N \not\models C'$ $\Rightarrow$ $I_N \models A$ and $I_N \not\models C'$
$\Rightarrow$ some $D = D' \vee A \in N$ produces A. As $\frac{D' \vee A \qquad \neg A \vee C'}{D' \vee C'}$, we infer that $D' \vee C' \in N$, and $C \succ D' \vee C'$ and $I_N \not\models D' \vee C'$
$\Rightarrow$ contradicts minimality of C.

Case 2: $C = C' \vee A \vee A$. Then $I_N \not\models A$ and $I_N \not\models C'$. Then $\frac{C' \vee A \vee A}{C' \vee A}$ yields a smaller exception $C' \vee A \in N$.
$\Rightarrow$ contradicts minimality of C.

Summary

- Refutational completeness of *Res*
- Model construction
  - Given: Set N of ground clauses; atom ordering $\succ$
  - Output: Candidate model $I_N^{\succ}$
- Model Existence Theorem

- productive clause
- minimal exceptions

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Lecture 6

Previously ...

- Basic resolution calculus for ground clauses
- Soundness
- Refutational completeness
- Model existence theorem
- Model construction, guided by $\succ$

Ordered Resolution with Well-Behaved Selection

- Motivation: Search space for *Res* is very large.
- Ideas for improvement:
  - In the completeness proof (Model Existence Theorem, Pty 12) one only needs to resolve upon and factor maximal atoms
    - $\Rightarrow$ if the calculus is restricted to inferences involving maximal atoms, the proof remains correct
    - $\Rightarrow$ ordering restrictions
  - In the generalised completeness proof, it does not really matter with which negative literal an inference is performed
    - $\Rightarrow$ choose a negative literal don't-care-nondeterministically
    - $\Rightarrow$ well-behaved selection

Selection Functions

- **Note:** Since in Part II we are exclusively interested in well-behaved selection, we drop the reference 'well-behaved' and talk only about 'selection'. Thus, 'selection' in this part is not the same as in Part I.
- A selection function is a mapping

$$S : C \quad \mapsto \quad \text{(multi-)set of occurrences of negative literals in } C$$

- Example of selection with selected literals indicated as $\boxed{L}$:

$$\boxed{\neg A} \vee \neg A \vee B$$

$$\boxed{\neg B_0} \vee \boxed{\neg B_1} \vee A$$

Maximality wrt ground & non-ground clauses

- In the completeness proof for the ground calculus, we talk about (strictly) maximal literals of ground clauses.
- General refutational completeness can be proved using refutational completeness of ground resolution, Herbrand's Theorem and
  - the 'lifting lemma': every ground refutation of a ground instance of N can be mapped step-wise to a non-ground refutation of N.

  (See optional material on website, for details about refutational completeness of general resolution.)
- Fact: In the non-ground calculus, we have to consider those literals that correspond to (strictly) maximal literals of ground instances.

Maximal and strictly maximal literals

- Let $\succ$ be a total and well-founded ordering on ground atoms.
- A ground literal L is called maximal wrt. a ground clause C iff for all L' in C: $L \succeq L'$.
- A ground literal L is called strictly maximal wrt. a ground clause C iff for all L' in C: $L \succ L'$.
- A non-ground literal L is [strictly] maximal wrt. a clause C iff there exists a ground substitution σ such that for all L' in C: $L\sigma \succeq L'\sigma$ $[L\sigma \succ L'\sigma]$.
- If L is [strictly] maximal wrt. a clause C then we say that L is [strictly] maximal in $L \vee C$.
- Notation: maximal literals indicated by $\underline{L}$

Resolution Calculus $Res_S^{\succ}$

- Let $\succ$ be an atom ordering and S a selection function.
- **General ordered resolution calculus with selection $Res_S^{\succ}$:**

$$\frac{C \vee A \qquad \neg B \vee D}{(C \vee D)\sigma} \qquad \text{(ordered resolution with selection)}$$

provided $\sigma = \mathrm{mgu}(A, B)$ and

(i) $A\sigma$ strictly maximal wrt. $C\sigma$;

(ii) nothing is selected in C by S;

(iii) either $\neg B$ is selected,
or else nothing is selected in $\neg B \vee D$ and $\neg B\sigma$ is maximal wrt. $D\sigma$.

- Note, variable standardisation needs to be applied to the premises before applying resolution.

Resolution Calculus $Res_S^{\succ}$ (cont'd)

- Ordered factoring rule:

$$\frac{C \vee A \vee B}{(C \vee A)\sigma} \qquad \text{(ordered factoring)}$$

provided $\sigma = \mathrm{mgu}(A, B)$ and

(i) $A\sigma$ is maximal wrt. $C\sigma$ and

(ii) nothing is selected in C.

- Idea:
  - Inferences restricted to $\succ$-maximal or S-selected literals
  - S overrides $\succ$

Special Instance: $Res_S^{\succ}$ for Propositional Logic

- For propositional and ground clauses the resolution inference simplifies to

$$\frac{C \vee A \qquad \neg A \vee D}{C \vee D}$$

provided

(i) A is strictly maximal wrt. C, i.e. $A \succ C$;

(ii) nothing is selected in C by S;

(iii) $\neg A$ is selected in $\neg A \vee D$,
or else nothing is selected in $\neg A \vee D$ and $\neg A$ is max. wrt. D

Special Instance: $Res_S^{\succ}$ for Propositional Logic (cont'd)

- Ordered factoring:

$$\frac{C \vee A \vee A}{C \vee A}$$

provided

(i) A is maximal wrt. C and

(ii) nothing is selected in C.

Search Spaces Become Smaller

- Example:

1.	$\underline{A} \vee B$	given
2.	$\underline{A} \vee \boxed{\neg B}$	given
3.	$\underline{\neg A} \vee B$	given
4.	$\underline{\neg A} \vee \boxed{\neg B}$	given
5.	$\underline{B} \vee \underline{B}$	Res 1, 3
6.	$\underline{B}$	Fact 5
7.	$\underline{\neg A}$	Res 6, 4
8.	$\underline{A}$	Res 6, 2
9.	$\bot$	Res 8, 7

we assume $A \succ B$ and S as indicated by $\boxed{L}$. The maximal literal in a clause is depicted in *red*.

- With this ordering and selection function the refutation proceeds strictly deterministically in this example. Generally, proof search will still be non-deterministic but the search space will be much smaller than with unrestricted resolution.

Exercise

Consider the following set N of clauses.

1. $\neg P(x) \vee P(f(x))$
2. $P(a)$

(i) Give a derivation for it under unrestricted resolution.

(ii) Define an ordering or selection function, or both, so that no inference is performed on N.

Exercise

Consider the following set N of clauses.

1. $\neg P(x) \vee P(f(x))$
2. $P(a)$

(i) Give a derivation for it under unrestricted resolution.

1.	$\neg P(x) \vee P(f(x))$	given
2.	$P(a)$	given
3.	$P(f(a))$	(1,2)
4.	$P(f(a))$	(1,3)
5.	$P(f(f(a)))$	(1,4)
	$\vdots$	

(ii) Define an ordering or selection function, or both, so that no inference is performed on N.

Exercise

Consider the following set N of clauses.

1. $\neg P(x) \vee P(f(x))$
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(i) Give a derivation for it under unrestricted resolution.

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4.	$P(f(a))$	(1,3)
5.	$P(f(f(a)))$	(1,4)
	$\vdots$	

(ii) Define an ordering or selection function, or both, so that no inference is performed on N.

1. $\neg P(x) \vee \underline{P(f(x))}$
2. $\underline{P(a)}$

Don't select any literals and use an ordering under which $P(f(x))$ is strictly maximal wrt. $P(x)$

Avoiding Rotation Redundancy

- From

$$\frac{\dfrac{C_1 \vee A \qquad C_2 \vee \neg A \vee B}{C_1 \vee C_2 \vee B} \qquad C_3 \vee \neg B}{C_1 \vee C_2 \vee C_3}$$

we can obtain by rotation

$$\frac{C_1 \vee A \qquad \dfrac{C_2 \vee \neg A \vee B \qquad C_3 \vee \neg B}{C_2 \vee \neg A \vee C_3}}{C_1 \vee C_2 \vee C_3}$$

another proof of the same clause.

- In large proofs many rotations are possible.
- However, if $A \succ B$, then the second proof does not fulfill the ordering restrictions.

Avoiding Rotation Redundancy (cont'd)

- **Conclusion**:
  In the presence of ordering restrictions (however one chooses $\succ$) no rotations are possible. In other words, orderings identify exactly one representative in any class of rotation-equivalent proofs.

Soundness and Refutational Completeness

Property 15

Let $\succ$ be an atom ordering and S a selection function such that $Res_S^{\succ}(N) \subseteq N$, i.e. N is saturated (wrt. $Res_S^{\succ}$). Then

$$N \models \bot \text{ iff } \bot \in N$$

Proof:

The "$\Leftarrow$" part is trivial. For the "$\Rightarrow$" part consider the propositional level only: Construct a candidate model $I_N^{\succ}$ as for unrestricted resolution, except that clauses C in N that have selected literals are not productive, even when they are false in I_C and when their maximal atom occurs only once and positively.

Summary

- ground and non-ground maximality of literal wrt. clause
- ordered resolution with selection $Res_S^{\succ}$
  - inferences limited by ordering $\succ$
  - inferences limited by selection function S
- soundness and refutational completeness

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Lecture 7

Previously ...

- Ways of limiting inferences & enhancing efficiency
  - ordering restriction $\succ$
  - selection function S
- Idea:
  - Inferences restricted to $\succ$-maximal or S-selected literals
  - S overrides $\succ$
- ground and non-ground maximality of literal wrt. clause

Application: Compactness of Propositional Logic

- A theoretical application of the refutational completeness of Res is compactness of propositional logic.
- Observe: If $N \vdash_{Cal} F$ then
  there exist $F_1, \dots, F_n \in N$ s.t. $F_1, \dots, F_n \vdash_{Cal} F$
  If $N \vdash_{Res} \bot$ then
  there exist $C_1, \dots, C_n \in N$ s.t. $C_1, \dots, C_n \vdash_{Res} \bot$
  (resembles compactness).

  Note, N can be an infinite set of formulae.
- Recall

$$N \text{ is unsatisfiable iff for any interpretation } I,\ I \not\models N.$$

$$I \not\models N \text{ iff for some } C \in N,\ I \not\models C$$